PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 120, Number 1, January 1994

CONSTRUCTING UV^k-MAPS BETWEEN SPHERES

STEVEN C. FERRY

(Communicated by James West)

ABSTRACT. The purpose of this note is to give a quick proof of an extremely counterintuitive theorem of Bestvina, Walsh, and Wilson. The theorem says, for example, that the degree 2 map $d_2: S^3 \to S^3$ is homotopic to a map such that $p^{-1}(x)$ is connected for each $x \in S^3$.

We will say that a map $p: M^n \to B$ from a manifold to a polyhedron is UV^k if, for every $b \in B$ and neighborhood U of b, there is a neighborhood V of b contained in U so that $\pi_l(p^{-1}(V)) \xrightarrow{0} \pi_l(p^{-1}(V))$ for all $l \le k$. Since B is a polyhedron, this is equivalent to saying that, for each contractible open $U \subset B$, $\pi_i(p^{-1}(U)) = 0$ for $0 \le i \le k$. The composition of UV^k-maps is UV^k. If p is UV^k for $k \ge 0$, then $p^{-1}(b)$ is connected for each $b \in B$. See [8] for a full discussion.

Theorem 1 (Bestvina-Walsh-Wilson [2, 9, 11]). *Every map $f: S^n \to S^n$ is homotopic to a $UV^{[(n-3)/2]}$-map.*

Proof. In [3] Černavskii gives an elementary construction in one (dense) page of a UV^k-map from D^n to D^{n+l} whenever $2k+3 \le n$. His theorem generalizes earlier work of Anderson [1] and Keldyš [7]. His idea is: Since the composition of UV^k-maps is UV^k, it suffices to construct a UV^k-map from D^n to D^{n+1}. Consider D^{n+1} to be the suspension of D^n, and let P and Q be the top and bottom pieces of the boundary of D^{n+1}. There is a "line field" consisting of vertical segments connecting P to Q (see Figure 1 on the next page).

Let S be the $k = [\frac{n-3}{2}]$-skeleton of a fine triangulation of D^n, and let T be a dual $(n-k-1)$-skeleton. Let P_1 consist of P together with "stalactites" consisting of segments from the line field running from P through points of S and stopping just short of Q. Let Q_1 consist of Q together with similarly defined "stalagmites" running from Q almost to P. There is a new line field of short segments connecting P_1 to Q_1. (The blown-up 2-dimensional picture in Figure 1 is accurate!) There is a D_1^n consisting of midpoints of these segments. Now perform the same operation on D_1^n, using the new line field.

Received by the editors June 5, 1992.
1991 *Mathematics Subject Classification*. Primary 57Q99; Secondary 55R05, 57N60.
Key words and phrases. Space-filling curve, UV^k-map, cell-like map.
The author was partially supported by National Science Foundation Grant DMS 9003746.

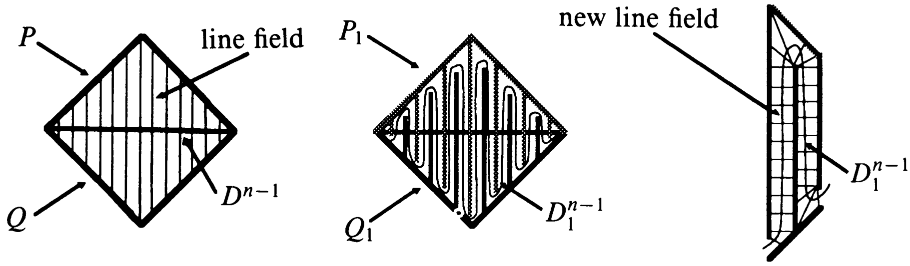

FIGURE 1

Performing this process repeatedly and taking the limit gives a map from D^n onto D^{n+1}. The limit map in UV^k. To show this, it suffices to show that every map $S^l \to D_i^n$ with small diameter extends to a map $D^{l+1} \to D_i^n$ with small diameter. A map $S^l \to D_i^n$, $l \leq k$, extends to a map $D^{l+1} \to D^{n+1}$ by convexity, whence it can be moved off of P_i and Q_i by general position and then contracted into D_i^n using the ith (very, very short!!) line field. See [3] for details.

The construction clearly extends to give a $UV^{[(n-3)/2]}$-map of M^n onto $M^n \times D^l$ for any PL manifold M. To complete the proof of Theorem 1, we need only invoke the following theorem from [5].

Theorem 2. *If M is a PL manifold, B is a polyhedron, and $f: M \to B$ is a map inducing isomorphisms on homotopy groups in dimensions $\leq k$ and an epimorphism in dimension $k+1$, then there is a UV^k-map $\overline{f}: M \times D^l \to B$ for some l so that $f \circ \operatorname{proj}$ is homotopic to $\overline{f}$.*

According to Theorem 2, if $f: S^n \to S^n$ is a map, there is a UV^{n-2}-map $\overline{f}: S^n \times D^l \to S^n$ which is homotopic to $f \circ \operatorname{proj}$. Composing $\overline{f}$ with a $UV^{[(n-3)/2]}$-map from S^n to $S^n \times D^l$ finishes the construction, proving Theorem 1. □

In proving Theorem 2, we will work with finite CW complexes built using PL cells and PL attaching maps. All of our spaces will be polyhedra, and all of the maps used in this part of the construction will be PL. We begin with an easy lemma.

Lemma 3. *Let K be a finite polyhedron, and let $i: K \to K \cup_f D^{r+2}$ be the inclusion. Then there exist a finite polyhedron K', a CE-PL map $c: K' \to K$, and a UV^r-map $i': K' \to K$ such that $i \circ c \simeq i'$. (A CE-PL map is a PL surjection with contractible point-inverses.)*

Proof. Let $K' = M(f)$ be the mapping cylinder of f, let $c: K' \to K$ be the mapping cylinder collapse, and let i' be the map which pinches the top of the mapping cylinder to the center point of the attached D^{r+2}. The only noncontractible point-inverse is an S^{r+1}. □

Proof of Theorem 2. By Whitehead's cell-trading lemma ([10, p. 246] or [4, 7.3]), there exist a polyhedron Z and CE-PL maps $c_1: Z \to M(f)$, $c_2: Z \to Q$, where Q is a finite PL cell complex obtained from M by attaching cells of dimension $\geq k+2$. We write $M = M_0 \subset M_1 \subset \cdots \subset M_t = Q$, where each M_{i+1} is obtained from M_i by attaching a PL r-cell, $r \geq k+2$. We will use

$$\begin{tikzcd}
L \arrow[d, "CE"'] \arrow[dr, "UV^k"] & & \\
L_1 \arrow[d, "CE"'] \arrow[dr, "UV^k"] & L_2 \arrow[d, "CE"'] \arrow[dr, "UV^k"] & \\
M \arrow[r] & M_{t-1} \arrow[r] & M_t = Q
\end{tikzcd}$$

FIGURE 2

$$\begin{tikzcd}
M \arrow[r] \arrow[rrr, bend left, "f"] & Q \arrow[r] & M(f) \arrow[r, "CE"] & B \\
L \arrow[u, "CE"] \arrow[ur, "UV^k"'] & Z \arrow[u, "CE"] \arrow[ur, "UV^k"'] & & \\
J \arrow[u, "CE"] \arrow[ur, "UV^k"'] & & & \\
M \times D^l \arrow[u, "CE"] \arrow[uuurrr, "\bar{f}"] & & &
\end{tikzcd}$$

FIGURE 3

induction on t to construct a finite polyhedron L, a CE-PL map $q: L \to M$, and a PL UV^k-map $p: L \to Q$ so that $i \circ q \simeq p$, where $i: M \to Q$ is the inclusion. Inductively, we assume that we have constructed L_1 with a CE map to M and a UV^k-map to M_{t-1}. By the lemma, there is an L_2 with a CE map to M_{t-1} and a UV^k-map to M_t. Taking the pullback in Figure 2 gives the required L.

Consider Figure 3. The first two rows have already been constructed. Construct J by taking the pullback. Embedding J in $M \times D^l$ for some l, we have a CE-PL regular neighborhood collapse from $M \times D^l$ to J. □

Remark. Lacher has shown that, if $f: S^n \to S^n$ is a map of degree $\neq \pm 1$, then f is not homotopic to a UV^k-map with $k > [\frac{n-3}{2}]$. In [6] the author has generalized Theorem 1 to the case where the homotopy fiber has finite, rather than trivial, skeleta through a range. That writeup contains more discussion as well.

ACKNOWLEDGMENTS

We would like to thank Mladen Bestvina, Robert Daverman, and John Walsh for a series of helpful e-mails. We would also like to thank the University of Chicago and the Institute for Advanced Study for hospitality and support while this note was being written.

REFERENCES

1. R. D. Anderson, *Open mappings of compact continua*, Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 347–349.
2. M. Bestvina, *Characterizing k-dimensional universal Menger compacta*, Mem. Amer. Math. Soc., no. 71, Amer. Math. Soc., Providence, RI, 1988.
3. A. V. Černavskii, *A generalization of Keldysh's construction of a monotone map of a cube onto a cube of higher dimension*, Russian Math. Surveys **40** (1985), 165–167.
4. M. M. Cohen, *A course in simple-homotopy theory*, Springer-Verlag, New York, 1970.
5. S. C. Ferry, *A stable converse to the Vietoris-Smale theorem with applications to shape theory*, Trans. Amer. Math. Soc. **261** (1980), 369–386.
6. ——, *Mapping manifolds to polyhedra*, preprint.
7. L. V. Keldyš, *A monotone map of a cube onto a cube of higher dimension*, Mat. Sb. **41** (1957), 129–158.
8. R. C. Lacher, *Cell-like mappings and their generalizations*, Bull. Amer. Math. Soc. **83** (1977), 495–552.
9. J. J. Walsh, *A general method of constructing UV^k-mappings on manifolds with applications to spheres*, preprint, 1974.
10. J. H. C. Whitehead, *Simplicial spaces, nucleii, and m-groups*, Proc. London Math. Soc. (3) **45** (1939), 243–327.
11. D. Wilson, *On constructing monotone and UV^1 mappings of arbitrary degree*, Duke Math. J. **40** (1973), 103–109.

DEPARTMENT OF MATHEMATICAL SCIENCES, STATE UNIVERSITY OF NEW YORK AT BINGHAMTON, BINGHAMTON, NEW YORK 13901

E-mail address: steve@math.binghamton.edu