INVERSE THEOREM FOR BEST POLYNOMIAL APPROXIMATION IN $L_p$, $0 < p < 1$

Z. DITZIAN, D. JIANG, AND D. LEVIATAN

(Communicated by J. Marshall Ash)

Abstract. A direct theorem for best polynomial approximation of a function in $L_p[-1, 1]$, $0 < p < 1$, has recently been established. Here we present a matching inverse theorem. In particular, we obtain as a corollary the equivalence for $0 < a < k$ between $E_n(f)_p = O(n^{-a})$ and $\omega^k_p(f, t)_p = O(t^a)$. The present result complements the known direct and inverse theorem for best polynomial approximation in $L_p[-1, 1]$, $1 \leq p \leq \infty$. Analogous results for approximating periodic functions by trigonometric polynomials in $L_p[-\pi, \pi]$, $0 < p \leq \infty$, are known.

1. Introduction

The rate of the best polynomial approximation in $L_p[-1, 1]$ is defined by

$$(1.1) \quad E_n(f)_p := \inf_{P_n \in \Pi_n} \|f - P_n\|_p, \quad 0 < p \leq \infty,$$

where $\Pi_n$ is the set of polynomials of degree $n$ and $\|g\|_p := (\int_{-1}^{1} |g(x)|^p \, dx)^{1/p}$, $0 < p < \infty$. (Note that $\|g\|_p$ is not a norm when $0 < p < 1$.) We believe that for estimating $E_n(f)_p$ the measure of smoothness $\omega^k_p(f, t)_p$ introduced by Ditzian and Totik [2] is the appropriate tool. Recall that

$$(1.2) \quad \omega^k_p(f, t)_p := \sup_{0 < h \leq t} \left( \int_{-1}^{1} |\Delta^k_{h\varphi(x)}f(x)|^p \, dx \right)^{1/p}$$

where

$$\Delta^k_{h\varphi(x)}f(x) := \begin{cases} \sum_{i=0}^{k} (-1)^i \binom{k}{i} f(x + \frac{k}{2} - i) h\varphi(x), & x + \frac{k}{2} h\varphi(x) \in [-1, i], \\ 0 & \text{otherwise.} \end{cases}$$

Received by the editors January 22, 1992 and, in revised form, April 24, 1992.

1991 Mathematics Subject Classification. Primary 41A27, 41A50.

Key words and phrases. Inverse theorems, best polynomial approximation, $L_p$ spaces, $0 < p < 1$.

The first author was supported by NSERC Grant A4816 of Canada.

The third author was supported by Binational Science Foundation Grant 89-00505.

© 1993 American Mathematical Society

0002-9939/93 $1.00 + .25 per page
The direct result for $E_n(f)_p$ is given by

(1.4) \[ E_n(f)_p \leq C(p)\omega^k_p(f, 1/n)_p, \quad \phi(x) := \sqrt{1 - x^2}. \]

For $1 \leq p \leq \infty$ (1.4) was proved by Ditzian and Totik [2, Chapter 7], and for $0 < p < 1$ it has recently been proved by DeVore, Leviatan, and Yu [1]. For $1 \leq p \leq \infty$ the direct result (1.4) has a matching inverse result (see [2, Chapter 7]). It is always important to have a matching inverse result for a known direct result. Recently, Tachev [8, 9] has proved a direct and inverse result for $E_n(f)_p$, $0 < p < 1$, using the measure of smoothness $\tau_k(f, \Delta_n(x))_p, \rho$, which was introduced by K. Ivanov. Since for $0 < p < 1$ the relation between $\tau_k(f, \Delta_n(x))_p, \rho$ and $\omega^k_p(f, 1/n)_p$ is not known and since it is our belief that the expression $\omega^k_p(f, t)_p$ is of simpler character, we present here an inverse result to match (1.4).

In most cases, inverse theorems are proved making use of the equivalence between the $K$-functional

\[ K_r(f, t') := \inf_{g^{(r)} \in X} (\|f - g\|_X + t'\|g^{(r)}\|_X) \]

and the modulus of smoothness $\omega^r(f, t)_X$. This equivalence is not valid for the space $X = L_p[-1, 1]$ when $0 < p < 1$. (For a discussion of the pathological behavior and phenomena in this space we refer the reader to the paper by Peetre [7] and especially §6 therein.) Therefore $\omega^k_p(f, t)_p$ cannot be equivalent to the appropriate $K$-functional when $0 < p < 1$. This adds to the interest one has in the following inverse theorem.

**Theorem 1.1.** For $f \in L_p[-1, 1]$, $0 < p < 1$, we have

(1.5) \[ \omega^k_p(f, t)_p \leq C t^k \left( \sum_{0 \leq n \leq 1/t} (n + 1)^{kp-1} E_n(f)_p^p \right)^{1/p} \]

and hence

(1.6) \[ E_n(f)_p = O(n^{-\alpha}) \iff \omega^k_p(f, t)_p = O(t^\alpha) \]

for $0 < \alpha < k$.

Note that Tachev [9] proved the analogue of (1.5) with $\tau_k(f, \Delta_n(x))_p, \rho$ and that this note is influenced by his work.

2. Some preliminary results

In this section we will prove a few lemmas crucial for obtaining our main result, that is, Theorem 1.1.

**Lemma 2.1.** For $f \in L_p[-1, 1]$, $0 < p < 1$,

(2.1) \[ \omega^k_p(f, t)_p \leq C(k)\|f\|_p. \]

**Proof.** Using $|f|^p \in L_1[-1, 1]$, (2.1) follows from the inequality

\[ \int |f(x + \nu h \phi(x))|^p \, dx \leq M \int |f(x)|^p \, dx, \quad \nu = \frac{k}{2} - j, \quad j = 0, \ldots, k \]

(cf. [2, p. 21]). $\square$
Lemma 2.2. For $P_n \in \Pi_n$, $1 \leq i \leq n$, and $\varphi(x) = \sqrt{1 - x^2}$ we have
\begin{equation}
\|\varphi^i P_n^{(i)}\|_p \leq (C(p))^{\frac{i!}{n!}} \|P_n\|_p
\end{equation}
where $C(p)$ is independent of $i$ and $n$.

As far as we know Lemma 2.2 is due to Tachev [9, Lemma 4]. As Tachev’s proof is inaccessible we give a short proof.

Proof. By induction, it suffices to show that
\begin{equation}
\|\varphi^{j+1} Q_n^i\|_p \leq C(p) n(j + 1) \|\varphi^j Q_n\|_p
\end{equation}
for $Q_n \in \pi_n$ and $0 \leq j < n$. To estimate $\varphi^{j+1} Q_n^i$, we write
\[ \varphi(x)^{2[j/2]} Q_n^i(x) = (\varphi(x)^{2[j/2]} Q_n)^i + [j/2] 2x \varphi(x)^{2[j/2]-1} Q_n(x) \]
(note that $[j/2] = 0$ when $j < 2$). We apply [6, Theorem 5] substituting there $W_n(x) = 1$, $W(x) = \varphi(x)^{2[j/2]}$, and $\pi_{n+2[j/2]} = \varphi^{2[j/2]} Q_n$ to obtain
\begin{equation}
\|\varphi^{2[j/2]} Q_n^i \varphi^{-2[j/2]+1}\|_p \leq C_1(p) (n + [j/2]) \|\varphi^j Q_n\|_p.
\end{equation}

Observe that $n + 2[j/2] \leq 2n$ and $C_1(p)$ depends on $W$ (which is either 1 or $\varphi$) but not on $j$. We now estimate $[j/2] 2x \varphi(x)^{j-1} Q_n$ using [6, Lemma 3] with $\alpha = \beta = (j - 2[j/2])$, $\gamma = 0$, and with $\pi_m = \varphi^{2[j/2]-2} Q_n$ ($m = n + 2[j/2] - 2 \leq 2n$) to obtain for some fixed $\delta > 0$
\[ j \|\varphi^{-j} Q_n\|_p = j \|\varphi^{2[j/2]-2} Q_n \varphi^{-2[j/2]+1}\|_p \]
\[ \leq 2 j \|\varphi^{-j} Q_n\|_{L_p[-1+\delta/n^2, 1-\delta/n^2]} \leq \frac{2j}{\sqrt{\delta}} \|\varphi Q_n\|_p \]
for $j \geq 2$. □

Remark. Lemma 2.2 can be deduced from the paper of Hille, Szegö, and Tamarkin [5, p. 731], but Nevai’s explicit results [6] are more amenable to a short proof. It was indicated to us by Nevai that we could have used Remez type and Markov-Bernstein type inequalities for generalized polynomials which are proved in a forthcoming paper of Erdélyi, Máté, and Nevai [4]. We decided against that in order not to have to introduce the new concept of generalized polynomials with which not many people are familiar.

Lemma 2.3. For $P_n \in \Pi_n$, $k = 1, 2, \ldots$, and $0 < p < 1$, we have
\begin{equation}
\omega_k^p(P_n, t)_p \leq C(nt)^k \|P_n\|_p
\end{equation}
where $C = C(p, k)$.

Proof. In view of Lemma 2.1 applied to $f = P_n$, we have to show (2.5) only for $0 \leq nt \leq L$. Using the Taylor series of $P_n$ and the identity $\sum_{i=0}^k \binom{k}{i} (-1)^i i^j = 0$, $0 \leq j < k$, we have
\[ \Delta_{h\varphi}^k P_n(x) = \sum_{i=0}^k \binom{k}{i} (-1)^i \frac{h^i}{i!} \varphi^i(x) P_n^{(i)}(x) = \sum_{i=0}^k \binom{k}{i} (-1)^i \sum_{j=k}^n \binom{k/2 - i}{j} \frac{h^j}{j!} \varphi^j(x) P_n^{(j)}(x). \]
Hence, by (2.2) and the choice \( L = 1/kC(p) \), we have

\[
\|\Delta h P_n\|_p \leq \sum_{i=0}^{k} \binom{k}{i} p \sum_{j=k}^{n} \frac{(|k/2 - i|h)^{j \rho}}{(j!)^{\rho}} \|\phi^i P_n^{(j)}\|_p
\]

\[
\leq C_1 \sum_{j=k}^{n} \frac{(kh/2)^{j \rho}}{(j!)^{\rho}} C(p)^{j \rho} (j!)^{\rho} n^{j \rho} \|P_n\|_p
\]

\[
\leq C_1 \|P_n\|_p^p \left( \frac{k}{2} C(p) \right)^{k \rho} \sum_{j=k}^{n} \left( \frac{k}{2} h C(p) n \right)^{(j-k) \rho}
\]

\[
\leq C_2 (nk)^{k \rho} \|P_n\|_p^p.
\]

In view of the definition of \( \omega^k(t, p) \), this completes the proof of (2.5). \( \square \)

3. Proof of the inverse result

Proof of Theorem 1.1. Let \( P_n \in \Pi_n \) be a polynomial of best approximation of \( f \). For \( t > 0 \) define \( l = l(t) \) by \( 2^l \leq t < 2^{l+1} \). Using Lemma 2.1 we have

\[
\omega^k(f, t, p) \leq \omega^k(f, 2^{-l}, p)
\]

(3.1) \[
\leq \omega^k(f - P_{2^l}, 2^{-l}, p + \omega^k(P_{2^l}, 2^{-l}, p)
\]

\[
\leq CE_2(f)_p + \omega^k(P_{2^l}, 2^{-l}, p).
\]

With the understanding \( P_{2^{-l}} := P_0 \), we can use Lemma 2.3 to obtain

\[
\omega^k(P_{2^l}, 2^{-l}, p) \leq \sum_{i=0}^{l} \omega^k(P_{2^l} - P_{2^{-l}}, 2^{-l}, p)
\]

(3.2) \[
\leq C \sum_{i=0}^{l} \left( 2^{-i-l} \right)^{kp} \|P_{2^l} - P_{2^{-l}}\|_p
\]

\[
\leq C_1 2^{-k \rho} \sum_{i=1}^{l} \left( 2^{-i-l} \right)^{kp} E_2(f)_p \quad \left( E_2(f)_p := E_0(f)_p \right)
\]

Since \( \sum_{i=1}^{l} \left( 2^{-i-l} \right)^{kp} E_2(f)_p \) is equivalent to the right-hand side of (1.5), inequalities (3.1) and (3.2) together with \( 2^{-i-kp} \sim i^{kp} \) complete the proof of our theorem. \( \square \)

4. Existence and estimates of \( f^{(k)} \)

Following results for \( p \geq 1 \) (see [3, Theorem 6.2]) we can prove

Theorem 4.1. Suppose \( f \in L_p[-1, 1] \), \( 0 < p < 1 \), and \( \sum_{n=0}^{\infty} (n + 1)^{k-1} E_n(f)_p < \infty \) for some positive integer \( k \). Then \( f^{(k)} \in L_p[-1, 1] \) and

\[
\|\phi^k(f^{(k)} - P_n^{(k)})\|_p \leq M \left( \sum_{m \geq n} (m + 1)^{k-1} E_m(f)_p \right)^{1/p}
\]

(4.1)
Proof. Suppose $P_m \in \Pi_m$ is the best approximant to $f$ in $L_p[-1, 1]$. By virtue of (1.4) we may write $\sum_{j=1}^{\infty} (P_{2^n} - P_{2^{n-1}}) = f - P_n$. By Lemma 2.2, we write

$$\left\| \varphi \left( \sum_{i=1}^{\infty} (P_{2^n} - P_{2^{n-1}}) \right) \right\|_p^p \leq C \sum_{i=1}^{\infty} (2^i n)^{ kp} \| P_{2^n} - P_{2^{n-1}} \|_p^p$$

$$\leq C_1 \sum_{i=0}^{\infty} (2^i n)^{ kp} E_{2^n}(f)_p^p$$

$$\leq C_2 \sum_{m \geq n} (m + 1)^{ kp - 1} E_m(f)_p^p,$$

which completes the proof. □

References

9. _____, A converse theorem for the algebraic approximation in $L_p[-1, 1]$ (0 < $p$ < 1), Serdica 17 (1991), 161–166.