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ENDOMORPHISM RINGS OF NONDEGENERATE MODULES

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ABSTRACT. Let ${}_RM$ be a left R-module whose Morita context is nondegenerate, $S = \mathrm{End}({}_RM)$, and $N = \mathrm{Hom}({}_RM, R)$. If ${}_RM$ is also nonsingular, then the main results of Khuri (Proc. Amer. Math. Soc. **96** (1986), 553–559) are the following: (1) S is left (right) strongly modular if and only if any element of S which has zero kernel in ${}_RM(N_R)$ has essential image in ${}_RM(N_R)$; (2) S is a left (right) Utumi ring if and only if every submodule ${}_RU$ of ${}_RM$ (U_R^* of N_R) such that $U^\perp = 0$ (${}^\perp U^* = 0$) is essential in ${}_RM(N_R)$. In this paper, we show that the same results hold in any nondegenerate Morita context without ${}_RM$ being nonsingular and that S is right nonsingular if and only if N_R is nonsingular.

1. PRELIMINARIES

Throughout this paper, R and S are associative rings with identity. The left and right annihilators in S of a subset K of S will be denoted by $l(K)$ and $r(K)$, respectively. The notation $l_M(K)$, $r_N(K)$, $r_S(U)$, $l_S(U^*)$ will be used for annihilators in ${}_RM$ of $K \subseteq S$, in N_R of $K \subseteq S$, in S of $U \subseteq {}_RM$, and in S of $U^* \subseteq N_R$, respectively. The notation $U \subseteq_e {}_RM$ will be used to indicate that U is an essential R-submodule of ${}_RM$. Recall that ${}_RM$ is said to be nonsingular if, for $m \in {}_RM$, $\mathrm{ann}_R(m) \subseteq_e R$, then $m = 0$.

Recall that a Morita context (R, M, N, S) consists of two rings R and S, two bimodules ${}_RM_S$ and ${}_SN_R$, and two bimodule homomorphisms $(-,-)$: $M \otimes_S N \to R$ and $[-,-]$: $N \otimes_R M \to S$ satisfying $m[n, m'] = (m, n)m'$ and $n(m, n') = [n, m]n'$ for all $m, m' \in M$ and $n, n' \in N$ with the images being I and J, respectively. I and J are both ideals and are called the trace ideals of the context.

(R, M, N, S) is said to be nondegenerate if the four modules ${}_RM$, M_S, ${}_SN$, N_R and the two pairings are faithful (the latter leading to the fact that $(m, N) = 0$ implies $m = 0$ and three analogous implications).

This is equivalent to the eight natural maps associated with the Morita context being injective (e.g., two of these maps are $m \mapsto (m, -)$ and $r \mapsto (n \mapsto nr) \in \mathrm{End}(N_R)$ for $m \in M$, $n \in N$, and $r \in R$). The standard context $(R, M, N = \mathrm{Hom}({}_RM, R), S = \mathrm{End}({}_RM))$ is nondegenerate if and only if ${}_RM$ is torsionless and faithful and ${}_RR$ is I-free; that is, $Ir \neq 0$ whenever

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$0 \neq r \in R$ [2]. In this case, the module ${}_RM$ is called nondegenerate; however, we will consistently work on a general Morita context instead of a standard one from a module.

Proposition 1. *(R, M, N, S) is nondegenerate if and only if all R-modules and all S-modules associated are I-free, resp., J-free.*

Proof. $\Rightarrow$ If $m \in M$ and $\mathrm{Im} = 0$, then $\mathrm{Im} = (M, N)m = M[N, m] = 0 \Rightarrow [N, m] = 0$ since M_S is faithful $\Rightarrow m = 0$ since $[N, -]$ is faithful. Similarly we can show that ${}_RR$, R_R, N_R are I-free and M_S, ${}_SN$, ${}_SS$, and S_S are J-free.

$\Leftarrow$ If $r \in R$ and $rM = 0$, then $(rM, N) = r(M, N) = 0 \Rightarrow r = 0$ since R_R is I-free $\Rightarrow {}_RM$ is faithful. If $s \in S$, $sN = 0$, then $[sN, M] = s[N, M] = 0 \Rightarrow s = 0$ since S_S is J-free $\Rightarrow {}_SN$ is faithful.

Similarly we can show that M_S, N_R, and the two pairings are faithful.

2. Main results

First we need the following useful lemma.

Lemma 2. *If ${}_RM$ is I-free and ${}_SS$ is J-free, then:*

(a) $K \subseteq_e H \subseteq {}_SS \Leftrightarrow MK \subseteq_e MH$.
(b) $U \subseteq_e U_1 \subseteq {}_RM \Leftrightarrow [N, U] \subseteq_e [N, U_1]$.
(c) $U \subseteq_e {}_RM \Leftrightarrow [N, U] \subseteq_e {}_SS$.

If N_R is I-free and S_S is J-free, then:

(a′) $K \subseteq_e H \subseteq S_S \Leftrightarrow KN \subseteq_e HN$.
(b′) $U^* \subseteq_e U_1^* \subseteq N_R \Leftrightarrow [U^*, M] \subseteq_e [U_1^*, M]$.
(c′) $U^* \subseteq_e N_R \Leftrightarrow [U^*, M] \subseteq_e S_S$.

Proof. We only prove (a′) and (b′); the rest of the proof is similar.

(a′) $\Rightarrow$ First we note that under the assumption, we have $[n, M] \neq 0$ whenever $0 \neq n \in N$ and ${}_SN$ is faithful. In fact, $[n, M] = 0 \Rightarrow [n, M]N = n(M, N) = 0 \Rightarrow n = 0$ since N_R is I-free, and $sN = 0 \Rightarrow [sN, M] = s[N, M] = 0 \Rightarrow s = 0$ since S_S is J-free. Now assume that $K \subseteq_e H \subseteq S_S$ and $0 \neq n = \sum_{i=1}^m h_i n_i \in HN$ with $h_i \in H$, $n_i \in N$. Then $0 \neq [n, M] = \sum_{i=1}^m h_i[n_i, M] \subseteq H \Rightarrow K \cap [n, M] \neq 0 \Rightarrow 0 \neq (K \cap [n, M])N \subseteq KN \cap [n, M]N = KN \cap n(M, N) \subseteq KN \cap nR \Rightarrow KN \subseteq_e HN$.

$\Leftarrow$ If $0 \neq h \in H$, then $0 \neq hN \Rightarrow 0 \neq KN \cap hN \Rightarrow 0 \neq [KN \cap hN, M] \subseteq [KN, M] \cap [hN, M] = K[N, M] \cap h[N, M] \subseteq K \cap hS \Rightarrow K \subseteq_e H$.

(b′) $\Rightarrow$ Let $0 \neq s \in [U_1^*, M]$. Then $0 \neq sN \subseteq [U_1^*, M]N \subseteq U_1^*(M, N) \subseteq U_1^* \Rightarrow 0 \neq U^* \cap sN \Rightarrow 0 \neq [U^* \cap sN, M] \subseteq [U^*, M] \cap [sN, M] = [U^*, M] \cap s[N, M] \subseteq [U^*, M] \cap sS \Rightarrow [U^*, M] \subseteq_e [U_1^*, M]$.

$\Leftarrow$ Let $0 \neq u \in U_1^*$. Then $[u, M] \neq 0 \Rightarrow [U^*, M] \cap [u, M] \neq 0 \Rightarrow 0 \neq ([U^*, M] \cap [u, M])N \subseteq [U^*, M]N \cap [u, M]N = U^*(M, N) \cap u(M, N) \subseteq U^* \cap uR \Rightarrow U^* \subseteq_e U_1^*$.

It is known that if (R, M, N, S) is nondegenerate and one of ${}_RR$, ${}_RM$, ${}_SN$, ${}_SS$ is nonsingular, then all of them are nonsingular [2, Proposition 14]. It seems unsure in [1] whether S_S nonsingular implies that N_R is nonsingular (see [1, §1]), since ${}_SS$ is, in general, only a proper subring of $\mathrm{End}(N_R)$. But, in fact, by the symmetry of a Morita context and the condition of nondegeneracy, the following is also true.

Proposition 3. (1) *If* (R, M, N, S) *is a Morita context and* N_R *is I-free and* S_S *is J-free, then* N_R *is nonsingular if and only if* S_S *is nonsingular.*

(2) *If* (R, M, N, S) *is nondegenerate, and if one of* R_R, M_S, N_R, S_S *is nonsingular, then all of them are nonsingular.*

Proof. (1) $\Rightarrow$ If $s \in S$, $K \subseteq_e S_S$ such that $sK = 0$, then $sKN = 0$, but $KN \subseteq_e N$ by Lemma 2(a′) $\Rightarrow sN = 0$ since N_R is nonsingular $\Rightarrow s = 0$, i.e., S_S is nonsingular.

$\Leftarrow$ We show that if N_R is not nonsingular, then neither is S_S. If $0 \neq n \in N$ such that $nP = 0$, $P \subseteq_e R_R$, then $[n, m] \neq 0$ for some $m \in M$ (since $[n, M] \neq 0$ from the proof of Lemma 2). If $n' \in N$ such that $[n, m]n' \neq 0$, then $0 \neq n(m, n') \Rightarrow (m, n') \neq 0 \Rightarrow P \cap (m, n')R \neq 0 \Rightarrow$ there is an $r \in R$ such that $(m, n')r \neq 0$, but $n(m, n')r = 0 \Rightarrow [n, m]n'r = 0$ but $n'r \neq 0$; otherwise, $(m, n')r = 0$, which means $U^* = \{n' \in N,\ [n, m]n' = 0\} \subseteq_e N_R \Rightarrow [U^*, M] \subseteq_e S_S$, but $[n, m][U^*, M] = 0 \Rightarrow S_S$ is not nonsingular.

(2) The proof is the same as Proposition 14 of [2].

Corollary 4. *Let* (R, M, N, S) *be nondegenerate. Then the following are equivalent*:

(1) S_S *is nonsingular.*
(2) N_R *is nonsingular.*
(3) *For any* $U^* \subseteq_e N_R \Rightarrow l_S(U^*) = 0$.

Proof. Combine [1, Proposition 5] and our previous proposition.

A ring S is called left strongly modular if, for $s \in S$, $l(s) = 0 \Rightarrow Ss \subseteq_e {}_SS$, and right strongly modular if $r(s) = 0 \Rightarrow sS \subseteq_e S_S$ [1]. Now we have

Theorem 5. *Let* (R, M, N, S) *be nondegenerate. Then*

(1) S *is left strongly modular if and only if, for each* $s \in S$, $l_M(s) = 0 \Rightarrow Ms \subseteq_e {}_RM$.
(2) S *is right strongly modular if and only if, for each* $s \in S$, $r_N(s) = 0 \Rightarrow sN \subseteq_e N_R$.

Proof. (1) $\Rightarrow$ If $l_M(s) = 0$ and $s's = 0$, then $Ms's = (Ms')s = 0 \Rightarrow Ms' = 0 \Rightarrow s' = 0$ since M_S is faithful $\Rightarrow l(s) = 0 \Rightarrow Ss \subseteq_e {}_SS$ by the assumption $\Rightarrow Ms = MSs \subseteq_e MS = M$ by Lemma 2(a).

$\Leftarrow$ If $l(s) = 0$ and $ms = 0$, $m \in M$, $s \in S$, then $[N, ms] = [N, m]s = 0 \Rightarrow [N, m] = 0 \Rightarrow m = 0$ since $[N, -]$ is faithful $\Rightarrow l_M(s) = 0 \Rightarrow Ms \subseteq_e {}_RM \Rightarrow [N, Ms] = [N, M]s \subseteq_e {}_SS$ by Lemma 2(c) $\Rightarrow Ss \subseteq_e {}_SS$.

(2) Similarly.

Recall that a ring S is called left Utumi if $K \subseteq {}_SS$ and $r(K) = 0 \Rightarrow K \subseteq_e {}_SS$; S is called right Utumi if $H \subseteq S_S$ and $l(H) = 0 \Rightarrow H \subseteq_e S_S$ [1]. If $U \subseteq {}_RM$, $U^* \subseteq N_R$, let $U^\perp = \{n \in N,\ (U, n) = 0\}$, ${}^\perp U^* = \{m \in M,\ (m, U^*) = 0\}$.

Lemma 6. *Let* (R, M, N, S) *be nondegenerate. Then*:

(1) $r_S(U) = r([N, U])$, $l_S(U^*) = l([U^*, M])$.
(2) $r_S(U) = 0 \Leftrightarrow U^\perp = 0$, $l_S(U^*) = 0 \Leftrightarrow {}^\perp U^* = 0$.
(3) $r(K) = r_S(MK)$, $l(K) = l_S(KN)$.

Proof. We only prove (2); the rest is straightforward.

If $r_S(U) = 0$ and $(U, n) = 0 \Rightarrow (U, n)M = 0 \Rightarrow U[n, M] = 0 \Rightarrow [n, M] = 0 \Rightarrow n = 0 \Rightarrow U^\perp = 0$. On the other hand, $U^\perp = 0$ and $Us = 0 \Rightarrow [N, Us] =$

$0 \Rightarrow [N, Us]N = N(Us, N) = N(U, sN) = 0 \Rightarrow (U, sN) = 0 \Rightarrow sN = 0 \Rightarrow s = 0 \Rightarrow r_S(U) = 0$.

If $l_S(U^*) = 0$ and $(m, U^*) = 0 \Rightarrow N(m, U^*) = [N, m]U^* = 0 \Rightarrow [N, m] = 0 \Rightarrow m = 0 \Rightarrow {}^{\perp}U^* = 0$. On the other hand, if ${}^{\perp}U^* = 0$ and $sU^* = 0 \Rightarrow (M, sU^*) = (Ms, U^*) = 0 \Rightarrow Ms = 0 \Rightarrow s = 0 \Rightarrow l_S(U^*) = 0$.

Theorem 7. *Let (R, M, N, S) be nondegenerate. Then the following are equivalent:*

(a) *S is left Utumi.*
(b) *For any $U \subseteq {}_RM$, $U^{\perp} = 0 \Rightarrow U \subseteq_e {}_RM$.*
(c) *For any $U \subseteq {}_RM$, $r_S(U) = 0 \Rightarrow U \subseteq_e {}_RM$.*

The following are also equivalent:

(a′) *S is right Utumi.*
(b′) *For any $U^* \subseteq N_R$, ${}^{\perp}U^* = 0 \Rightarrow U^* \subseteq_e N_R$.*
(c′) *For any $U^* \subseteq N_R$, $l_S(U^*) = 0 \Rightarrow U^* \subseteq_e N_R$.*

Proof. (a) ⇒ (b) By Lemma 6(1), (2), $U^{\perp} = 0 \Rightarrow r_S(U) = 0 \Rightarrow r([N, U]) = 0 \Rightarrow [N, U] \subseteq_e {}_SS$ since S is left Utumi $\Rightarrow U \subseteq_e {}_RM$ by Lemma 2(c).

(b) ⇒ (a) By Lemma 6(3), if $K \subseteq {}_SS$ and $r(K) = 0 \Rightarrow r_S(MK) = 0 \Rightarrow MK \subseteq_e {}_RM \Rightarrow K \subseteq_e y_SS$ by Lemma 2(a).

(b) ⇔ (c) follows from Lemma 6(2).

(a′) ⇔ (b′) ⇔ (c′) come from the symmetry.

In [3, Theorem 3.3], it is shown that a right and left nonsingular ring S has isomorphic maximal left and right quotient rings if and only if S is both right and left Utumi. Therefore, we have the following:

Corollary 8. *Let (R, M, N, S) be nondegenerate and ${}_RM$, N_R nonsingular. Then S has isomorphic maximal left and right quotient rings if and only if one of* (a), (b), (c) *and one of* (a′), (b′), (c′) *in Theorem 7 hold.*

Finally, we observe that there is no difference between the roles of R and S in a nondegenerate Morita context in the situation under consideration, so we can have all versions related to R, M_S, and ${}_SN$ of all the results in this paper simply from symmetry.

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