PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 120, Number 1, January 1994

INTEGRABILITY OF SUPERHARMONIC FUNCTIONS AND SUBHARMONIC FUNCTIONS

HIROAKI AIKAWA

(Communicated by J. Marshall Ash)

Dedicated to Professor Ohtsuka on the occasion of his seventieth birthday

ABSTRACT. We apply the coarea formula to obtain integrability of superharmonic functions and nonintegrability of subharmonic functions. The results involve the Green function. For a certain domain, say Lipschitz domain, we estimate the Green function and restate the results in terms of the distance from the boundary.

1. INTRODUCTION

Let D be a domain in $\mathbb{R}^n$ with $n \geq 2$. Integrability of superharmonic functions, subharmonic functions and harmonic functions on D has been considered by many authors [2, 3, 7–10, 12–16]. In this paper we shall apply the coarea formula to obtain integrability of superharmonic functions and nonintegrability of subharmonic functions. Our results involve the Green function for D. Let D be a regular domain with Green function $G(x, y)$. Let $x_0 \in D$ and write $g(x) = G(x, x_0)$. Our main theorem is the following.

Theorem. *Let $\varphi(t)$ be a nonnegative function on $(0, \infty)$. Let $c_2 = 2\pi$ and $c_n = (n-2)\sigma_n$ for $n \geq 3$ where σ_n is the surface measure of a unit sphere.*

(i) *If u is a superharmonic function on D, then*

$$\int_D u(x)\varphi(g(x))|\nabla g(x)|^2\, dx \leq c_n u(x_0) \int_0^\infty \varphi(t)\, dt.$$

(ii) *If s is a subharmonic function on D, then*

$$\int_D s(x)\varphi(g(x))|\nabla g(x)|^2\, dx \geq c_n s(x_0) \int_0^\infty \varphi(t)\, dt.$$

Our Theorem has the following corollaries. Let D be a proper subdomain and put $\delta(x) = \operatorname{dist}(x, \partial D)$. Since $g(x)$ is a positive harmonic function on $D \setminus \{x_0\}$, it is easy to see that $|\nabla g(x)| \leq Mg(x)/\delta(x)$ for x apart from a

Received by the editors January 6, 1992.
1991 *Mathematics Subject Classification*. Primary 31B05.
Key words and phrases. Superharmonic function, subharmonic function, Green function, coarea formula.

neighborhood of x_0. Let $x_1 \in D$. By the Harnack principle there is a constant $M > 1$ such that $M^{-1} g(x) \leq G(x, x_1) \leq M g(x)$ for x close to the boundary. Hence we have

Corollary 1. *Let φ satisfy the doubling condition:*

$$\sup_{T \leq t \leq 2T} \varphi(t) \leq M \inf_{T \leq t \leq 2T} \varphi(t) \tag{1}$$

for $T > 0$ with $M \geq 1$ independent of T. Suppose

$$\int_0^T \varphi(t)\, dt = \infty \quad \text{for some } T > 0. \tag{2}$$

If s is a nonnegative nonzero subharmonic function on D, then for any compact subset E of D

$$\int_{D \setminus E} s(x) \varphi(g(x)) \frac{g(x)^2}{\delta(x)^2}\, dx = \infty.$$

For some domains we can estimate $g(x)$ by $\delta(x)$. By $B(x, r)$ we denote the open ball with radius r and center at x. We say that D is uniformly Δ-regular if there is a constant ε_1, $0 < \varepsilon_1 < 1$, such that, for all $x \in \partial D$ and all r, $0 < r < r_0$,

$$w_{x,r} \leq 1 - \varepsilon_1 \quad \text{on } B(x, r/2) \cap D, \tag{3}$$

where $w_{x,r}$ is the harmonic measure of $\partial B(x, r) \cap D$ in the region $B(x, r) \cap D$ [1, Definition 2]. It is known that D is uniformly Δ-regular if and only if it satisfies the capacity density condition (CDC) (see [6, 18]). For such a domain there are constants β, $0 < \beta \leq 1$, and $M > 0$ such that

$$g(x) \leq M \delta(x)^\beta \tag{4}$$

for x close to ∂D (see [1]). Letting $\varphi(t) = 1/t$ in Corollary 1, we obtain

Corollary 2. *Let D be a uniformly Δ-regular domain and let β be as above. If s is a nonnegative nonzero subharmonic function on D, then for any compact subset E of D*

$$\int_{D \setminus E} s(x) \delta(x)^{\beta - 2}\, dx = \infty. \tag{5}$$

Let $T_\psi = \{x : x_n > |x| \cos \psi\}$. This is a cone with vertex at the origin and aperture ψ. It is not so difficult to see that there is a positive harmonic function u_ψ on T_ψ such that $u_\psi = 0$ on ∂T_ψ; such a function u_ψ is unique up to a multiplicative constant and is homogeneous of degree $\alpha = \alpha_n(\psi) > 0$, i.e., $u_\psi(rx) = r^\alpha u_\psi(x)$ for $r > 0$. This constant $\alpha_n(\psi)$ is referred to as the maximal order of barriers (see [7, p. 271]). It is known that α_n is strictly decreasing; $\alpha_n(\pi/2) = 1$; $\lim_{\psi \to 0} \alpha_n(\psi) = \infty$; $\lim_{\psi \to \pi} \alpha_n(\psi) = 0$ (for $n \geq 3$); $\alpha_2(\psi) = \pi/(2\psi)$; and $\alpha_4(\psi) = \pi/\psi - 1$.

We say that D satisfies the exterior cone condition with aperture ψ if there exists $\rho > 0$ such that for each $y \in \partial D$ there is a truncated cone of radius ρ with vertex at y and aperture ψ lying outside D. It is easy to see that if D satisfies the exterior cone condition with aperture ψ, then D is uniformly Δ-regular and the constant β in (4) can be taken as $\alpha_n(\pi - \psi)$. Estimating the derivative of a certain conformal mapping, Masumoto [8] proved this result for the plane case. (Note that the constant θ in his notation is related to ψ as $\pi\theta = 2\psi$.)

Corollary 3. *Let D satisfy the exterior cone condition with aperture ψ and let $\beta = \alpha_n(\pi - \psi)$. If s is a nonnegative nonzero subharmonic function on D, then* (5) *holds for any compact subset E of D.*

Our Theorem also yields integrability of superharmonic functions. For this purpose we need to give a lower bound of $|\nabla g(x)|$. It is, in general, difficult to obtain the bound; the pointwise estimate $|\nabla g(x)| \geq Mg(x)/\delta(x)$ does not hold. However, we can prove that $|\nabla g(x)| \geq Mg(x)/\delta(x)$ in a certain sense (see Lemma 1 below) for an NTA domain introduced by [5]. An NTA domain is uniformly Δ-regular. As a result we have

Corollary 4. *Let D be an NTA domain. Let φ satisfy* (1). *Suppose*

$$\int_0^T \varphi(t)\,dt < \infty \quad \text{for } T > 0.$$

Then every nonnegative superharmonic function u on D satisfies

$$\int_{D \setminus B(x_0, r_1)} u(x)\varphi(g(x))\frac{g(x)^2}{\delta(x)^2}\,dx < \infty$$

for any $r_1 > 0$.

We say that a bounded domain D is k-Lipschitz if D and ∂D are given locally by a Lipschitz function whose Lipschitz constant is at most k. If D is k-Lipschitz for some $k > 0$, then we say that D is a Lipschitz domain. A Lipschitz domain is an NTA domain. Let D be a k-Lipschitz domain and let α_n be the maximal order of barriers as before. Then it is known that

$$g(x) \geq M\delta(x)^\alpha \quad \text{for } x \in D,$$
$$g(x) \leq M\delta(x)^\beta \quad \text{for } x \in D \setminus B(x_0, r_2),$$

where $\alpha = \alpha_n(\psi)$, $\beta = \alpha_n(\pi - \psi)$, and $\psi = \arctan(1/k)$, $0 < \psi < \pi/2$ [7, Proposition 2]. We remark that $0 < \beta < 1 < \alpha$. Since $\int_0 t^{\varepsilon - 1}\,dt < \infty$ for $\varepsilon > 0$, we obtain the following corollary, which improves [7, Theorem 8] with $p = 1$.

Corollary 5. *Let D be a k-Lipschitz domain and let α and β be as above.*

(i) *If u is a nonnegative superharmonic function on D, then for $\varepsilon > 0$*

$$\int_D u(x)\delta(x)^{\varepsilon + \alpha - 2}\,dx < \infty.$$

(ii) *If s is a nonnegative nonzero subharmonic function on D, then for any compact subset E of D*

$$\int_{D \setminus E} s(x)\delta(x)^{\beta - 2}\,dx = \infty.$$

For the plane case Masumoto [9] proved a result more general than the above (i). He also informed us that Stegenga and Ullrich [13] recently proved the integrability of superharmonic functions on a Hölder domain and a John domain. However, our method is completely different and gives sharp exponents for at least Lipschitz domains (see also Corollary 6 below). Another advantage of the use of the coarea formula is that it enables us to deal with superharmonic functions and subharmonic functions, simultaneously.

Recently, Maeda pointed out that $\alpha_n(\psi) = 2$ for $\cos\psi = 1/\sqrt{n}$. Hence we have the following: *If* $0 < k < 1/\sqrt{n-1}$, *then every nonnegative superharmonic function on a* k*-Lipschitz domain* D *is integrable over* D.

The plan of this paper is as follows. We prove our Theorem in the next section, and Corollary 4 in §3. Other corollaries are almost straightforward. In §4 we give some L^p-integrability results.

I would like to thank Professors Fumi-Yuki Maeda, Makoto Masumoto, and Noriaki Suzuki for valuable conversations.

2. Proof of the Theorem

Our main tool for the proof of the Theorem is the coarea formula. For the reader's convenience we state it below. For a proof see, e.g., [11, pp. 37–39].

Lemma (Coarea formula). *Let* Ω *be an open set in* $\mathbb{R}^n$. *Suppose* ψ *is a Borel measurable function on* Ω *and* f *is a smooth function on* Ω. *Then*

$$\int_\Omega \psi(x)|\nabla f(x)|\,dx = \int_0^\infty dt \int_{\{x\in\Omega\,:\,|f(x)|=t\}} \psi(x)\,d\sigma(x),$$

where σ *is the surface measure.*

Proof of Theorem. Let $D_t = \{x \in D : g(x) > t\}$ and let $G_t(x, y)$ be the Green function for D_t. Obviously $x_0 \in D_t$ and $D_t \uparrow D$ as $t \downarrow 0$. By the Sard theorem (see, e.g., [11, Corollary, p. 35]), we find a set $\mathscr{E}$ of linear measure 0 in $(0, \infty)$ such that $\partial D_t = \{x \in D : g(x) = t\}$ is smooth for $t \notin \mathscr{E}$. Observe that $G_t(x, x_0) = g(x) - t$, and hence the harmonic measure ω_t at x_0 for D_t is given by

$$c_n \frac{d\omega_t}{d\sigma} = -\frac{\partial G_t(\cdot, x_0)}{\partial n} = -\frac{\partial g}{\partial n} = |\nabla g| \quad \text{on } \partial D_t$$

for $t \notin \mathscr{E}$, since the outward normal n on ∂D_t is equal to $-\nabla g/|\nabla g|$. Hence the Poisson integral formula becomes

$$u(x_0) \geq \frac{1}{c_n}\int_{\partial D_t} u|\nabla g|\,d\sigma \quad \text{for a superharmonic function } u \text{ on } D,$$

$$s(x_0) \leq \frac{1}{c_n}\int_{\partial D_t} s|\nabla g|\,d\sigma \quad \text{for a subharmonic function } s \text{ on } D.$$

Let us invoke the coarea formula with $\psi = u\varphi(g)|\nabla g|$, $f = g$, and $\Omega = D$. Then

$$\int_D u(x)\varphi(g(x))|\nabla g(x)|^2\,dx = \int_0^\infty \varphi(t)\,dt \int_{\partial D_t} u|\nabla g|\,d\sigma \leq c_n u(x_0) \int_0^\infty \varphi(t)\,dt.$$

Thus (i) follows. Similarly, we obtain (ii) by letting $\psi = s\varphi(g)|\nabla g|$, $f = g$, and $\Omega = D$. The Theorem is proved.

Remark. Letting $\varphi(t) = 1/T$ for $0 < t < T$, we obtain the following:

(i) If u is a superharmonic function on D, then

$$u(x_0) \geq \frac{1}{c_n} \limsup_{T\to 0} \frac{1}{T} \int_{\{x\in D\,:\,0<g(x)<T\}} u(x)|\nabla g(x)|^2\,dx.$$

(ii) If s is a subharmonic function on D, then

$$s(x_0) \leq \frac{1}{c_n} \liminf_{T \to 0} \frac{1}{T} \int_{\{x \in D : 0 < g(x) < T\}} s(x) |\nabla g(x)|^2 \, dx.$$

3. Proof of Corollary 4

Let $\{Q_j\}$ be the Whitney decomposition of D. We may assume that $x_0 \in Q_0$ and we shall omit this cube Q_0 in the succeeding argument. For each Whitney cube Q_j we let x_j be the center of Q_j, r_j the diameter of Q_j, and $t_j = \operatorname{dist}(Q_j, \partial D)$. We write M for a positive constant independent of Q_j, whose value may change from one occurrence to the next. If $M^{-1} f \leq g \leq M f$ for two positive quantities f and g, then we write $f \approx g$. Observe that $r_j \approx t_j$. Let $x_j^* \in \partial D$ be a point such that $\operatorname{dist}(x_j^*, Q_j) = t_j$. It is well known that the boundary Harnack principle holds for an NTA domain [5, Lemma 4.10]. In view of (3) and the boundary Harnack principle, we can find a constant $M_1 > 1$ such that

$$g(x) \leq (1 - \varepsilon_1) \inf_{Q_j} g \quad \text{for } x \in D \cap B(x_j^*, t_j/M_1).$$

Hence we can find a constant M_2 with the following property: Let $\widetilde{Q}_j$ be the cube with the same center x_j as Q_j but expanded M_2 times. Then

(i) there is a point $x' \in \widetilde{Q}_j$ such that $g(x') \leq (1 - \varepsilon_1) \inf_{Q_j} g$;

(ii) $\operatorname{dist}(\widetilde{Q}_j, \partial D) \approx r_j$.

Observe that only a finite number of cubes $\widetilde{Q}_0, \dots, \widetilde{Q}_{j_0}$ meet $\widetilde{Q}_0$. We shall omit these cubes in the succeeding argument. Let $g(x_j) = g_j$. By the Harnack inequality $g(x) \approx g_j$ for $x \in Q_j$. By $m(E)$ we denote the Lebesgue measure of E.

Lemma 1. *Let $\widetilde{Q}_j$ be as above. There exists a positive constant ε_2 such that*

$$E_j = \{x \in \widetilde{Q}_j : |\nabla g(x)| \geq \varepsilon_2 g_j / r_j\}$$

satisfies $m(E_j) \geq M m(\widetilde{Q}_j)$.

Proof. Observe that $|\nabla g| \leq M g_j / r_j$ on $\widetilde{Q}_j$. Let $x' \in \widetilde{Q}_j$ be as before the lemma. For $x \in Q_j$ we let l be the line segment connecting x' and x. Note that $l \subset \widetilde{Q}_j$ and $|l| \leq M r_j$ uniformly for $x \in Q_j$, where $|l|$ is the length of l. We have for $\varepsilon > 0$

$$\begin{aligned}
\varepsilon_1 g_j &\leq M \varepsilon_1 g(x) \leq M(g(x) - g(x')) \leq M \int_l |\nabla g| \, ds \\
&\leq M \int_{\{y \in l : |\nabla g(y)| \leq \varepsilon g_j / r_j\}} \varepsilon \frac{g_j}{r_j} \, ds + M \int_{\{y \in l : |\nabla g(y)| \geq \varepsilon g_j / r_j\}} M \frac{g_j}{r_j} \, ds \\
&\leq M |l| \varepsilon \frac{g_j}{r_j} + M \frac{g_j}{r_j} \left| \left\{ y \in l : |\nabla g(y)| \geq \varepsilon \frac{g_j}{r_j} \right\} \right| \\
&\leq M_3 \varepsilon g_j + M \frac{g_j}{r_j} \left| \left\{ y \in l : |\nabla g(y)| \geq \varepsilon \frac{g_j}{r_j} \right\} \right|.
\end{aligned}$$

Letting $\varepsilon > 0$ so small that $M_3\varepsilon \le \varepsilon_1/2$, we obtain

$$\frac{\varepsilon_1}{2} g_j \le M \frac{g_j}{r_j} \left| \left\{ y \in l : |\nabla g(y)| \ge \varepsilon \frac{g_j}{r_j} \right\} \right| ,$$

whence $|\{y \in l : |\nabla g(y)| \ge \varepsilon g_j/r_j\}| \ge M r_j$. This inequality holds for any $x \in Q_j$. Therefore, Fubini's theorem asserts that for $\varepsilon_2 = \varepsilon$

$$E_j = \{y \in \widetilde{Q}_j : |\nabla g(y)| \ge \varepsilon_2 g_j/r_j\}$$

satisfies $m(E_j) \ge M m(\widetilde{Q}_j)$.

Lemma 2. *Let Q be a cube in $\mathbb{R}^n$. For $M_4 > 1$ we let Q^* be the cube with the same center as Q but expanded M_4 times. Suppose E is a measurable subset of Q. If u is a nonnegative superharmonic function on Q^*, then*

$$\frac{1}{m(Q)} \int_Q u\, dx \le \frac{M_5}{m(E)} \int_E u\, dx ,$$

where M_5 depends only on the dimension and M_4.

Proof. Let $G^*(x, y)$ be the Green function for Q^*. Taking the balayage over Q, if necessary, we may assume that u is a Green potential $\int G^*(\cdot, y)\, d\mu(y)$ with measure μ on the closure of Q. Let r be the diameter of Q. Then it is easy to see that

$$M r^{2-n} m(E) \le \int_E G^*(x, y)\, dx \le \int_Q G^*(x, y)\, dx \le M r^2$$

uniformly for y in the closure of Q, where M depends only on the dimension and M_4. Hence Fubini's theorem yields

$$\frac{1}{m(Q)} \int_Q u\, dx \le \frac{M r^2}{m(Q)} \|\mu\| = M r^{2-n} \|\mu\| \le \frac{M}{m(E)} \int_E u\, dx.$$

The lemma is proved.

Proof of Corollary 4. By (1) and the Harnack inequality we have

$$\varphi(g(x)) g(x)^2 \delta(x)^{-2} \approx \varphi(g_j) g_j^2 r_j^{-2}$$

for $x \in \widetilde{Q}_j$. Let us apply Lemma 2 to $Q = \widetilde{Q}_j$ and $E = E_j$. Then we obtain from Lemma 1 that

$$\begin{aligned} \int_{\widetilde{Q}_j} u\varphi(g) g^2 \delta^{-2}\, dx &\le M\varphi(g_j) g_j^2 r_j^{-2} \int_{\widetilde{Q}_j} u\, dx \le M\varphi(g_j) g_j^2 r_j^{-2} \int_{E_j} u\, dx \\ &\le M \int_{E_j} u\varphi(g_j) |\nabla g|^2\, dx \le M \int_{\widetilde{Q}_j} u\varphi(g) |\nabla g|^2\, dx. \end{aligned}$$

Note that the multiplicity of $\widetilde{Q}_j$ is bounded, i.e., $\sum \chi_{\widetilde{Q}_j} \le N$. Summing up the integral over $\widetilde{Q}_j$, we obtain from the Theorem that

$$\int_{D \setminus (\widetilde{Q}_0 \cup \cdots \cup \widetilde{Q}_{J_0})} u\varphi(g) g^2 \delta^{-2}\, dx \le M \int_{D \setminus \widetilde{Q}_0} u\varphi(g) |\nabla g|^2\, dx \le M u(x_0) < \infty.$$

Since $u\varphi(g) g^2 \delta^{-2}$ is integrable on any compact subset of $D \setminus \{x_0\}$, we obtain the corollary.

4. L^p-INTEGRABILITY

In this section we shall prove the following corollary, which gives an answer to Problems 3.32 and 3.34 raised by Armitage and Gardiner in [4].

Corollary 6. *Let D be a k-Lipschitz domain and let α and β be as in Corollary 5.*

(i) *If u is a nonnegative superharmonic function on D, then*

$$\int_D u(x)^p\,dx < \infty$$

for $0 < p < \min\{n/(n+\alpha-2),\, 1/(\alpha-1)\}$.

(ii) *Let $0 < p \le 1$. If s is a nonnegative nonzero subharmonic function on D, then for any compact subset E of D*

$$\int_{D\setminus E} \frac{s(x)^p}{\delta(x)^{n-np+(2-\beta)p}}\,dx = \infty.$$

We remark that $0 < \beta < 1 < \alpha$ and

$$\min\{n/(n+\alpha-2),\, 1/(\alpha-1)\} = \begin{cases} n/(n+\alpha-2) & \text{if } 1 < \alpha \le 2, \\ 1/(\alpha-1) & \text{if } \alpha > 2. \end{cases}$$

If s is a nonnegative subharmonic function, then so is s^p for $p > 1$. Thus the case $p > 1$ is irrelevant for nonintegrability of subharmonic functions. It is easy to see that the bound of p in Corollary 6 is sharp. In particular, the bound given in [7] is improved. For the plane case Masumoto [8, 9] gave the same bound of p. Observe that if D is a bounded Lipschitz domain, then for $\varepsilon < 1$

$$\int_D \delta(x)^{-\varepsilon}\,dx < \infty. \tag{6}$$

In view of Corollary 5, we obtain that Corollary 6(i) follows from the next lemma.

Lemma 3. *Let D be a bounded domain and suppose that u is a nonnegative superharmonic function on D.*

(i) *If $0 \le \gamma < 1$ and $0 < p \le n/(n-\gamma)$, then*

$$\left(\int_D u(x)^p\,dx\right)^{1/p} \le M \int_D u(x)\delta(x)^{-\gamma}\,dx. \tag{7}$$

(ii) *Suppose (6) holds for some $\varepsilon > 0$. If $\gamma < 0$ and $0 < p < \varepsilon/(\varepsilon-\gamma)$, then (7) holds.*

Proof. For (i) we may assume that $p \ge 1$. Let $\{Q_j\}$ be the Whitney decomposition of D. By an argument similar to Lemma 2 we have

$$\left(\int_{Q_j} u(x)^p dx\right)^{1/p} \le M(\operatorname{diam} D)^{\gamma-n+p/n} \int_{Q_j} u(x)\delta(x)^{-\gamma}\,dx,$$

since $\gamma - n + p/n \ge 0$ by assumption. Summing up the above integral, we obtain (7).

We note that $p < \varepsilon/(\varepsilon - \gamma) < 1$ for (ii). Using the Hölder inequality with $1/p > 1$, we obtain

$$\int_D u(x)^p\,dx \le \left(\int_D u(x)\delta(x)^{-\gamma}dx\right)^p \left(\int_D \delta(x)^{p\gamma/(1-p)}dx\right)^{1-p}.$$

Since $0 < p < \varepsilon/(\varepsilon - \gamma)$ implies $p\gamma/(1-p) > -\varepsilon$, we obtain (7). The lemma is proved.

For the proof of Corollary 6(ii), it is sufficient to show the following lemma.

Lemma 4. *Let* φ *satisfy* (1) *and let* $0 < p < 1$. *Suppose* s *is a nonnegative subharmonic function on* D. *Then*

$$\begin{aligned}&\int_{D\setminus B(x_0,r)} s(x)\varphi(g(x))\frac{g(x)^2}{\delta(x)^2}\,dx \\ &\qquad \le M\left(\int_{D\setminus B(x_0,r)} \frac{s(x)^p\varphi(g(x))^p g(x)^{2p}}{\delta(x)^{n-np+2p}}\,dx\right)^{1/p}\end{aligned} \tag{8}$$

for $0 < r < \delta(x_0)/2$.

Proof. We denote by I the integral in the right-hand side of (8). Put $k = r/(2\delta(x_0))$. As observed in [14, Proof of Theorem 2], we have

$$s(x)^p \le M\delta(x)^{-n}\int_{B(x,k\delta(x))} s(y)^p\,dy, \tag{9}$$

where M depends only on p, k, and the dimension. For $x \in D\setminus B(x_0, r)$ we have

$$\delta(x) \le |x - x_0| + \delta(x_0) \le \left(1 + \frac{\delta(x_0)}{r}\right)|x - x_0| < \frac{3}{4k}|x - x_0|,$$

so that $g = G(\cdot, x_0)$ is harmonic on $B(x, \frac{4}{3}k\delta(x))$. Hence (1), (9), and the Harnack inequality yield

$$\begin{aligned}s(x)^p &\le M\delta(x)^{(2-n)p}\varphi(g(x))^{-p}g(x)^{-2p}\int_{B(x,k\delta(x))}\frac{s(y)^p\varphi(g(y))^p g(y)^{2p}}{\delta(y)^{n-np+2p}}\,dy\\ &\le M\delta(x)^{(2-n)p}\varphi(g(x))^{-p}g(x)^{-2p}I.\end{aligned}$$

Therefore

$$s(x) = s(x)^p\cdot s(x)^{1-p} \le Ms(x)^p(\delta(x)^{(2-n)p}\varphi(g(x))^{-p}g(x)^{-2p}I)^{(1-p)/p}.$$

Substituting this inequality in the left-hand side of (8), we obtain the lemma.

Note added in proof

Professor Peter Lindqvist informed me that he proved the integrability of a positive supersolution of certain nonlinear equations, such as the p-Laplace equation (J. Analyse Math. **62** (to appear)). He used the BMO-norm estimate and an argument similar to [12, 13].

REFERENCES

1. A. Ancona, *On strong barriers and an inequality of Hardy for domains in* $\mathbb{R}^n$, J. London Math. Soc. (2) **34** (1986), 274–290.
2. D. H. Armitage, *On the global integrability of superharmonic functions in balls*, J. London Math. Soc. (2) **4** (1971), 365–373.
3. ——, *Further result on the global integrability of superharmonic functions*, J. London Math. Soc. (2) **6** (1972), 109–121.
4. D. A. Brannan and W. K. Hayman, *Research problems in complex analysis*, Bull. London Math. Soc. **21** (1989), 1–35.
5. D. S. Jerison and C. E. Kenig, *Boundary behavior of harmonic functions in non-tangentially accessible domains*, Adv. in Math. **46** (1982), 80–147.
6. P. Jones and T. Wolff, *Hausdorff dimension of harmonic measures in the plane*, Acta Math. **161** (1988), 131–144.
7. F.-Y. Maeda and N. Suzuki, *The integrability of superharmonic functions on Lipschitz domains*, Bull. London Math. Soc. **21** (1989), 270–278.
8. M. Masumoto, *A distorsion theorem for conformal mappings with an application to subharmonic functions*, Hiroshima Math. J. **20** (1990), 341–350.
9. ——, *Integrability of superharmonic functions on plane domains*, J. London Math. Soc. (2) **45** (1992), 62–78.
10. ——, *Integrability of superharmonic functions on Hölder domains of the plane*, Proc. Amer. Math. Soc. **117** (1993), 1083–1088.
11. V. G. Maz'ya, *Sobolev spaces*, Springer, New York, 1985.
12. W. Smith and D. A. Stegenga, *Sobolev imbedding and integrability of harmonic functions on Hölder domains*, Potential Theory (Proc. Internat. Conf. Potential Theory, Nagoya 1990) (M. Kishi, ed.), Walter de Gruyter, Berlin, 1992, pp. 303–313.
13. D. A. Stegenga and D. C. Ullrich, *Superharmonic functions in Hölder domains*, preprint.
14. N. Suzuki, *Nonintegrability of harmonic functions in a domain*, Japan. J. Math. **16** (1990), 269–278.
15. ——, *Nonintegrability of superharmonic functions*, Proc. Amer. Math. Soc. **113** (1991), 113–115.
16. ——, *Note on the integrability of superharmonic functions*, preprint.
17. K.-O. Widman, *Inequalities for the Green function and boundary continuity of the gradient of solutions of elliptic differential equations*, Math. Scand. **21** (1967), 17–37.
18. J.-M. Wu, *Boundary density and the Green function*, Michigan Math. J. **38** (1991), 175–189.

DEPARTMENT OF MATHEMATICS, GUNMA UNIVERSITY, KIRYU 376, JAPAN

Current address: Department of Mathematics, Faculty of Science, Kumamoto University, Kumamoto 860, Japan

E-mail address: aikawa@kumamoto-u.ac.jp