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A CHARACTERIZATION OF THE SECOND DUAL OF $C_0(S, A)$

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ABSTRACT. Let S be a locally compact Hausdorff space, and let A be a Banach space. The space of the continuous functions from S to A vanishing at infinity is denoted by $C_0(S, A)$. Let $MW(S, A^*)$ be the space of the representing measures of all the bounded linear functionals on $C_0(S, A)$. For $\mu \in MW(S, A^*)$ let

$$L_\infty(|\mu|, A^{**}, A^*) = \{f: S \to A^{**} | f(\cdot)x^* \in L_\infty(|\mu|)\ \forall x^* \in A^*\}.$$

The second dual of $C_0(S, A)$ is characterized in the general case by means of certain elements in the product linear space $\prod\{L_\infty(|\mu|, A^{**}, A^*): \mu \in MW(S, A^*)\}$.

1. Introduction

Let A be a Banach space, S be a locally compact Hausdorff space, and $\mathscr{B}(S)$ be the σ-algebra of all the Borel sets of S. The space of the continuous functions from S to A vanishing at infinity endowed with the uniform norm is denoted by $C_0(S, A)$. The second dual of $C_0(S, A)$ is considered for the case where S is compact and the dual A^* has the Radon-Nikodym property in [3], for the case where S is locally compact and A is a Banach algebra with a positive cone satisfying certain conditions in [4]. Recently a characterization of $C_0^{**}(S, A)$ by means of the "generalized functions" was given in [5] by the authors in the case where A^* and A^{**} have the Radon-Nikodym property. The purpose of this paper is to give a characterization of $C_0^{**}(S, A)$ in general by means of the elements in $\mathrm{gl}(S, A^{**})$ (see Definition 3.1).

Let $T: C_0(S, A) \to \mathbb{C}$ be a bounded linear functional, and let $M(S)$ be the space of all the bounded regular Borel measures on S. Then, since $\mathbb{C}$ is reflexive, T is weakly compact. There is a unique representing measure $m: \mathscr{B}(S) \to A^*$ such that $m(\cdot)x \in M(S)$ for all $x \in A$, $T(f) = \int f\, dm$ for all $f \in C_0(S, A)$ and $\|T\| = \tilde{m}(S)$, the semivariation of m (see, e.g., [2, 4]). Note that the total variation $|m|$ and $\tilde{m}$ are the same in this case [8, p. 54]. Let $MW(S, A^*)$ be the set of the representing measures m of all such functionals T. Since $|m|(S) = \tilde{m}(S) = \|T\|$ is finite, we see that $|m| \in M(S)$ [2, Theorem

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2.8] and that

$$C_0^*(S, A) = MW(S, A^*).$$

For $\mu \in MW(S, A^*)$ let $L_1(|\mu|, A^*)$ be the space of all the equivalence classes of A^*-valued Bochner integrable functions defined on S. Measures defined by densities in $L_1(|\mu|, A^*)$ are studied in §2. It is shown in Theorem 2.2 that the linear span M of all such measures in $MW(S, A^*)$ is contained between $M(S) \otimes A^*$ and its closure $[M(S) \otimes A^*]^-$ with respect to the semivariation norm. The second dual $C_0^{**}(S, A)$ is then characterized in §3.

2. Measures defined by densities in $L_1(|\mu|, A^*)$

Let $\varphi \in L_1(|\mu|, A^*)$, and define $\nu = \varphi|\mu| \in MW(S, A^*)$ by the Riesz Representation Theorem

$$\int g\, d\nu = \int \langle g, \varphi \rangle\, d|\mu| \qquad (g \in C_0(S, A)).$$

It follows that, for $E \in \mathscr{B}(S)$,

$$\nu(E)x = \int \langle \chi_E x, \varphi \rangle\, d|\mu| \qquad (x \in A)$$

and

$$(*) \qquad \nu(E) = \int_E \varphi\, d|\mu|.$$

Also $|\nu| = |\varphi|\mu|| = |\varphi|\,|\mu|$ by Dinculeanu [8, Theorem 6, p. 186]. Hence

$$(**) \qquad \|\nu\| = |\nu|(S) = (|\varphi|\,|\mu|)(S) = \int |\varphi|\, d|\mu| = \|\varphi\|_{1,|\mu|}.$$

We see from $(*)$ that $\nu \ll |\mu|$ (see [7, Definition 3, p. 11]). Therefore $\varphi \to \nu$ is a linear isometry from $L_1(|\mu|, A^*)$ into $\{\nu \in MW(S, A^*) : \nu \ll |\mu|\}$.

Definition 2.1. Denote by M the linear span in $MW(S, A^*)$ of all the measures defined by densities in $L_1(|\mu|, A^*)$ for all $\mu \in MW(S, A^*)$. That is,

$$M = \operatorname{span}\left\langle \bigcup \{L_1(|\mu|, A^*)|\mu| : \mu \in MW(S, A^*)\}\right\rangle.$$

We shall show in Theorem 2.2 that this subspace M is large enough for us to apply the Hahn-Banach Extension Theorem to characterize $C_0^{**}(S, A)$. Let us recall first a result from Cambern and Grein [3, Corollary 2] that $M(S) \otimes A^*$ can be embedded in $MW(S, A^*)$ in such a way that $m \otimes x^*$ corresponds to $m(\cdot)x^*$ for all $m \in M(S)$ and $x^* \in A^*$.

Theorem 2.2. *Let $[M(S) \otimes A^*]^-$ be the closure of $M(S) \otimes A^*$ with respect to the semivariation norm of $MW(S, A^*)$. Then*

$$[M(S) \otimes A^*]^- \supseteq M \supseteq M(S) \otimes A^*.$$

Proof. We show first that $M \supseteq M(S) \otimes A^*$. Let $m \in M(S)$ and $x^* \in A^*$. Then, by the Radon-Nikodym Theorem for scalar measures, $m = \phi|m|$ for some $\phi \in L_1(|m|)$. For any $\sum_{i=1}^N f_i \otimes x_i \in C_0(S) \otimes A$,

$$(m \otimes x^*)\left(\sum_1^N f_i \otimes x_i\right) = \sum_1^N m(f_i)x^*(x_i) = \sum_1^N \phi|m|(f_i) \cdot x^*(x_i)$$
$$= \sum_1^N \int \langle f_i x_i, \phi x^* \rangle \, d|m| = \sum_1^N (\phi x^*)|m|(f_i x_i)$$
$$= (\phi x^*)|m| \left(\sum_1^N f_i \otimes x_i\right).$$

Since $C_0(S, A) = C_0(S)\check{\otimes}A$ with the least cross norm, we see that $m \otimes x^* = (\phi x^*)|m| \in M$, so $M \supseteq M(S) \otimes A^*$.

Next we shall prove that $(L_1(|\mu|) \otimes A^*)|\mu| \subset M(S) \otimes A^*$. Recall that $(L_1(|\mu|)\hat{\otimes}A^*) = L_1(|\mu|, A^*)$, where the closure on the left is with respect to the greatest cross norm (see, e.g., Diestel and Uhl [7, Example 10, p. 228]). For $\sum_{i=1}^N f_i \otimes x_i \in C_0(S) \otimes A$, $\sum_{j=1}^M (\phi_j \otimes x_j^*) \in L_1(|\mu|) \otimes A^*$,

$$\sum_{j=1}^M (\phi_j \otimes x_j^*)|\mu| \left(\sum_{i=1}^N f_i \otimes x_i\right) = \int \left\langle \sum_{i=1}^N f_i \otimes x_i, \ \sum_{j=1}^M \phi_j \otimes x_j^* \right\rangle d|\mu|$$
$$= \sum_{j=1}^M \int \left\langle \sum_{i=1}^N f_i \otimes x_i, \ \phi_j \otimes x_j^* \right\rangle d|\mu| = \sum_{j=1}^M \sum_{i=1}^N \int \langle f_i, \phi_j \rangle x_j^*(x_i) \, d|\mu|$$
$$= \sum_{j=1}^M \sum_{i=1}^N \int \langle f_i, \phi_j \rangle \, d|\mu| \cdot x_j^*(x_i) = \sum_{j=1}^M \sum_{i=1}^N (\phi_j|\mu| \otimes x_j^*)(f_i \otimes x_i)$$
$$= \sum_{j=1}^M (\phi_j|\mu| \otimes x_j^*) \left(\sum_{i=1}^N f_i \otimes x_i\right).$$

By taking the limit process we see that

$$\sum_{j=1}^M (\phi_j \otimes x_j^*)|\mu| = \sum_{j=1}^M (\phi_j|\mu| \otimes x_j^*).$$

Thus

$$(L_1(|\mu|) \otimes A^*)|\mu| \subset M(S) \otimes A^* \subseteq MW(S, A^*).$$

Finally let $\varphi \in L_1(|\mu|, A^*)$, and let

$$\varphi_j = \sum_{j=1}^{N_j} \phi_i \otimes x_i^* \in L_1(|\mu|) \otimes A^*$$

be such that $\varphi_j \to \varphi$ in $L_1(|\mu|, A^*)$. For $f \in C_0(S, A)$, $\|f\| \leq 1$,

$$\left| \int f \, d\varphi_j|\mu| - \int f \, d\varphi|\mu| \right| = \left| \int \langle f, \varphi_j - \varphi \rangle \, d|\mu| \right|$$
$$= \left| \int f \, d(\varphi_j - \varphi)|\mu| \right| \leq \|f\| \cdot \|\varphi_j - \varphi\|_{1, |\mu|}$$
$$\leq \|\varphi_j - \varphi\|_{1, |\mu|} \to 0.$$

That is, $\varphi_j|\mu| \to \varphi|\mu|$ in $MW(S, A^*)$ with semivariation (= total variation) norm. Hence $M \subseteq [M(S) \otimes A^*]^-$. This completes the proof of the theorem.

Remark 2.3. It is worthwhile to note that if S is compact then $[M(S) \otimes A^*]^- = MW(S, A^*)$ if S is dispersed or A^* has the Radon-Nikodym property (see [3, Theorem 1]).

3. Characterization of the second dual

For $\mu \in MW(S, A^*)$ let $L_\infty(|\mu|)$ be the Banach space of all the equivalence classes of bounded Borel measurable (real-valued) functions on S with essential supremum norm (see, e.g., [11, p. 85]). Denote

$$L_\infty(|\mu|, A^{**}, A^*) = \{f: S \to A^{**} | f(\cdot)x^* \in L_\infty(|\mu|)\ \forall x^* \in A^*\}.$$

For details of the definition of these spaces see [10, Definition 3, p. 75 and §5, p. 78]. Functions in $L_\infty(|\mu|, A^{**}, A^*)$ are identified by the equivalence relation $f \equiv g(w)$, which means by definition

$$f(\cdot)x^* = g(\cdot)x^* \qquad (x^* \in A^*)$$

$|\mu|$-a.e. (the null set can depend on x^*) [10, p. 76].

Note that functions in $L_\infty(|\mu|, A^{**}, A^*)$ need not be $|\mu|$-measurable (hence not necessarily Bochner integrable) unless for example when A^{**} is separable [10, Corollary to Theorem 10, p. 73], in which case, $L_\infty(|\mu|, A^{**}, A^*) = L_\infty(|\mu|, A^{**})$, where $L_\infty(|\mu|, A^{**})$ stands for the space of the equivalence classes of A^{**}-valued Bochner integrable functions defined on S that are $|\mu|$-essentially bounded.

The space $L_\infty(|\mu|, A^{**}, A^*)$ is a Banach space with norm

$$N_\infty(g) = \inf\{\alpha > 0 : \{s : |g(s)| > \alpha\} \text{ is } |\mu|\text{-null}\}$$

[10, p. 74]. Note that, by the uniform boundedness principle, we have

$$N_\infty(g) \le \|g\| = \sup\{\|g(s)\| : s \in S\} \le \infty$$

[10, p. 75].

It is also shown in [10, Chapter VII, §4, Theorem 7, and Corollary] that

$$L_1(|\mu|, A^*)^* \cong L_\infty(|\mu|, A^{**}, A^*)$$

is an isometric isomorphism under the correspondence

$$(g, \varphi) = \int \langle g(s), \varphi(s)\rangle\, d|\mu|(s)$$

for $\varphi \in L_1(|\mu|, A^*)$, $g \in L_\infty(|\mu|, A^{**}, A^*)$.

Definition 3.1. Consider the product linear space $\prod\{L_\infty(|\mu|, A^{**}, A^*) : \mu \in MW(S, A^*)\}$. Let $f = (f_\mu)_{\mu \in MW(S, A^*)}$ be an element in this space satisfying the following two conditions:

(1) $\|f\|_\infty = \sup\{N_\infty(f_\mu) : \mu \in MW(S, A^*)\} < \infty$ and
(2) if $\mu, \nu \in M(S, A^*)$ are such that $|\mu| \ll |\nu|$, then $f_\mu \equiv f_\nu(w)$ with respect to $|\mu|$.

The set of all such elements f on S is denoted by $\mathrm{gl}(S, A^{**})$. With the norm $\|f\|_\infty$, $\mathrm{gl}(S, A^{**})$ is a Banach space.

Lemma 3.2. *Let $\mu, \nu \in MW(S, A^*)$. If $|\mu| \ll |\nu|$ with $|\mu| = \omega|\nu|$ for some $\omega \in L_1(|\nu|)$ and $\varphi \in L_1(|\mu|, A^*)$, $\psi = \omega\varphi$, then $\varphi|\mu| = \psi|\nu|$.*

Proof. Since $|\mu| \ll |\nu|$, by the Radon-Nikodym Theorem, there is $\omega \in L_1(|\nu|)$ such that $|\mu| = \omega|\nu|$ and

$$\int g\, d|\mu| = \int g\omega\, d|\nu| \qquad (g \in L_1(|\mu|)).$$

Fix $\varphi \in L_1(|\mu|, A^*)$ and let $\psi = \omega\varphi$. Then $\psi \in L_1(|\nu|, A^*)$. Let $f \in C_0(S, A)$

$$\begin{aligned}\int f\, d\varphi|\mu| &= \int \langle f, \varphi\rangle\, d|\mu| = \int \langle f, \varphi\rangle \omega\, d|\nu| \\ &= \int \langle f, \omega\varphi\rangle\, d|\nu| \quad (\omega \text{ is scalar-valued}) \\ &= \int \langle f, \psi\rangle\, d|\nu| = \int f\, d\psi|\nu|.\end{aligned}$$

Hence $\varphi|\mu| = \psi|\nu|$.

Theorem 3.3. *For each $F \in C_0^{**}(S, A)$, there is a (unique) $f \in \mathrm{gl}(S, A^{**})$ such that*

$$F(\varphi|\mu|) = \int \langle f_\mu, \varphi\rangle\, d|\mu| \qquad (\varphi \in L_1(|\mu|, A^*))$$

for all $\mu \in MW(S, A^*)$.

Proof. For each $F \in C_0^{**}(S, A)$, consider the bounded linear functional

$$\varphi \to F(\varphi|\mu|) \qquad (\varphi \in L_1(|\mu|, A^*)).$$

From the result of Tulcea and Tulcea [10, Chapter VII, §4] mentioned earlier, there is $f_\mu \in L_\infty(|\mu|, A^{**}, A^*) = L_1(|\mu|, A^*)^*$ such that

$$F(\varphi|\mu|) = \int \langle f_\mu, \varphi\rangle\, d|\mu| \qquad (\varphi \in L_1(|\mu|, A^*)).$$

Hence we have an $f = (f_\mu)_{\mu\in MW(S, A^*)} \in \mathrm{gl}(S, A^{**})$ if we can show properties (1) and (2) of Definition 3.1. Now

(1) For each $\mu \in MW(S, A^*)$, since $L_1(|\mu|, A^*)^* \cong L_\infty(|\mu|, A^{**}, A^*)$ is an isometric isomorphism and $\|\varphi|\mu|\| = \|\varphi\|_{1, |\mu|}$ (see $(**)$ in §2),

$$\begin{aligned}N_\infty(f_\mu) &= \sup\left\{\left|\int \langle f_\mu, \varphi\rangle\, d|\mu| : \right| : \|\varphi\|_{1, |\mu|} \le 1\right\} \\ &= \sup\left\{\int |F(\varphi|\mu|)| : \|\varphi\|_{1, |\mu|} \le 1\right\} \\ &\le \sup\{|F(\nu)| : \|\nu\| \le 1\} \\ &\le \sup\{\|F\|\,\|\nu\| : \|\nu\| \le 1\} \le \|F\| < \infty.\end{aligned}$$

(2) Suppose $|\mu| \ll |\nu|$ and $\varphi \in L_1(|\mu|, A^*)$. By the Riesz Representation Theorem, there is $\omega \in L_1(|\nu|)$ such that $|\mu| = \omega|\nu|$. Let $\psi = \omega\varphi$. Then, from

Lemma 3.2, $\varphi|\mu| = \psi|\nu|$. Hence, for all $\varphi \in L_1(|\mu|, A^*)$,

$$\begin{aligned}\int \langle f_\mu, \varphi\rangle\, d|\mu| &= F(\varphi|\mu|) = F(\psi|\nu|)\\ &= \int \langle f_\nu, \psi\rangle\, d|\nu| = \int \langle f_\nu, \omega\varphi\rangle\, d|\nu|\\ &= \int \langle f_\nu, \varphi\rangle \omega\, d|\nu| = \int \langle f_\nu, \varphi\rangle\, d|\mu|.\end{aligned}$$

Hence $f_\mu \equiv f_\nu(w)$. Thus $f = (f_\mu) \in \mathrm{gl}(S, A^{**})$.

The uniqueness is due to the fact that functions in $L_\infty(|\mu|, A^{**}, A^*)$ are identified by the equivalence relation $f \equiv g(w)$. This completes the proof of the theorem.

For $f_\mu \in L_\infty(|\mu|, A^{**}, A^*)$, define F_μ on $L_1(|\mu|, A^*)|\mu| \subseteq MW(S, A^*)$ by

$$F_\mu(\varphi|\mu|) = \int \langle f_\mu, \varphi\rangle\, d|\mu| \qquad (\varphi \in L_1(|\mu|, A^*)).$$

Lemma 3.4. *Let $\mu, \nu \in MW(S, A^*)$ with $|\mu| \ll |\nu|$. Then F_μ is linear and bounded and $F_\mu = F_\nu$ on $L_1(|\mu|, A^*)|\mu|$.*

Proof. Clearly F_μ is linear and bounded because

$$\begin{aligned}\|F_\mu\| &= \sup\{|F_\mu(\varphi|\mu|)| : \|\varphi|\mu|\,\| \le 1\}\\ &\le \sup\left\{\left|\int \langle f_\mu, \varphi\rangle\, d|\mu|\right| : \|\varphi\|_{1,|\mu|} \le 1\right\}\\ &\le \sup\{N_\infty(f_\mu)\cdot\|\varphi\|_{1,|\mu|} : \|\varphi\|_{1,|\mu|} \le 1\}\\ &\le N_\infty(f_\mu) < \infty.\end{aligned}$$

Since $|\mu| \ll |\nu|$, there is some $\omega \in L_1(|\nu|)$ such that $|\mu| = \omega|\nu|$. Let $\psi = \omega\varphi$. Then

$$\begin{aligned}F_\nu(\psi|\nu|) &= \int \langle f_\nu, \psi\rangle\, d|\nu| = \int \langle f_\nu, \varphi\rangle \omega\, d|\nu|\\ &= \int \langle f_\nu, \varphi\rangle\, d|\mu| = \int \langle f_\mu, \varphi\rangle\, d|\mu| = F_\mu(\varphi|\mu|),\end{aligned}$$

since $f_\mu \equiv f_\nu(w)$ and $L_\infty(|\mu|, A^{**}, A^*) = L_1(|\mu|, A^*)^*$ is an isometry. Now, since $\psi|\nu| = \varphi|\mu|$ by Lemma 3.2,

$$F_\mu(\varphi|\mu|) = F_\nu(\psi|\nu|) = F_\nu(\varphi|\mu|).$$

Consequently, $F_\mu = F_\nu$ on $L_1(|\mu|, A^*)|\mu|$.

Theorem 3.5. *For all $\mu \in MW(S, A)$ and $f = (f_\mu)_\mu \in \mathrm{gl}(S, A^{**})$, there is an $F \in C_0^{**}(S, A)$, such that*

$$F(\varphi|\mu|) = \int \langle f_\mu, \varphi\rangle\, d|\mu| \qquad (\varphi \in L_1(|\mu|, A^*)).$$

Proof. We shall define a bounded linear functional F on the linear span M defined in Definition 2.1 such that

$$F(\varphi|\mu|) = \int \langle f_\mu, \varphi\rangle\, d|\mu|$$

for $\varphi \in L_1(|\mu|, A^*)$ and $\mu \in MW(S, A^*)$. In general, an element ϕ in M is of the form

$$\phi = \sum_{i=1}^{m} \varphi_i |\mu_i|,$$

where $\varphi_i \in L_1(|\mu_i|, A^*)$ and $\mu_i \in MW(S, A^*)$. Define

$$F(\phi) = \sum_{i=1}^{m} F_{\mu_i}(\varphi_i|\mu_i|).$$

We shall show that F is independent of the special representation of ϕ. Suppose also $\phi = \sum_{j=1}^{n} \psi_j |\nu_j|$. Pick $x^* \in A^*$ with $\|x^*\| = 1$, and put

$$\tau = \left(\sum_{i=1}^{m} |\mu_i| + \sum_{j=1}^{n} |\nu_j|\right) x^*.$$

Then clearly $\tau \in MW(S, A^*)$ and $|\mu_i| \ll |\tau|$, $|\nu_j| \ll |\tau|$ for $1 \le i \le m$, $1 \le j \le n$. Thus $|\mu_i| = \omega_i|\tau|$, $|\nu_j| = \eta_j|\tau|$ for some ω_i, $\eta_j \in L_1(|\tau|)$. Thus

$$\begin{aligned} \phi &= \sum_{i=1}^{m} \varphi_i |\mu_i| = \sum_{i=1}^{m} \varphi_i \omega_i |\tau| \\ &= \left(\sum_{i=1}^{m} \varphi_i \omega_i\right) |\tau| \in L_1(|\tau|, A^*)|\tau| \end{aligned}$$

by Lemma 3.2. Let $\sum_{i=1}^{m} \varphi_i \omega_i = g$. Similarly, we have

$$\phi = \sum_{j=1}^{n} \psi_j |\nu_j| = h|\tau|$$

for some $h \in L_1(|\tau|, A^*)$. Since $\phi = g|\tau| = h|\tau|$, we see from $(*)$ in §2 that $g = h|\tau|$ a.e. Since they are elements in $L_1(|\tau|, A^*)$, $g = h$. Now

$$\begin{aligned} \sum_{i=1}^{m} F_{\mu_i}(\varphi_i|\mu_i|) &= \sum_{i=1}^{m} \int \langle f_{\mu_i}, \varphi_i \rangle \, d|\mu_i| \\ &= \sum_{i=1}^{m} \int \langle f_\tau, \varphi_i \rangle \, d|\mu_i| \qquad (|\mu_i| \ll |\tau|) \\ &= \sum_{i=1}^{m} \int \langle f_\tau, \varphi_i \rangle \omega_i \, d|\tau| = \sum_{i=1}^{m} \int \langle f_\tau, \varphi_i \omega_i \rangle \, d|\tau| \\ &= \int \left\langle f_\tau, \sum_{i=1}^{m} \int \varphi_i \omega_i \right\rangle d|\tau| = \int \langle f_\tau, g \rangle \, d|\tau| = F_\tau(g|\tau|). \end{aligned}$$

With similar argument we also have

$$\sum_{j=1}^{n} F_{\nu_j}(\psi_j|\nu_j|) = F_\tau(h|\tau|).$$

Since $g = h$, we see

$$\sum_{i=1}^{m} F_{\mu_i}(\varphi_i|\mu_i|) = \sum_{j=1}^{m} F_{\nu_j}(\psi_j|\nu_j|).$$

Thus F is well defined on M.

Additivity of F on M follows from definition. Also, for $\lambda \in \mathbb{C}$,

$$F\left(\lambda \sum_{i=1}^{n} \varphi_i|\mu_i|\right) = F\left(\sum_{i=1}^{n} \lambda\varphi_i|\mu_i|\right)$$
$$= \sum_{i=1}^{n} F_{\mu_i}(\lambda\varphi_i|\mu_i|) = \lambda \sum_{i=1}^{n} F_{\mu_i}(\varphi_i|\mu_i|),$$

so F is linear.

By definition

$$F(\varphi|\mu|) = \int \langle f_\mu, \varphi\rangle \, d|\mu| \qquad (\varphi \in L_1(|\mu|, A^*)).$$

We shall show that F is bounded on M. Clearly

$$|F_\mu(\varphi|\mu|)| \le N_\infty(f_\mu)\|\varphi\|_{1,|\mu|} = N_\infty(f_\mu)\|\varphi|\mu|\,\|.$$

So, for $f = (f_\mu) \in \mathrm{gl}(S, A^{**})$,

$$\|F_\mu\| \le N_\infty(f_\mu) \le \|f\|_\infty < \infty.$$

That is, F_μ is bounded on $L_1(|\mu|, A^*)|\mu|$. Next suppose $\phi = \sum_{i=1}^{m} \varphi_i|\mu_i| \in M$. Let $\gamma = (\sum_{i=1}^{m} |\mu_i|)x^*$ for some $x^* \in A^*$, $\|x^*\| = 1$. Then $\gamma \in MW(S, A^*)$ and $|\mu_i| \ll |\gamma|$ for each i. Let $|\mu_i| = \omega_i|\gamma|$ for some $\omega_i \in L_1(|\gamma|)$, and let $\omega = \sum_{i=1}^{n} \varphi_i\omega_i$. Then $\phi = \omega|\gamma|$ and

$$|F(\phi)| = |F_\tau(\phi)| \le \|F_\tau\|\,\|\phi\| \le \|f\|_\infty\|\phi\|.$$

That is, F is bounded on M with $\|F\| \le \|f\|_\infty$. It follows that F has a unique extension (also denoted by F) to the closed linear span of M with $\|F\| \le \|f\|_\infty$ and then to $MW(S, A^*)^* = C_0^{**}(S, A)$ by the Hahn-Banach Extension Theorem (the extension need not be unique). This completes the proof of the theorem.

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