

## ON NONNEGATIVE COSINE POLYNOMIALS WITH NONNEGATIVE INTEGRAL COEFFICIENTS

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**ABSTRACT.** We show that there exist  $p_0 > 0$  and  $p_1, \dots, p_N$  nonnegative integers, such that

$$0 \leq p(x) = p_0 + p_1 \cos x + \dots + p_N \cos Nx$$

and  $p_0 \ll s^{1/3}$  for  $s = \sum_{j=0}^N p_j$ , improving on a result of Odlyzko who showed the existence of such a polynomial  $p$  that satisfies  $p_0 \ll (s \log s)^{1/3}$ . Our result implies an improvement of the best known estimate for a problem of Erdős and Szekeres. As our method is nonconstructive, we also give a method for constructing an infinite family of such polynomials, given one good "seed" polynomial.

### 1. INTRODUCTION

We consider nonnegative cosine polynomials of the form

$$0 \leq p(x) = p_0 + p_1 \cos x + p_2 \cos 2x + \dots + p_N \cos Nx, \quad x \in [0, 2\pi],$$

where  $p_j \geq 0$ . We also write  $\hat{p}(0) = p_0$ . Notice that  $p(0) = \sum_{j=0}^N p_j$  is the maximum of  $p(x)$ . We are interested in estimating the size of

$$M(s) = \inf_{p(0) \geq s} \hat{p}(0)$$

for  $s \rightarrow \infty$ . That is, we want to find polynomials of the above form for which  $p_0 = \frac{1}{2\pi} \int_0^{2\pi} p(x) dx$  is small compared to the maximum of  $p(x)$ . In what follows  $C$  denotes an arbitrary positive constant and  $a \ll b$  means  $a \leq Cb$  for some  $C$ .

If no more restrictions are imposed on the cosine polynomial  $p(x)$  then  $M(s) = 0$  for all  $s$ . This is because the Fejér kernel

$$K_A(x) = \sum_{j=-A}^A \left(1 - \frac{|j|}{A+1}\right) e^{ijx} = 1 + \sum_{j=1}^A 2 \left(1 - \frac{j}{A+1}\right) \cos jx$$

has constant coefficient 1, has  $K_A(0) \gg A$ , and is nonnegative.

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If we restrict the coefficients  $p_1, \dots, p_N$  to be either 0 or 1, we have the classical *cosine problem*, about which we know that for some  $\varepsilon > 0$

$$(1) \quad 2^{\log^\varepsilon s} \ll M(s) \ll s^{1/2}.$$

The upper bound in (1) is easily proved by considering the polynomial

$$(2) \quad f(x) = \left( \sum_1^A \cos 3^j x \right)^2$$

$$(3) \quad = A + \frac{1}{2} \sum_{j=1}^A \cos(2 \cdot 3^j x) + \sum_{\substack{k,l=1 \\ k>l}}^A (\cos(3^k + 3^l)x + \cos(3^k - 3^l)x).$$

All cosines in (3) have distinct frequencies. Define  $f_1(x) = f(x) + \frac{1}{2}A - \frac{1}{2} \sum_{j=1}^A \cos(2 \cdot 3^j x)$ . Then  $f_1(x) \geq 0$ ,  $f_1(0) \gg A^2$ ,  $\widehat{f_1}(0) \ll A$ , and  $f_1$  has nonconstant coefficients which are either 0 or 1. The lower bound in (1) is much harder to prove and is due to Bourgain [3]. Earlier, Roth [8] had obtained  $M(s) \gg (\log s / \log \log s)^{1/2}$ .

From this point on, we will study the case of  $p_1, \dots, p_N$  being arbitrary nonnegative integers. This case was studied by Odlyzko [7] who showed that

$$(4) \quad M(s) \ll (s \log s)^{1/3}.$$

The method is the following. Consider the nonnegative polynomial

$$q(x) = \alpha K_A(x) = q_0 + q_1 \cos x + \dots + q_A \cos Ax,$$

whose coefficients are not necessarily integers ( $\alpha > 0$ ). We modify  $q$  so that its nonconstant coefficients are integers, by adding to it a random polynomial

$$r(x) = r_1 \cos x + \dots + r_A \cos Ax.$$

The coefficients  $r_j$  are independent random variables which take values such that  $r_j + q_j$  is always an integer. A theorem of Salem and Zygmund [9] guarantees that  $\|r\|_\infty$  is small with high probability, and the nonnegative polynomial  $p(x) = q(x) + r(x) + \|r\|_\infty$  achieves (4) when  $\alpha$  is appropriately chosen as a function of  $A$ .

Odlyzko studied this problem in connection with a problem posed by Erdős and Szekeres [4]. The problem is to estimate

$$E(n) = \inf \max_{|z|=1} \left| \prod_{k=1}^n (1 - z^{a_k}) \right|$$

where  $a_1, \dots, a_n$  may be any positive integers. The inequality

$$(5) \quad \log E(n) \ll M(n) \log(n)$$

holds (see [7]), so that Odlyzko's result implies  $\log E(n) \ll n^{1/3} \log^{4/3} n$ .

In this paper we replace the random modification in Odlyzko's argument with a more careful modification, based, again, partly on randomization. We use a recent theorem of Spencer [10] which in some cases does better than the Salem-Zygmund theorem. We show in §3 that, when  $p_1, \dots, p_N$  are restricted to be

nonnegative integers, we have  $M(s) \ll s^{1/3}$ . By (5) this implies  $\log E(n) \ll n^{1/3} \log n$ . Our method is similar to that used by Beck [2] on a different problem, posed by Littlewood.<sup>1</sup>

Both the Salem-Zygmund theorem and Spencer theorem are nonconstructive. In §4 we give a deterministic procedure which, given a polynomial  $p(x)$  with nonnegative integral Fourier coefficients (in other words,  $p_j$  is a nonnegative even integer, for  $j \geq 1$ ) and with  $\widehat{p}(0) \leq (p(0))^\alpha$ , for some  $\alpha > 0$ , produces a sequence of polynomials  $p = p^{(0)}, p^{(1)}, p^{(2)}, \dots$ , such that  $\deg p^{(n)} \rightarrow \infty$ ,  $p^{(n)}(0) \rightarrow \infty$ , and  $(p^{(n)})^\wedge(0) \leq (p^{(n)}(0))^\alpha$ . This shows  $M(s) \leq Cs^{1/\alpha}$ , with  $C$  dependent on the initial  $p$  only.

2. BOUNDS ON RANDOM TRIGONOMETRIC POLYNOMIALS

In [7] the following classical theorem was used to estimate the size of a random polynomial.

**Theorem 1** (Salem and Zygmund [9; 5, p. 69]). *Let  $f_1(x), \dots, f_n(x)$  be trigonometric polynomials of degree at most  $m$  and  $\xi_1, \dots, \xi_n$  be independent random variables, which satisfy  $\mathbf{E}e^{\lambda\xi_j} \leq e^{\lambda^2/2}$  for all  $j$  and  $\lambda > 0$  (subnormal random variables). Write*

$$f(x) = \sum_{j=1}^n \xi_j f_j(x).$$

Then, for some  $C > 0$ ,

$$\Pr \left( \|f\|_\infty \geq C \left( \sum_{j=1}^n \|f_j\|_\infty^2 \log m \right)^{1/2} \right) \leq \frac{1}{m^2}.$$

Theorem 1 was used in [7] to change the coefficients of a polynomial to integers without a big loss:

**Corollary 1.** *Let  $p(x) = p_0 + \sum_{j=1}^N p_j \cos jx$  and define the random polynomial  $r(x)$  so that  $p(x) + r(x)$  always has integral coefficients (except perhaps the constant coefficient):*

$$r(x) = \sum_{j=1}^N \xi_j \cos jx$$

with  $\xi_j = 0$  if  $p_j$  is an integer, else

$$\xi_j = \begin{cases} \lfloor p_j \rfloor - p_j & \text{with probability } \lceil p_j \rceil - p_j, \\ \lceil p_j \rceil - p_j & \text{with probability } p_j - \lfloor p_j \rfloor. \end{cases}$$

Then  $\Pr (\|r\|_\infty \ll (N \log N)^{1/2}) \rightarrow 1$ , as  $N \rightarrow \infty$ .

*Proof of Corollary 1.* The above defined  $\xi_j$  are subnormal (see, e.g., [1, p. 235, Lemma A.6]). Theorem 1 can now be applied.  $\square$

<sup>1</sup>After this paper was submitted for publication the author learned that the method employed for the proof of the basic result has also appeared in [6].

The following theorem of Spencer [10] is sometimes better than the Salem-Zygmund theorem, though unfortunately only in the symmetric case  $\xi_j = \pm 1$  (Rademacher random variables).

**Theorem 2** (Spencer [10]). *Let  $a_{ij}$ ,  $i = 1, \dots, n_1$ ,  $j = 1, \dots, n_2$ , be such that  $|a_{ij}| \leq 1$ . Then there are signs  $\varepsilon_1, \dots, \varepsilon_{n_2} \in \{-1, 1\}$  such that, for all  $i$ ,*

$$(6) \quad \left| \sum_{j=1}^{n_2} \varepsilon_j a_{ij} \right| \leq C n_1^{1/2}.$$

Notice there is no dependence of the bound on  $n_2$ .

**Corollary 2.** *Let  $f_1(x), \dots, f_n(x)$ ,  $\|f_j\|_\infty \leq C$ , be trigonometric polynomials of degree at most  $m$ . Then there is a choice of signs  $\varepsilon_1, \dots, \varepsilon_n \in \{-1, 1\}$  such that*

$$\left\| \sum_{j=1}^n \varepsilon_j f_j \right\|_\infty \leq C m^{1/2}. \quad \square$$

*Proof of Corollary 2.* For  $i = 1, \dots, 10m$ ,  $j = 1, \dots, n$ , define  $a_{ij} = f_j(x_i)$ , where  $x_i = i \frac{2\pi}{10m}$ . Let  $\varepsilon_1, \dots, \varepsilon_n$  be the sequence of signs given by Theorem 2 for the matrix  $a_{ij}$  and write  $f = \sum_{j=1}^n \varepsilon_j f_j$ . There is  $x_0 \in [0, 2\pi]$  such that  $|f(x_0)| = \|f\|_\infty$ . For some  $k$  we have  $|x_k - x_0| \leq \frac{2\pi}{10m}$ . By Bernstein's inequality,  $\|f'\|_\infty \leq m\|f\|_\infty$ , we get

$$|f(x_0) - f(x_k)| \leq \frac{2\pi}{10m} \|f'\|_\infty \leq \frac{2\pi}{10} |f(x_0)|$$

which, since  $\frac{2\pi}{10} < 1$ , implies

$$\|f\|_\infty = |f(x_0)| \leq C |f(x_k)| = C \left| \sum_{j=1}^n \varepsilon_j a_{kj} \right| \leq C m^{1/2},$$

and the proof is complete.  $\square$

Corollary 2 is better than Theorem 1 only when  $m/\log m = o(n)$ . It is a strictly symmetric result and cannot directly be applied to modify a polynomial  $p(x)$  so that it has integral coefficients, as we need to do in our case. We show in §3 that a sequence of applications of Corollary 2 is needed.

### 3. PROOF OF THE INEQUALITY $M(s) \ll s^{1/3}$

Since Corollary 2 only allows us to choose random signs, we cannot use it directly (as we used the Salem-Zygmund theorem) to modify the coefficients of a polynomial to integers, while controlling the size of the change. In this section we show how to modify the coefficients little by little to achieve the same result.

Let  $\alpha > 0$  and define

$$a(x) = \alpha K_A(x) = \sum_{j=0}^A a_j \cos jx.$$

Suppose  $\varepsilon > 0$  and the nonnegative integer  $k_0$  is such that for some nonnegative integers  $b_j$

$$|a_j - b_j 2^{-k_0}| \leq \varepsilon \quad \text{for all } j = 1, \dots, A.$$

We shall define a finite sequence of polynomials

$$a^{(0)}(x) = a_0 + \sum_{j=1}^A b_j 2^{-k_0} \cos jx, \quad a^{(1)}(x), \dots, \quad a^{(k_0)}(x)$$

inductively, so that if  $a^{(k)}(x) = a_0 + \sum_{j=1}^A a_j^{(k)} \cos jx$  then, for each  $j = 1, \dots, A$ ,

$$(7) \quad a_j^{(k)} = b_j^{(k)} 2^{k-k_0}$$

for some nonnegative integers  $b_j^{(k)}$ . We define inductively the coefficients of  $a^{(k+1)}$  as follows. If  $b_j^{(k)}$ ,  $j > 0$ , is even then  $a_j^{(k+1)} = a_j^{(k)}$ . Else define

$$(8) \quad a_j^{(k+1)} = a_j^{(k)} + \varepsilon_j^{(k)} 2^{k-k_0}$$

where  $\varepsilon_j^{(k)} \in \{-1, 1\}$  are such that

$$(9) \quad \left\| \sum_{b_j^{(k)} \text{ odd}} \varepsilon_j^{(k)} \cos jx \right\| \leq CA^{1/2}.$$

The existence of the signs  $\varepsilon_j^{(k)}$  is guaranteed by Corollary 2. Notice that (8) implies the preservation of (7) by the inductive definition. We deduce from (9) that

$$(10) \quad \|a^{(k+1)} - a^{(k)}\|_\infty \leq C 2^{k-k_0} A^{1/2}.$$

The polynomial  $a^{(k_0)}$  has integral coefficients (except perhaps for the constant coefficient). Summing (10) we get

$$\|a - a^{(k_0)}\|_\infty \leq \|a - a^{(0)}\|_\infty + \|a^{(0)} - a^{(k_0)}\|_\infty \leq A\varepsilon + CA^{1/2}.$$

Choose  $\varepsilon = 1/A$  to get  $\|a - a^{(k_0)}\|_\infty \leq CA^{1/2}$ . On the other hand, the coefficients of  $a$  and  $a^{(k_0)}$  differ by at most 1, and this implies that for the nonnegative polynomial  $p(x) = a^{(k_0)}(x) + \|a - a^{(k_0)}\|_\infty$  we have

$$(11) \quad p(0) \geq a(0) - A \geq C\alpha A - A,$$

$$(12) \quad \widehat{p}(0) = \alpha + \|a - a^{(k_0)}\|_\infty \leq \alpha + CA^{1/2}.$$

Select  $\alpha = A^{1/2}$  to get  $\widehat{p}(0) \ll A^{1/2}$  and  $p(0) \gg A^{3/2}$ . Since  $p$  has integral coefficients, we have exhibited a polynomial that achieves  $M(s) \ll s^{1/3}$ , and the proof is complete.  $\square$

*Remark on cosine sums.* Applying the method of the preceding proof on the coefficients of the Fejér kernel  $K_A(x)$ , one ends up with a nonnegative polynomial

of degree at most  $A$ , which is of the form

$$p(x) = p_0 + 2 \sum_{j=1}^k \cos \lambda_j x$$

where  $\lambda_j \in \{1, \dots, A\}$  are distinct. We have  $\|K_A - p\|_\infty \ll A^{1/2}$  which, since  $p_0 = \frac{1}{2\pi} \int_0^{2\pi} p(x) dx$ , implies

$$p_0 \ll A^{1/2} \quad \text{and} \quad p(0) \gg A.$$

Thus  $p$  is a new example of a cosine sum that achieves the upper bound in (1). It is not as simple as the one mentioned in the introduction but the spectrum of it is much denser:  $\frac{1}{2}A + O(A^{1/2})$  cosines with frequencies from 1 to  $A$ .

Since the Dirichlet kernel

$$(13) \quad D_A(x) = \sum_{j=-A}^A e^{ijx}$$

$$(14) \quad = 1 + 2 \sum_{j=1}^A \cos jx$$

$$(15) \quad = \frac{\sin(A + \frac{1}{2})x}{\sin \frac{x}{2}}$$

has a minimum asymptotically equal to  $-\frac{4}{3\pi}A$ , it is conceivable that one may be able to raise the above number of cosine from  $\frac{1}{2}A + O(A^{1/2})$  to

$$\left(1 - \frac{2}{3\pi}\right)A + o(A).$$

In other words, since

$$\min_x \sum_{j=1}^A \cos jx = -\frac{2}{3\pi}A + o(A),$$

one must remove at least  $\frac{2}{3\pi}A$  cosines from the above sum in order to make its minimum  $o(A)$  in absolute value.

*Note added in proof.* The author has now proved that  $\frac{1}{2}A + o(A)$  is best possible.

#### 4. THE CONSTRUCTION

Suppose we are given a polynomial  $p(x) \geq 0$  of degree  $d$ , whose nonconstant coefficients are even nonnegative integers, which satisfies

$$\widehat{p}(0) \leq (p(0))^\alpha$$

for some  $\alpha > 0$ . Define the infinite sequence of nonnegative polynomials  $p = p^{(1)}, p^{(2)}, p^{(3)}, \dots$ , with the recursive formula

$$(16) \quad p^{(k+1)}(x) = p^{(k)}((d+1)x) \cdot p(x).$$

Since  $p$  has even nonconstant coefficients, the Fourier coefficients of all  $p^{(k)}$  are nonnegative integers. The spectrum of the first factor in (16) is supported by the multiples of  $d + 1$ , and that of the second factor is supported by the interval  $[-d, d]$ . This implies that  $(p^{(k+1)})^\wedge(0) = (p^{(k)})^\wedge(0)\widehat{p}(0)$ . We obviously have  $p^{(k+1)}(0) = p^{(k)}(0)p(0)$ . We conclude that for all  $k \geq 0$

$$(p^{(k)})^\wedge(0) = (\widehat{p}(0))^k \quad \text{and} \quad p^{(k)}(0) = (p(0))^k,$$

and consequently

$$(p^{(k)})^\wedge(0) \leq (p^{(k)}(0))^\alpha.$$

So, if  $s$  is a power of  $p(0)$ , we have  $M(s) \leq s^\alpha$ , and for any  $s$  we have  $M(s) \leq Cs^\alpha$ , where  $C = (p(0))^\alpha$ .

As an example we give

$$p(x) = 4 + 4 \cos x + 2 \sum_{j=2}^{10} \cos jx$$

which can be checked numerically to be positive and has constant coefficient  $\widehat{p}(0) = 4$  and  $p(0) = 26$ . This gives  $\alpha = \log 4 / \log 26 = 0.42549 \dots$

In view of the above construction, finding a single polynomial  $p$  with  $\widehat{p}(0) \leq (p(0))^\alpha$ , with  $\alpha < \frac{1}{3}$ , will prove that the result in this paper is not the best possible. The above example was actually found by a computer, but if no more insight is gained into how these good "seed" polynomials look, the computing time grows dramatically as we increase the degree of the polynomial.

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