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INVERSE THEOREM FOR BEST POLYNOMIAL APPROXIMATION IN L_p, $0 < p < 1$

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ABSTRACT. A direct theorem for best polynomial approximation of a function in $L_p[-1, 1]$, $0 < p < 1$, has recently been established. Here we present a matching inverse theorem. In particular, we obtain as a corollary the equivalence for $0 < \alpha < k$ between $E_n(f)_p = O(n^{-\alpha})$ and $\omega_\varphi^k(f, t)_p = O(t^\alpha)$. The present result complements the known direct and inverse theorem for best polynomial approximation in $L_p[-1, 1]$, $1 \le p \le \infty$. Analogous results for approximating periodic functions by trigonometric polynomials in $L_p[-\pi, \pi]$, $0 < p \le \infty$, are known.

1. INTRODUCTION

The rate of the best polynomial approximation in $L_p[-1, 1]$ is defined by

$$E_n(f)_p := \inf_{P_n \in \Pi_n} \|f - P_n\|_p, \qquad 0 < p \le \infty, \tag{1.1}$$

where Π_n is the set of polynomials of degree n and $\|g\|_p := (\int_{-1}^1 |g(x)|^p\,dx)^{1/p}$, $0 < p < \infty$. (Note that $\|g\|_p$ is not a norm when $0 < p < 1$.) We believe that for estimating $E_n(f)_p$ the measure of smoothness $\omega_\varphi^k(f, t)_p$ introduced by Ditzian and Totik [2] is the appropriate tool. Recall that

$$\omega_\varphi^k(f, t)_p := \sup_{0<h\le t} \left(\int_{-1}^t |\Delta_{h\varphi(x)}^k f(x)|^p\,dx \right)^{1/p} \tag{1.2}$$

where
(1.3)

$$\Delta_{h\varphi(x)}^k f(x) := \begin{cases} \sum_{i=0}^k (-1)^i \binom{k}{i} f\left(x + \left(\frac{k}{2} - i\right) h\varphi(x)\right), & x \pm \frac{k}{2} h\varphi(x) \in [-1, 1], \\ 0 & \text{otherwise}. \end{cases}$$

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The direct result for $E_n(f)_p$ is given by

$$E_n(f)_p \le C(p)\omega_\varphi^k(f, 1/n)_p, \qquad \varphi(x) := \sqrt{1-x^2}. \tag{1.4}$$

For $1 \le p \le \infty$ (1.4) was proved by Ditzian and Totik [2, Chapter 7], and for $0 < p < 1$ it has recently been proved by DeVore, Leviatan, and Yu [1]. For $1 \le p \le \infty$ the direct result (1.4) has a matching inverse result (see [2, Chapter 7]). It is always important to have a matching inverse result for a known direct result. Recently, Tachev [8, 9] has proved a direct and inverse result for $E_n(f)_p$, $0 < p < 1$, using the measure of smoothness $\tau_k(f, \Delta_n(x))_{p,p}$, which was introduced by K. Ivanov. Since for $0 < p < 1$ the relation between $\tau_k(f, \Delta_n(x))_{p,p}$ and $\omega_\varphi^k(f, 1/n)_p$ is not known and since it is our belief that the expression $\omega_\varphi^k(f, t)_p$ is of simpler character, we present here an inverse result to match (1.4).

In most cases, inverse theorems are proved making use of the equivalence between the K-functional

$$K_r(f, t^r) := \inf_{g^{(r)} \in X} (\|f-g\|_X + t^r \|g^{(r)}\|_X)$$

and the modulus of smoothness $\omega^r(f, t)_X$. This equivalence is not valid for the space $X = L_p[-1, 1]$ when $0 < p < 1$. (For a discussion of the pathological behavior and phenomena in this space we refer the reader to the paper by Peetre [7] and especially §6 therein.) Therefore $\omega_\varphi^k(f, t)_p$ cannot be equivalent to the appropriate K-functional when $0 < p < 1$. This adds to the interest one has in the following inverse theorem.

Theorem 1.1. *For $f \in L_p[-1, 1]$, $0 < p < 1$, we have*

$$\omega_\varphi^k(f, t)_p \le Ct^k \left(\sum_{0 \le n \le 1/t} (n+1)^{kp-1} E_n(f)_p^p \right)^{1/p} \tag{1.5}$$

and hence

$$E_n(f)_p = O(n^{-\alpha}) \Leftrightarrow \omega_\varphi^k(f, t)_p = O(t^\alpha) \tag{1.6}$$

for $0 < \alpha < k$.

Note that Tachev [9] proved the analogue of (1.5) with $\tau_k(f, \Delta_n(x))_{p,p}$ and that this note is influenced by his work.

2. Some preliminary results

In this section we will prove a few lemmas crucial for obtaining our main result, that is, Theorem 1.1.

Lemma 2.1. *For $f \in L_p[-1, 1]$, $0 < p < 1$,*

$$\omega_\varphi^k(f, t)_p \le C(k)\|f\|_p. \tag{2.1}$$

Proof. Using $|f|^p \in L_1[-1, 1]$, (2.1) follows from the inequality

$$\int |f(x + \nu h\varphi(x))|^p \, dx \le M \int |f(x)|^p \, dx, \qquad \nu = \frac{k}{2} - j, \; j = 0, \dots, k$$

(cf. [2, p. 21]). □

Lemma 2.2. *For* $P_n \in \Pi_n$, $1 \le i \le n$, *and* $\varphi(x) = \sqrt{1-x^2}$ *we have*

$$\|\varphi^i P_n^{(i)}\|_p \le (C(p))^i i! n^i \|P_n\|_p \tag{2.2}$$

where $C(p)$ *is independent of* i *and* n.

As far as we know Lemma 2.2 is due to Tachev [9, Lemma 4]. As Tachev's proof is inaccessible we give a short proof.

Proof. By induction, it suffices to show that

$$\|\varphi^{j+1} Q_n'\|_p \le C(p) n (j+1) \|\varphi^j Q_n\|_p \tag{2.3}$$

for $Q_n \in \pi_n$ and $0 \le j < n$. To estimate $\varphi^{j+1} Q_n'$, we write

$$\varphi(x)^{2[j/2]} Q_n'(x) = (\varphi(x)^{2[j/2]} Q_n)' + [j/2] 2x \varphi(x)^{2([j/2]-1)} Q_n(x)$$

(note that $[j/2] = 0$ when $j < 2$). We apply [6, Theorem 5] substituting there $W_n(x) = 1$, $W(x) = \varphi(x)^{j-2[j/2]}$, and $\pi_{n+2[j/2]} = \varphi^{2[j/2]} Q_n$ to obtain

$$\|(\varphi^{2[j/2]} Q_n)' \varphi^{j-2[j/2]+1}\|_p \le C_1(p)(n + [j/2]) \|\varphi^j Q_n\|_p . \tag{2.4}$$

Observe that $n + 2[j/2] \le 2n$ and $C_1(p)$ depends on W (which is either 1 or φ) but not on j. We now estimate $[j/2] 2x \varphi(x)^{j-1} Q_n$ using [6, Lemma 3] with $\alpha = \beta = (j - 2[j/2])p$, $\gamma = 0$, and with $\pi_m = \varphi^{2[j/2]-2} Q_n$ $(m = n + 2[j/2] - 2 \le 2n)$ to obtain for some fixed $\delta > 0$

$$\begin{aligned} j \|\varphi^{j-1} Q_n\|_p &= j \|(\varphi^{2[j/2]-2} Q_n) \varphi^{j-2[j/2]+1}\|_p \\ &\le 2j \|\varphi^{j-1} Q_n\|_{L_p[-1+\delta/n^2,\ 1-\delta/n^2]} \le \frac{2jn}{\sqrt{\delta}} \|\varphi^j Q_n\|_p \end{aligned}$$

for $j \ge 2$. □

Remark. Lemma 2.2 can be deduced from the paper of Hille, Szegö, and Tamarkin [5, p. 731], but Nevai's explicit results [6] are more amenable to a short proof. It was indicated to us by Nevai that we could have used Remez type and Markov-Bernstein type inequalities for generalized polynomials which are proved in a forthcoming paper of Erdélyi, Máté, and Nevai [4]. We decided against that in order not to have to introduce the new concept of generalized polynomials with which not many people are familiar.

Lemma 2.3. *For* $P_n \in \Pi_n$, $k = 1, 2, \ldots$, *and* $0 < p < 1$, *we have*

$$\omega_\varphi^k(P_n, t)_p \le C(nt)^k \|P_n\|_p \tag{2.5}$$

where $C = C(p, k)$.

Proof. In view of Lemma 2.1 applied to $f = P_n$, we have to show (2.5) only for $0 \le nt \le L$. Using the Taylor series of P_n and the identity $\sum_{i=0}^k \binom{k}{i} (-1)^i i^j = 0$, $0 \le j < k$, we have

$$\begin{aligned} \Delta_{h\varphi}^k P_n(x) &= \sum_{i=0}^k \binom{k}{i} (-1)^i P_n\left(x + \left(\frac{k}{2} - i\right) h\varphi(x)\right) \\ &= \sum_{i=0}^k \binom{k}{i} (-1)^i \sum_{j=k}^n \frac{(k/2 - i)^j h^j}{j!} \varphi^j(x) P_n^{(j)}(x) . \end{aligned}$$

Hence, by (2.2) and the choice $L = 1/kC(p)$, we have

$$\begin{aligned}\|\Delta_{h\varphi}^k P_n\|_p^p &\le \sum_{i=0}^{k}\binom{k}{i}^p \sum_{j=k}^{n} \frac{(|k/2-i|h)^{jp}}{(j!)^p}\|\varphi^i P_n^{(j)}\|_p^p \\ &\le C_1 \sum_{j=k}^{n} \frac{(kh/2)^{jp}}{(j!)^p} C(p)^{jp}(j!)^p n^{jp}\|P_n\|_p^p \\ &\le C_1\|P_n\|_p^p (nh)^{kp}\left(\frac{k}{2}C(p)\right)^{kp}\sum_{j=k}^{n}\left(\frac{k}{2}hC(p)n\right)^{(j-k)p} \\ &\le C_2(nk)^{kp}\|P_n\|_p^p .\end{aligned}$$

In view of the definition of $\omega_\varphi^k(f, t)_p$, this completes the proof of (2.5). □

3. Proof of the inverse result

Proof of Theorem 1.1. Let $P_n \in \Pi_n$ be a polynomial of best approximation of f. For $t > 0$ define $l = l(t)$ by $2^l \le t < 2^{l+1}$. Using Lemma 2.1 we have

$$\begin{aligned}\omega_\varphi^k(f, t)_p^p &\le \omega_\varphi^k(f, 2^{-l})_p^p \\ &\le \omega_\varphi^k(f - P_{2^l}, 2^{-l})_p^p + \omega_\varphi^k(P_{2^l}, 2^{-l})_p^p \\ &\le CE_{2^l}(f)_p + \omega_\varphi^k(P_{2^l}, 2^{-l})_p^p .\end{aligned} \tag{3.1}$$

With the understanding $P_{2^{-1}} := P_0$, we can use Lemma 2.3 to obtain

$$\begin{aligned}\omega_\varphi^k(P_{2^l}, 2^{-l})_p^p &\le \sum_{i=0}^{l}\omega_\varphi^k(P_{2^i} - P_{2^{i-1}}, 2^{-l})_p^p \\ &\le C\sum_{i=0}^{l}(2^{i-l})^{kp}\|P_{2^i} - P_{2^{i-1}}\|_p^p \\ &\le C_1 2^{-lkp}\sum_{i=-1}^{l-1} 2^{ikp}E_{2^i}(f)_p^p \qquad (E_{2^{-1}}(f)_p := E_0(f)_p) .\end{aligned} \tag{3.2}$$

Since $\sum_{i=-1}^{\infty} 2^{ikp}E_{2^i}(f)_p^p$ is equivalent to the right-hand side of (1.5), inequalities (3.1) and (3.2) together with $2^{-lkp} \sim t^{kp}$ complete the proof of our theorem. □

4. Existence and estimates of $f^{(k)}$

Following results for $p \ge 1$ (see [3, Theorem 6.2]) we can prove

Theorem 4.1. *Suppose* $f \in L_p[-1, 1]$, $0 < p < 1$, *and* $\sum_{n=0}^{\infty}(n+1)^{kp-1}E_n(f)_p^p < \infty$ *for some positive integer* k. *Then* $f^{(k)} \in L_p[-1, 1]$ *and*

$$\|\varphi^k(f^{(k)} - P_n^{(k)})\|_p \le M\left(\sum_{m\ge n}(m+1)^{kp-1}E_m(f)_p^p\right)^{1/p} . \tag{4.1}$$

Proof. Suppose $P_m \in \Pi_m$ is the best approximant to f in $L_p[-1, 1]$. By virtue of (1.4) we may write $\sum_{i=1}^{\infty}(P_{2^i n} - P_{2^{i-1}n}) = f - P_n$. By Lemma 2.2, we write

$$\left\| \varphi^k \left(\sum_{i=1}^{\infty} (P_{2^i n} - P_{2^{i-1}n}) \right)^{(k)} \right\|_p^p \leq C \sum_{i=1}^{\infty} (2^i n)^{kp} \| P_{2^i n} - P_{2^{i-1}n} \|_p^p$$
$$\leq C_1 \sum_{i=0}^{\infty} (2^i n)^{kp} E_{2^i n}(f)_p^p$$
$$\leq C_2 \sum_{m \geq n} (m+1)^{kp-1} E_m(f)_p^p,$$

which completes the proof. □

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