A CHARACTERIZATION OF THE SECOND DUAL OF $C_0(S, A)$

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Abstract. Let $S$ be a locally compact Hausdorff space, and let $A$ be a Banach space. The space of the continuous functions from $S$ to $A$ vanishing at infinity is denoted by $C_0(S, A)$. Let $MW(S, A^*)$ be the space of the representing measures of all the bounded linear functionals on $C_0(S, A)$. For $\mu \in MW(S, A^*)$ let

$$L_\infty(|\mu|, A^{**}, A^*) = \{ f: S \to A^{**} | f(\cdot)x^* \in L_\infty(|\mu|) \ \forall x^* \in A^* \}.$$

The second dual of $C_0(S, A)$ is characterized in the general case by means of certain elements in the product linear space $\prod\{L_\infty(|\mu|, A^{**}, A^*) : \mu \in MW(S, A^*)\}$.

1. Introduction

Let $A$ be a Banach space, $S$ be a locally compact Hausdorff space, and $B(S)$ be the $\sigma$-algebra of all the Borel sets of $S$. The space of the continuous functions from $S$ to $A$ vanishing at infinity endowed with the uniform norm is denoted by $C_0(S, A)$. The second dual of $C_0(S, A)$ is considered for the case where $S$ is compact and the dual $A^*$ has the Radon-Nikodym property in [3], for the case where $S$ is locally compact and $A$ is a Banach algebra with a positive cone satisfying certain conditions in [4]. Recently a characterization of $C_0^{**}(S, A)$ by means of the "generalized functions" was given in [5] by the authors in the case where $A^*$ and $A^{**}$ have the Radon-Nikodym property. The purpose of this paper is to give a characterization of $C_0^{**}(S, A)$ in general by means of the elements in $gl(S, A^{**})$ (see Definition 3.1).

Let $T: C_0(S, A) \to \mathbb{C}$ be a bounded linear functional, and let $M(S)$ be the space of all the bounded regular Borel measures on $S$. Then, since $\mathbb{C}$ is reflexive, $T$ is weakly compact. There is a unique representing measure $m: B(S) \to A^*$ such that $m(\cdot)x \in M(S)$ for all $x \in A$, $T(f) = \int f \, dm$ for all $f \in C_0(S, A)$ and $\|T\| = \hat{m}(S)$, the semivariation of $m$ (see, e.g., [2, 4]). Note that the total variation $|m|$ and $\hat{m}$ are the same in this case [8, p. 54]. Let $MW(S, A^*)$ be the set of the representing measures $m$ of all such functionals $T$. Since $|m|(S) = \hat{m}(S) = \|T\|$ is finite, we see that $|m| \in M(S)$ [2, Theorem 1.1].

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2.8] and that

\[ C_0(S, A) = MW(S, A^*) . \]

For \( \mu \in MW(S, A^*) \) let \( L_1(\mu, A^*) \) be the space of all the equivalence classes of \( A^* \)-valued Bochner integrable functions defined on \( S \). Measures defined by densities in \( L_1(\mu, A^*) \) are studied in §2. It is shown in Theorem 2.2 that the linear span \( M \) of all such measures in \( MW(S, A^*) \) is contained between \( M(S) \otimes A^* \) and its closure \([M(S) \otimes A^*]^-\) with respect to the semivariation norm. The second dual \( C_0^{**}(S, A) \) is then characterized in §3.

2. MEASURES DEFINED BY DENSITIES IN \( L_1(\mu, A^*) \)

Let \( \varphi \in L_1(\mu, A^*) \), and define \( \nu = \varphi |\mu| \in MW(S, A^*) \) by the Riesz Representation Theorem

\[ \int g \, d\nu = \int \langle g, \varphi \rangle \, d|\mu| \quad (g \in C_0(S, A)) . \]

It follows that, for \( E \in \mathcal{B}(S) \),

\[ \nu(E)x = \int (\chi_E x , \varphi) \, d|\mu| \quad (x \in A) \]

and

\[ \nu(E) = \int_E \varphi \, d|\mu| . \]

Also \( |\nu| = |\varphi| |\mu| = |\varphi| |\mu| \) by Dinculeanu [8, Theorem 6, p. 186]. Hence

\[ \|\nu\| = |\nu|(S) = |\varphi| |\mu|(S) = \int |\varphi| \, d|\mu| = \|\varphi\| \|\mu| . \]

We see from (*) that \( \nu \ll |\mu| \) (see [7, Definition 3, p. 11]). Therefore \( \varphi \rightarrow \nu \) is a linear isometry from \( L_1(\mu, A^*) \) into \( \{ \nu \in MW(S, A^*) : \nu \ll |\mu| \} \).

Definition 2.1. Denote by \( M \) the linear span in \( MW(S, A^*) \) of all the measures defined by densities in \( L_1(\mu, A^*) \) for all \( \mu \in MW(S, A^*) \). That is,

\[ M = \text{span} \left( \bigcup \{ L_1(\mu, A^*)|\mu| : \mu \in MW(S, A^*) \} \right) . \]

We shall show in Theorem 2.2 that this subspace \( M \) is large enough for us to apply the Hahn-Banach Extension Theorem to characterize \( C_0^{**}(S, A) \). Let us recall first a result from Cambern and Grein [3, Corollary 2] that \( M(S) \otimes A^* \) can be embedded in \( MW(S, A^*) \) in such a way that \( m \otimes x^* \) corresponds to \( m(\cdot)x^* \) for all \( m \in M(S) \) and \( x^* \in A^* \).

Theorem 2.2. Let \( [M(S) \otimes A^*]^-\) be the closure of \( M(S) \otimes A^* \) with respect to the semivariation norm of \( MW(S, A^*) \). Then

\[ [M(S) \otimes A^*]^- \supseteq M \supseteq M(S) \otimes A^* . \]

Proof. We show first that \( M \supseteq M(S) \otimes A^* \). Let \( m \in M(S) \) and \( x^* \in A^* \). Then, by the Radon-Nikodym Theorem for scalar measures, \( m = \phi|m| \) for
some \( \phi \in L_1(|\mu|) \). For any \( \sum_{i=1}^{N} f_i \otimes x_i \in C_0(S) \otimes A \),
\[
(m \otimes x^*) \left( \sum_{i=1}^{N} f_i \otimes x_i \right) = \sum_{i=1}^{N} m(f_i)x^*(x_i) = \sum_{i=1}^{N} \phi|m|(f_i) \cdot x^*(x_i)
\]
\[
= \sum_{i=1}^{N} \int \langle f_i, x_i, \phi x^* \rangle \, d|m| = \sum_{i=1}^{N} (\phi x^*)|m|(f_i x_i)
\]
\[
= (\phi x^*)|m| \left( \sum_{i=1}^{N} f_i \otimes x_i \right).
\]

Since \( C_0(S, A) = C_0(S) \hat{\otimes} A \) with the least cross norm, we see that \( m \otimes x^* = (\phi x^*)|m| \in M \), so \( M \supseteq M(S) \otimes A^* \).

Next we shall prove that \( (L_1(|\mu|) \otimes A^*)|\mu| \subset M(S) \otimes A^* \). Recall that \( (L_1(|\mu|) \hat{\otimes} A^*) = L_1(|\mu|, A^*) \), where the closure on the left is with respect to the greatest cross norm (see, e.g., Diestel and Uhl [7, Example 10, p. 228]). For \( \sum_{i=1}^{N} f_i \otimes x_i \in C_0(S) \otimes A \), \( \sum_{j=1}^{M} (\phi_j \otimes x^*_j) \in L_1(|\mu|) \otimes A^* \),
\[
\sum_{j=1}^{M} (\phi_j \otimes x^*_j)|\mu| \left( \sum_{i=1}^{N} f_i \otimes x_i \right) = \int \left( \sum_{i=1}^{N} f_i \otimes x_i, \sum_{j=1}^{M} \phi_j \otimes x^*_j \right) \, d|\mu|
\]
\[
= \sum_{j=1}^{M} \int \left( \sum_{i=1}^{N} f_i \otimes x_i, \phi_j \otimes x^*_j \right) \, d|\mu| = \sum_{j=1}^{M} \sum_{i=1}^{N} \int \langle f_i, \phi_j \rangle x_j^*(x_i) \, d|\mu|
\]
\[
= \sum_{j=1}^{M} \sum_{i=1}^{N} \int \langle f_i, \phi_j \rangle \, d|\mu| \cdot x_j^*(x_i) = \sum_{j=1}^{M} \sum_{i=1}^{N} (\phi_j|\mu| \otimes x_j^*)(f_i \otimes x_i)
\]
\[
= \sum_{j=1}^{M} (\phi_j|\mu| \otimes x_j^*) \left( \sum_{i=1}^{N} f_i \otimes x_i \right).
\]

By taking the limit process we see that
\[
\sum_{j=1}^{M} (\phi_j \otimes x^*_j)|\mu| = \sum_{j=1}^{M} (\phi_j|\mu| \otimes x_j^*).
\]

Thus
\[
(L_1(|\mu|) \otimes A^*)|\mu| \subset M(S) \otimes A^* \subseteq MW(S, A^*).
\]

Finally let \( \phi \in L_1(|\mu|, A^*) \), and let
\[
\varphi_j = \sum_{j=1}^{N_j} \phi_i \otimes x^*_i \in L_1(|\mu|) \otimes A^*
\]
be such that \( \varphi_j \to \phi \) in \( L_1(|\mu|, A^*) \). For \( f \in C_0(S, A) \), \( \|f\| \leq 1 \),
\[
\left| \int f \, d\varphi_j|\mu| - \int f \, d\varphi|\mu| \right| = \left| \int \langle f, \varphi_j - \varphi \rangle \, d|\mu| \right|
\]
\[
= \left| \int f \, d(\varphi_j - \varphi)|\mu| \right| \leq \|f\| \cdot \|\varphi_j - \varphi\|_{1, |\mu|}
\]
\[
\leq \|\varphi_j - \varphi\|_{1, |\mu|} \to 0.
\]
That is, \( \phi_j|\mu| \rightarrow \phi|\mu| \) in \( MW(S, A^*) \) with semivariation (= total variation) norm. Hence \( M \subseteq [M(S) \otimes A^*]^- \). This completes the proof of the theorem.

**Remark 2.3.** It is worthwhile to note that if \( S \) is compact then \( [M(S) \otimes A^*]^- = MW(S, A^*) \) if \( S \) is dispersed or \( A^* \) has the Radon-Nikodym property (see [3, Theorem 1]).

### 3. Characterization of the second dual

For \( \mu \in MW(S, A^*) \) let \( L_\infty(|\mu|) \) be the Banach space of all the equivalence classes of bounded Borel measurable (real-valued) functions on \( S \) with essential supremum norm (see, e.g., [11, p. 85]). Denote

\[
L_\infty(|\mu|, A^{**}, A^*) = \{ f : S \rightarrow A^{**} | f(\cdot)x^* \in L_\infty(|\mu|) \ \forall x^* \in A^* \}.
\]

For details of the definition of these spaces see [10, Definition 3, p. 75 and §5, p. 78]. Functions in \( L_\infty(|\mu|, A^{**}, A^*) \) are identified by the equivalence relation \( f \equiv g(w) \), which means by definition

\[
f(\cdot)x^* = g(\cdot)x^* \quad (x^* \in A^*)
\]

\(|\mu|\)-a.e. (the null set can depend on \( x^* \)) [10, p. 76].

Note that functions in \( L_\infty(|\mu|, A^{**}, A^*) \) need not be \(|\mu|\)-measurable (hence not necessarily Bochner integrable) unless for example when \( A^{**} \) is separable [10, Corollary to Theorem 10, p. 73], in which case, \( L_\infty(|\mu|, A^{**}, A^*) = L_\infty(|\mu|, A^{**}) \), where \( L_\infty(|\mu|, A^{**}) \) stands for the space of the equivalence classes of \( A^{**}\)-valued Bochner integrable functions defined on \( S \) that are \(|\mu|\)-essentially bounded.

The space \( L_\infty(|\mu|, A^{**}, A^*) \) is a Banach space with norm

\[
N_\infty(g) = \inf\{ \alpha > 0 : \{ s : |g(s)| > \alpha \} \text{ is } |\mu|\text{-null} \}
\]

[10, p. 74]. Note that, by the uniform boundedness principle, we have

\[
N_\infty(g) \leq \| g \| = \sup\{ \| g(s) \| : s \in S \} \leq \infty
\]

[10, p. 75].

It is also shown in [10, Chapter VII, §4, Theorem 7, and Corollary] that

\[
L_1(|\mu|, A^*)^* = L_\infty(|\mu|, A^{**}, A^*)
\]

is an isometric isomorphism under the correspondence

\[
(g, \varphi) = \int \langle g(s), \varphi(s) \rangle d|\mu|(s)
\]

for \( \varphi \in L_1(|\mu|, A^*) \), \( g \in L_\infty(|\mu|, A^{**}, A^*) \).

**Definition 3.1.** Consider the product linear space \( \prod\{ L_\infty(|\mu|, A^{**}, A^*) : \mu \in MW(S, A^*) \} \). Let \( f = (f_\mu)_{\mu \in MW(S, A^*)} \) be an element in this space satisfying the following two conditions:

1. \( \| f \|_\infty = \sup\{ N_\infty(f_\mu) : \mu \in MW(S, A^*) \} < \infty \) and
2. if \( \mu, \nu \in M(S, A^*) \) are such that \( |\mu| \ll |\nu| \), then \( f_\mu \equiv f_\nu(w) \) with respect to \( |\mu| \).
The set of all such elements $f$ on $S$ is denoted by $\text{gl}(S, A^{**})$. With the norm $\|f\|_\infty$, $\text{gl}(S, A^{**})$ is a Banach space.

**Lemma 3.2.** Let $\mu, \nu \in MW(S, A^*)$. If $|\mu| \ll |\nu|$ with $|\mu| = \omega|\nu|$ for some $\omega \in L_1(|\nu|)$ and $\varphi \in L_1(|\mu|, A^*)$, $\psi = \omega \varphi$, then $\varphi|\mu| = \psi|\nu|$.

**Proof.** Since $|\mu| \ll |\nu|$, by the Radon-Nikodym Theorem, there is $\omega \in L_1(|\nu|)$ such that $|\mu| = \omega|\nu|$ and

$$
\int g \, d|\mu| = \int g \omega \, d|\nu| \quad (g \in L_1(|\mu|)).
$$

Fix $\varphi \in L_1(|\mu|, A^*)$ and let $\psi = \omega \varphi$. Then $\psi \in L_1(|\nu|, A^*)$. Let $f \in C_0(S, A)$

$$
\int f \, d\varphi|\mu| = \int (f, \varphi) \, d|\mu| = \int (f, \varphi) \omega \, d|\nu| = \int (f, \varphi \omega) \, d|\nu| \quad (\omega \text{ is scalar-valued})
$$

$$
= \int (f, \psi \varphi) \, d|\nu| = \int f \, d\psi|\nu|.
$$

Hence $\varphi|\mu| = \psi|\nu|$.

**Theorem 3.3.** For each $F \in C_0^{**}(S, A)$, there is a (unique) $f \in \text{gl}(S, A^{**})$ such that

$$
F(\varphi|\mu|) = \int (f, \varphi) \, d|\mu| \quad (\varphi \in L_1(|\mu|, A^*))
$$

for all $\mu \in MW(S, A^*)$.

**Proof.** For each $F \in C_0^{**}(S, A)$, consider the bounded linear functional

$$
\varphi \mapsto F(\varphi|\mu|) \quad (\varphi \in L_1(|\mu|, A^*)).
$$

From the result of Tulcea and Tulcea [10, Chapter VII, §4] mentioned earlier, there is $f_\mu \in L_\infty(|\mu|, A^{**}, A^*) = L_1(|\mu|, A^*)^*$ such that

$$
F(\varphi|\mu|) = \int (f_\mu, \varphi) \, d|\mu| \quad (\varphi \in L_1(|\mu|, A^*)).
$$

Hence we have an $f = (f_\mu)_{\mu \in MW(S, A^*)} \in \text{gl}(S, A^{**})$ if we can show properties (1) and (2) of Definition 3.1. Now

1. For each $\mu \in MW(S, A^*)$, since $L_1(|\mu|, A^*)^* \cong L_\infty(|\mu|, A^{**}, A^*)$ is an isometric isomorphism and $\|\varphi|\mu|\| = \|\varphi\|_1,|\mu|$ (see (**) in §2),

$$
N_\infty(f_\mu) = \sup \left\{ \int (f_\mu, \varphi) \, d|\mu| : \|\varphi\|_1,|\mu| \leq 1 \right\}
$$

$$
= \sup \left\{ \int |F(\varphi|\mu|)| : \|\varphi\|_1,|\mu| \leq 1 \right\}
$$

$$
\leq \sup \{ |F(\nu)| : \|\nu\| \leq 1 \}
$$

$$
\leq \sup \{ \|F\| \|\nu\| : \|\nu\| \leq 1 \} \leq \|F\| < \infty.
$$

2. Suppose $|\mu| \ll |\nu|$ and $\varphi \in L_1(|\mu|, A^*)$. By the Riesz Representation Theorem, there is $\omega \in L_1(|\nu|)$ such that $|\mu| = \omega|\nu|$. Let $\psi = \omega \varphi$. Then, from
Lemma 3.2. \( \varphi|\mu| = \psi|\nu| \). Hence, for all \( \varphi \in L_1(|\mu|, A^*) \),

\[
\int \langle f_\mu, \varphi \rangle d|\mu| = F(\varphi|\mu|) = F(\psi|\nu|)
\]

\[
= \int \langle f_\nu, \psi \rangle d|\nu| = \int \langle f_\nu, \omega \varphi \rangle d|\nu|
\]

\[
= \int \langle f_\nu, \varphi \omega \rangle d|\nu| = \int \langle f_\nu, \varphi \rangle d|\mu|.
\]

Hence \( f_\mu \equiv f_\nu(w) \). Thus \( f = (f_\mu) \in \text{gl}(S, A^{**}) \).

The uniqueness is due to the fact that functions in \( L_\infty(|\mu|, A^{**}, A^*) \) are identified by the equivalence relation \( f \equiv g(w) \). This completes the proof of the theorem.

For \( f_\mu \in L_\infty(|\mu|, A^{**}, A^*) \), define \( F_\mu \) on \( L_1(|\mu|, A^*)|\mu| \subseteq MW(S, A^*) \) by

\[
F_\mu(\varphi|\mu|) = \int \langle f_\mu, \varphi \rangle d|\mu| \quad (\varphi \in L_1(|\mu|, A^*)). 
\]

Lemma 3.4. Let \( \mu, \nu \in MW(S, A^*) \) with \( |\mu| \ll |\nu| \). Then \( F_\mu \) is linear and bounded and \( F_\mu = F_\nu \) on \( L_1(|\mu|, A^*)|\mu| \).

Proof. Clearly \( F_\mu \) is linear and bounded because

\[
||F_\mu|| = \sup \{|F_\mu(\varphi|\mu|) : ||\varphi|| \leq 1\}
\]

\[
\leq \sup \left\{ \left| \int \langle f_\mu, \varphi \rangle d|\mu| \right| : ||\varphi||_1, |\mu| \leq 1 \right\}
\]

\[
\leq \sup \{N_\infty(f_\mu)||\varphi||_1, |\mu| : ||\varphi||_1, |\mu| \leq 1\}
\]

\[
\leq N_\infty(f_\mu) < \infty.
\]

Since \( |\mu| \ll |\nu| \), there is some \( \omega \in L_1(|\nu|) \) such that \( |\mu| = \omega|\nu| \). Let \( \psi = \omega \varphi \). Then

\[
F_\nu(\psi|\nu|) = \int \langle f_\nu, \psi \rangle d|\nu| = \int \langle f_\nu, \varphi \rangle \omega d|\nu|
\]

\[
= \int \langle f_\nu, \varphi \rangle d|\mu| = \int \langle f_\mu, \varphi \rangle d|\mu| = F_\mu(\varphi|\mu|),
\]

since \( f_\mu \equiv f_\nu(w) \) and \( L_\infty(|\mu|, A^{**}, A^*) = L_1(|\mu|, A^*)^* \) is an isometry. Now, since \( \psi|\nu| = \varphi|\mu| \) by Lemma 3.2,

\[
F_\mu(\varphi|\mu|) = F_\nu(\psi|\nu|) = F_\nu(\varphi|\mu|).
\]

Consequently, \( F_\mu = F_\nu \) on \( L_1(|\mu|, A^*)|\mu| \).

Theorem 3.5. For all \( \mu \in MW(S, A) \) and \( f = (f_\mu)_\mu \in \text{gl}(S, A^{**}) \), there is an \( F \in C_0^{**}(S, A) \), such that

\[
F(\varphi|\mu|) = \int \langle f_\mu, \varphi \rangle d|\mu| \quad (\varphi \in L_1(|\mu|, A^*)).
\]

Proof. We shall define a bounded linear functional \( F \) on the linear span \( M \) defined in Definition 2.1 such that

\[
F(\varphi|\mu|) = \int \langle f_\mu, \varphi \rangle d|\mu|.
for \( \phi \in L_1(\vert \mu \vert, A^*) \) and \( \mu \in MW(S, A^*) \). In general, an element \( \phi \) in \( M \) is of the form

\[
\phi = \sum_{i=1}^{m} \varphi_i \vert \mu_i \vert ,
\]

where \( \varphi_i \in L_1(\vert \mu_i \vert, A^*) \) and \( \mu_i \in MW(S, A^*) \). Define

\[
F(\phi) = \sum_{i=1}^{m} F_{\mu_i}(\varphi_i \vert \mu_i \vert) .
\]

We shall show that \( F \) is independent of the special representation of \( \phi \). Suppose also \( \phi = \sum_{j=1}^{n} \psi_j \vert \nu_j \vert \). Pick \( x^* \in A^* \) with \( \|x^*\| = 1 \), and put

\[
\tau = \left( \sum_{i=1}^{m} \vert \mu_i \vert + \sum_{j=1}^{n} \vert \nu_j \vert \right) x^* .
\]

Then clearly \( \tau \in MW(S, A^*) \) and \( \vert \mu_i \vert \ll \vert \tau \vert , \, \vert \nu_j \vert \ll \vert \tau \vert \) for \( 1 \leq i \leq m , \, 1 \leq j \leq n \). Thus \( \vert \mu_i \vert = \omega_i \vert \tau \vert , \, \vert \nu_j \vert = \eta_j \vert \tau \vert \) for some \( \omega_i , \eta_j \in L_1(\vert \tau \vert) \). Thus

\[
\phi = \sum_{i=1}^{m} \varphi_i \omega_i \vert \tau \vert = \sum_{i=1}^{m} \varphi_i \omega_i \vert \tau \vert
\]

by Lemma 3.2. Let \( \sum_{i=1}^{m} \varphi_i \omega_i = g \). Similarly, we have

\[
\phi = \sum_{j=1}^{n} \psi_j \nu_j = h \vert \tau \vert
\]

for some \( h \in L_1(\vert \tau \vert, A^*) \). Since \( \phi = g \vert \tau \vert = h \vert \tau \vert \), we see from (*) in §2 that \( g = h \vert \tau \vert \) a.e. Since they are elements in \( L_1(\vert \tau \vert, A^*) \), \( g = h \). Now

\[
\sum_{i=1}^{m} F_{\mu_i}(\varphi_i \vert \mu_i \vert) = \sum_{i=1}^{m} \int (f_{\mu_i}, \varphi_i) d \vert \mu_i \vert
\]

\[
= \sum_{i=1}^{m} \int (f_\tau, \varphi_i) d \vert \mu_i \vert \quad (\vert \mu_i \vert \ll \vert \tau \vert)
\]

\[
= \sum_{i=1}^{m} \int (f_\tau, \varphi_i \omega_i) d \vert \tau \vert = \sum_{i=1}^{m} \int (f_\tau, \varphi_i \omega_i) d \vert \tau \vert
\]

\[
= \int \left( f_\tau, \sum_{i=1}^{m} \varphi_i \omega_i \right) d \vert \tau \vert = \int (f_\tau, g) d \vert \tau \vert = F_\tau(g \vert \tau \vert) .
\]

With similar argument we also have

\[
\sum_{j=1}^{n} F_{\nu_j}(\psi_j \vert \nu_j \vert) = F_\tau(h \vert \tau \vert) .
\]
Since $g = h$, we see
\[ \sum_{i=1}^{m} F_{\mu_i}(\varphi_i|\mu_i|) = \sum_{j=1}^{m} F_{\nu_j}(\psi_j|\nu_j|). \]
Thus $F$ is well defined on $M$.

Additivity of $F$ on $M$ follows from definition. Also, for $\lambda \in \mathbb{C}$,
\[
F\left(\lambda \sum_{i=1}^{n} \varphi_i|\mu_i|\right) = F\left(\sum_{i=1}^{n} \lambda \varphi_i|\mu_i|\right)
= \sum_{i=1}^{n} F_{\mu_i}(\lambda \varphi_i|\mu_i|) = \lambda \sum_{i=1}^{n} F_{\mu_i}(\varphi_i|\mu_i|),
\]
so $F$ is linear.

By definition
\[
F(\varphi|\mu|) = \int (f_{\mu}, \varphi) \, d|\mu| \quad (\varphi \in L_1(|\mu|, A^*)).
\]

We shall show that $F$ is bounded on $M$. Clearly
\[
|F_{\mu}(\varphi|\mu|)| \leq N_\infty(f_{\mu}) \|\varphi\|_1, |\mu| = N_{\infty}(f_{\mu}) \|\varphi|\mu|\|.
\]
So, for $f = (f_{\mu}) \in gl(S, A^{**})$,
\[
\|F_{\mu}\| \leq N_\infty(f_{\mu}) \leq \|f\|_\infty < \infty.
\]
That is, $F_{\mu}$ is bounded on $L_1(|\mu|, A^*)|\mu|$. Next suppose $\phi = \sum_{i=1}^{m} \varphi_i|\mu_i| \in M$. Let $\gamma = (\sum_{i=1}^{m} |\mu_i|) x^*$ for some $x^* \in A^*$, $\|x^*\| = 1$. Then $\gamma \in MW(S, A^*)$ and $|\mu_i| \leq |\gamma|$ for each $i$. Let $|\mu_i| = \omega_i|\gamma|$ for some $\omega_i \in L_1(|\gamma|)$, and let $\omega = \sum_{i=1}^{n} \varphi_i\omega_i$. Then $\phi = \omega|\gamma|$ and
\[
|F(\phi)| = |F_{\gamma}(\phi)| \leq \|F_{\gamma}\| \|\phi\| \leq \|f\|_\infty \|\phi\|.
\]
That is, $F$ is bounded on $M$ with $\|F\| \leq \|f\|_\infty$. It follows that $F$ has a unique extension (also denoted by $F$) to the closed linear span of $M$ with $\|F\| \leq \|f\|_\infty$ and then to $MW(S, A^*)^* = C_{0}^{**}(S, A)$ by the Hahn-Banach Extension Theorem (the extension need not be unique). This completes the proof of the theorem.

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