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ON NONNEGATIVE COSINE POLYNOMIALS WITH NONNEGATIVE INTEGRAL COEFFICIENTS

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ABSTRACT. We show that there exist $p_0 > 0$ and $p_1, \ldots, p_N$ nonnegative integers, such that

$$0 \le p(x) = p_0 + p_1 \cos x + \cdots + p_N \cos Nx$$

and $p_0 \ll s^{1/3}$ for $s = \sum_{j=0}^N p_j$, improving on a result of Odlyzko who showed the existence of such a polynomial p that satisfies $p_0 \ll (s \log s)^{1/3}$. Our result implies an improvement of the best known estimate for a problem of Erdős and Szekeres. As our method is nonconstructive, we also give a method for constructing an infinite family of such polynomials, given *one* good "seed" polynomial.

1. INTRODUCTION

We consider nonnegative cosine polynomials of the form

$$0 \le p(x) = p_0 + p_1 \cos x + p_2 \cos 2x + \cdots + p_N \cos Nx \ , \qquad x \in [0, 2\pi],$$

where $p_j \ge 0$. We also write $\widehat{p}(0) = p_0$. Notice that $p(0) = \sum_{j=0}^N p_j$ is the maximum of $p(x)$. We are interested in estimating the size of

$$M(s) = \inf_{p(0) \ge s} \widehat{p}(0)$$

for $s \to \infty$. That is, we want to find polynomials of the above form for which $p_0 = \frac{1}{2\pi} \int_0^{2\pi} p(x)\, dx$ is small compared to the maximum of $p(x)$. In what follows C denotes an arbitrary positive constant and $a \ll b$ means $a \le Cb$ for some C.

If no more restrictions are imposed on the cosine polynomial $p(x)$ then $M(s) = 0$ for all s. This is because the Fejér kernel

$$K_A(x) = \sum_{j=-A}^{A} \left(1 - \frac{|j|}{A+1}\right) e^{ijx} = 1 + \sum_{j=1}^{A} 2\left(1 - \frac{j}{A+1}\right) \cos jx$$

has constant coefficient 1, has $K_A(0) \gg A$, and is nonnegative.

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If we restrict the coefficients $p_1, \dots, p_N$ to be either 0 or 1, we have the classical *cosine problem*, about which we know that for some $\varepsilon > 0$

$$2^{\log^\varepsilon s} \ll M(s) \ll s^{1/2}. \tag{1}$$

The upper bound in (1) is easily proved by considering the polynomial

$$f(x) = \left(\sum_1^A \cos 3^j x\right)^2 \tag{2}$$

$$= A + \frac{1}{2}\sum_{j=1}^A \cos(2\cdot 3^j x) + \sum_{\substack{k,l=1\\k>l}}^A (\cos(3^k+3^l)x + \cos(3^k-3^l)x). \tag{3}$$

All cosines in (3) have distinct frequencies. Define $f_1(x) = f(x) + \frac{1}{2}A - \frac{1}{2}\sum_{j=1}^A \cos(2\cdot 3^j x)$. Then $f_1(x) \ge 0$, $f_1(0) \gg A^2$, $\widehat{f_1}(0) \ll A$, and f_1 has nonconstant coefficients which are either 0 or 1. The lower bound in (1) is much harder to prove and is due to Bourgain [3]. Earlier, Roth [8] had obtained $M(s) \gg (\log s/\log\log s)^{1/2}$.

From this point on, we will study the case of $p_1, \dots, p_N$ being arbitrary nonnegative integers. This case was studied by Odlyzko [7] who showed that

$$M(s) \ll (s\log s)^{1/3}. \tag{4}$$

The method is the following. Consider the nonnegative polynomial

$$q(x) = \alpha K_A(x) = q_0 + q_1\cos x + \cdots + q_A \cos Ax,$$

whose coefficients are not necessarily integers ($\alpha > 0$). We modify q so that its nonconstant coefficients are integers, by adding to it a random polynomial

$$r(x) = r_1 \cos x + \cdots + r_A \cos Ax.$$

The coefficients r_j are independent random variables which take values such that $r_j + q_j$ is always an integer. A theorem of Salem and Zygmund [9] guarantees that $\|r\|_\infty$ is small with high probability, and the nonnegative polynomial $p(x) = q(x) + r(x) + \|r\|_\infty$ achieves (4) when α is appropriately chosen as a function of A.

Odlyzko studied this problem in connection with a problem posed by Erdős and Szekeres [4]. The problem is to estimate

$$E(n) = \inf \max_{|z|=1} \left|\prod_{k=1}^n (1 - z^{a_k})\right|$$

where $a_1, \dots, a_n$ may be any positive integers. The inequality

$$\log E(n) \ll M(n)\log(n) \tag{5}$$

holds (see [7]), so that Odlyzko's result implies $\log E(n) \ll n^{1/3}\log^{4/3} n$.

In this paper we replace the random modification in Odlyzko's argument with a more careful modification, based, again, partly on randomization. We use a recent theorem of Spencer [10] which in some cases does better than the Salem-Zygmund theorem. We show in §3 that, when $p_1, \dots, p_N$ are restricted to be nonnegative integers, we have $M(s) \ll s^{1/3}$. By (5) this implies $\log E(n) \ll n^{1/3} \log n$. Our method is similar to that used by Beck [2] on a different problem, posed by Littlewood.[1]

Both the Salem-Zygmund theorem and Spencer theorem are nonconstructive. In §4 we give a deterministic procedure which, given a polynomial $p(x)$ with nonnegative integral Fourier coefficients (in other words, p_j is a nonnegative even integer, for $j \geq 1$) and with $\widehat{p}(0) \leq (p(0))^\alpha$, for some $\alpha > 0$, produces a sequence of polynomials $p = p^{(0)}, p^{(1)}, p^{(2)}, \ldots$, such that $\deg p^{(n)} \to \infty$, $p^{(n)}(0) \to \infty$, and $(p^{(n)})\widehat{\ }(0) \leq (p^{(n)}(0))^\alpha$. This shows $M(s) \leq Cs^{1/\alpha}$, with C dependent on the initial p only.

2. Bounds on random trigonometric polynomials

In [7] the following classical theorem was used to estimate the size of a random polynomial.

Theorem 1 (Salem and Zygmund [9; 5, p. 69]). *Let $f_1(x), \ldots, f_n(x)$ be trigonometric polynomials of degree at most m and $\xi_1, \ldots, \xi_n$ be independent random variables, which satisfy $\mathbf{E}e^{\lambda\xi_j} \leq e^{\lambda^2/2}$ for all j and $\lambda > 0$ (subnormal random variables). Write*

$$f(x) = \sum_{j=1}^{n} \xi_j f_j(x).$$

Then, for some $C > 0$,

$$\Pr\left(\|f\|_\infty \geq C \left(\sum_{j=1}^{n} \|f_j\|_\infty^2 \log m \right)^{1/2} \right) \leq \frac{1}{m^2}.$$

Theorem 1 was used in [7] to change the coefficients of a polynomial to integers without a big loss:

Corollary 1. *Let $p(x) = p_0 + \sum_{j=1}^{N} p_j \cos jx$ and define the random polynomial $r(x)$ so that $p(x) + r(x)$ always has integral coefficients (except perhaps the constant coefficient):*

$$r(x) = \sum_{j=1}^{N} \xi_j \cos jx$$

with $\xi_j = 0$ if p_j is an integer, else

$$\xi_j = \begin{cases} \lfloor p_j \rfloor - p_j & \text{with probability } \lceil p_j \rceil - p_j, \\ \lceil p_j \rceil - p_j & \text{with probability } p_j - \lfloor p_j \rfloor. \end{cases}$$

Then $\Pr\left(\|r\|_\infty \ll (N \log N)^{1/2}\right) \to 1$, as $N \to \infty$.

Proof of Corollary 1. The above defined ξ_j are subnormal (see, e.g., [1, p. 235, Lemma A.6]). Theorem 1 can now be applied. □

[1]After this paper was submitted for publication the author learned that the method employed for the proof of the basic result has also appeared in [6].

The following theorem of Spencer [10] is sometimes better than the Salem-Zygmund theorem, though unfortunately only in the symmetric case $\xi_j = \pm 1$ (Rademacher random variables).

Theorem 2 (Spencer [10]). *Let* a_{ij}, $i = 1, \ldots, n_1$, $j = 1, \ldots, n_2$, *be such that* $|a_{ij}| \le 1$. *Then there are signs* $\varepsilon_1, \ldots, \varepsilon_{n_2} \in \{-1, 1\}$ *such that, for all* i,

$$\left|\sum_{j=1}^{n_2} \varepsilon_j a_{ij}\right| \le C n_1^{1/2}. \tag{6}$$

Notice there is no dependence of the bound on n_2.

Corollary 2. *Let* $f_1(x), \ldots, f_n(x)$, $\|f_j\|_\infty \le C$, *be trigonometric polynomials of degree at most* m. *Then there is a choice of signs* $\varepsilon_1, \ldots, \varepsilon_n \in \{-1, 1\}$ *such that*

$$\left\|\sum_{j=1}^{n} \varepsilon_j f_j\right\|_\infty \le C m^{1/2}. \quad \square$$

Proof of Corollary 2. For $i = 1, \ldots, 10m$, $j = 1, \ldots, n$, define $a_{ij} = f_j(x_i)$, where $x_i = i\frac{2\pi}{10m}$. Let $\varepsilon_1, \ldots, \varepsilon_n$ be the sequence of signs given by Theorem 2 for the matrix a_{ij} and write $f = \sum_{j=1}^n \varepsilon_j f_j$. There is $x_0 \in [0, 2\pi]$ such that $|f(x_0)| = \|f\|_\infty$. For some k we have $|x_k - x_0| \le \frac{2\pi}{10m}$. By Bernstein's inequality, $\|f'\|_\infty \le m\|f\|_\infty$, we get

$$|f(x_0) - f(x_k)| \le \frac{2\pi}{10m}\|f'\|_\infty \le \frac{2\pi}{10}|f(x_0)|$$

which, since $\frac{2\pi}{10} < 1$, implies

$$\|f\|_\infty = |f(x_0)| \le C|f(x_k)| = C\left|\sum_{j=1}^{n} \varepsilon_j a_{kj}\right| \le C m^{1/2},$$

and the proof is complete. $\square$

Corollary 2 is better than Theorem 1 only when $m/\log m = o(n)$. It is a strictly symmetric result and cannot directly be applied to modify a polynomial $p(x)$ so that it has integral coefficients, as we need to do in our case. We show in §3 that a sequence of applications of Corollary 2 is needed.

3. Proof of the inequality $M(s) \ll s^{1/3}$

Since Corollary 2 only allows us to choose random signs, we cannot use it directly (as we used the Salem-Zygmund theorem) to modify the coefficients of a polynomial to integers, while controlling the size of the change. In this section we show how to modify the coefficients little by little to achieve the same result.

Let $\alpha > 0$ and define

$$a(x) = \alpha K_A(x) = \sum_{j=0}^{A} a_j \cos jx.$$

Suppose $\varepsilon > 0$ and the nonnegative integer k_0 is such that for some nonnegative integers b_j

$$|a_j - b_j 2^{-k_0}| \le \varepsilon \quad \text{for all } j = 1, \dots, A.$$

We shall define a finite sequence of polynomials

$$a^{(0)}(x) = a_0 + \sum_{j=1}^{A} b_j 2^{-k_0} \cos jx, \ a^{(1)}(x), \ \dots, \ a^{(k_0)}(x)$$

inductively, so that if $a^{(k)}(x) = a_0 + \sum_{j=1}^{A} a_j^{(k)} \cos jx$ then, for each $j = 1, \dots, A$,

$$a_j^{(k)} = b_j^{(k)} 2^{k-k_0} \tag{7}$$

for some nonnegative integers $b_j^{(k)}$. We define inductively the coefficients of $a^{(k+1)}$ as follows. If $b_j^{(k)}$, $j > 0$, is even then $a_j^{(k+1)} = a_j^{(k)}$. Else define

$$a_j^{(k+1)} = a_j^{(k)} + \varepsilon_j^{(k)} 2^{k-k_0} \tag{8}$$

where $\varepsilon_j^{(k)} \in \{-1, 1\}$ are such that

$$\left\| \sum_{b_j^{(k)} \text{ odd}} \varepsilon_j^{(k)} \cos jx \right\| \le C A^{1/2}. \tag{9}$$

The existence of the signs $\varepsilon_j^{(k)}$ is guaranteed by Corollary 2. Notice that (8) implies the preservation of (7) by the inductive definition. We deduce from (9) that

$$\|a^{(k+1)} - a^{(k)}\|_\infty \le C 2^{k-k_0} A^{1/2}. \tag{10}$$

The polynomial $a^{(k_0)}$ has integral coefficients (except perhaps for the constant coefficient). Summing (10) we get

$$\|a - a^{(k_0)}\|_\infty \le \|a - a^{(0)}\|_\infty + \|a^{(0)} - a^{(k_0)}\|_\infty \le A\varepsilon + C A^{1/2}.$$

Choose $\varepsilon = 1/A$ to get $\|a - a^{(k_0)}\|_\infty \le C A^{1/2}$. On the other hand, the coefficients of a and $a^{(k_0)}$ differ by at most 1, and this implies that for the nonnegative polynomial $p(x) = a^{(k_0)}(x) + \|a - a^{(k_0)}\|_\infty$ we have

$$p(0) \ge a(0) - A \ge C\alpha A - A, \tag{11}$$

$$\widehat{p}(0) = \alpha + \|a - a^{(k_0)}\|_\infty \le \alpha + C A^{1/2}. \tag{12}$$

Select $\alpha = A^{1/2}$ to get $\widehat{p}(0) \ll A^{1/2}$ and $p(0) \gg A^{3/2}$. Since p has integral coefficients, we have exhibited a polynomial that achieves $M(s) \ll s^{1/3}$, and the proof is complete. □

Remark on cosine sums. Applying the method of the preceding proof on the coefficients of the Fejér kernel $K_A(x)$, one ends up with a nonnegative polynomial of degree at most A, which is of the form

$$p(x) = p_0 + 2\sum_{j=1}^{k} \cos \lambda_j x$$

where $\lambda_j \in \{1, \ldots, A\}$ are distinct. We have $\|K_A - p\|_\infty \ll A^{1/2}$ which, since $p_0 = \frac{1}{2\pi}\int_0^{2\pi} p(x)\,dx$, implies

$$p_0 \ll A^{1/2} \quad \text{and} \quad p(0) \gg A.$$

Thus p is a new example of a cosine sum that achieves the upper bound in (1). It is not as simple as the one mentioned in the introduction but the spectrum of it is much denser: $\frac{1}{2}A + O(A^{1/2})$ cosines with frequencies from 1 to A.

Since the Dirichlet kernel

$$D_A(x) = \sum_{j=-A}^{A} e^{ijx} \tag{13}$$

$$= 1 + 2\sum_{j=1}^{A} \cos jx \tag{14}$$

$$= \frac{\sin(A + \frac{1}{2})x}{\sin \frac{x}{2}} \tag{15}$$

has a minimum asymptotically equal to $-\frac{4}{3\pi}A$, it is conceivable that one may be able to raise the above number of cosine from $\frac{1}{2}A + O(A^{1/2})$ to

$$\left(1 - \frac{2}{3\pi}\right)A + o(A).$$

In other words, since

$$\min_x \sum_{j=1}^{A} \cos jx = -\frac{2}{3\pi}A + o(A),$$

one must remove at least $\frac{2}{3\pi}A$ cosines from the above sum in order to make its minimum $o(A)$ in absolute value.

Note added in proof. The author has now proved that $\frac{1}{2}A + o(A)$ is best possible.

4. The construction

Suppose we are given a polynomial $p(x) \geq 0$ of degree d, whose nonconstant coefficients are even nonnegative integers, which satisfies

$$\widehat{p}(0) \leq (p(0))^\alpha$$

for some $\alpha > 0$. Define the infinite sequence of nonnegative polynomials $p = p^{(1)}, p^{(2)}, p^{(3)}, \ldots$, with the recursive formula

$$p^{(k+1)}(x) = p^{(k)}((d+1)x) \cdot p(x). \tag{16}$$

Since p has even nonconstant coefficients, the Fourier coefficients of all $p^{(k)}$ are nonnegative integers. The spectrum of the first factor in (16) is supported by the multiples of $d+1$, and that of the second factor is supported by the interval $[-d, d]$. This implies that $(p^{(k+1)})^\frown(0) = (p^{(k)})^\frown(0)\widehat{p}(0)$. We obviously have $p^{(k+1)}(0) = p^{(k)}(0)p(0)$. We conclude that for all $k \geq 0$

$$(p^{(k)})^\frown(0) = (\widehat{p}(0))^k \quad \text{and} \quad p^{(k)}(0) = (p(0))^k,$$

and consequently

$$(p^{(k)})^\frown(0) \leq (p^{(k)}(0))^\alpha.$$

So, if s is a power of $p(0)$, we have $M(s) \leq s^\alpha$, and for any s we have $M(s) \leq Cs^\alpha$, where $C = (p(0))^\alpha$.

As an example we give

$$p(x) = 4 + 4\cos x + 2\sum_{j=2}^{10} \cos jx$$

which can be checked numerically to be positive and has constant coefficient $\widehat{p}(0) = 4$ and $p(0) = 26$. This gives $\alpha = \log 4/\log 26 = 0.42549\ldots$.

In view of the above construction, finding a single polynomial p with $\widehat{p}(0) \leq (p(0))^\alpha$, with $\alpha < \frac{1}{3}$, will prove that the result in this paper is not the best possible. The above example was actually found by a computer, but if no more insight is gained into how these good "seed" polynomials look, the computing time grows dramatically as we increase the degree of the polynomial.

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