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TRANSLATION INVARIANTS FOR PERIODIC DENJOY-CARLEMAN CLASSES

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ABSTRACT. The Denjoy-Carleman classes in real $C^\infty(\mathbb{R}/\mathbb{Z})$ on which the derivative sequence $f^{(n)}(x)$ at any point is a complete set of invariants are exactly the ones on which the integrals of products of derivatives $f^{(n_1)}\cdots f^{(n_r)}$ are a complete set of invariants up to translation.

The object of this note is to combine the Adler-Konheim complete set of invariants on the space of real integrable functions on a locally compact abelian group, up to translation, with the Denjoy-Carleman theorem on "quasi-analytic" classes of $C^\infty(\mathbb{R}/\mathbb{Z})$ to give the result (Theorem 3) in the abstract.

Theorem 1 (Adler-Konheim [1]). *Let G be a locally compact abelian group. The functions φ_r in $L^1(G^r)$ for $r = 1, 2, 3, \ldots$ defined for f in real $L^1(G)$ by*

$$\varphi_r(\mathbf{y}) = \int_G f(x+y_1)\cdots f(x+y_r)\,dx$$

are a complete set of invariants up to translation. If the Fourier transform of f has no zeros, then f is determined up to translation by $\varphi_1, \varphi_2, \varphi_3$.

For f in $C^\infty(\mathbb{R}/\mathbb{Z})$ we define functions f_r on $\mathbb{N}^r$ (with $\mathbb{N} = \{0, 1, 2, \ldots\}$) for $r = 1, 2, 3, \ldots$ by

$$f_r(\mathbf{n}) = \int_0^1 f^{(n_1)}(x)\cdots f^{(n_r)}(x)\,dx.$$

These functions are not a complete set of invariants for $C^\infty(\mathbb{R}/\mathbb{Z})$ up to translation since, e.g., $g(x)+g(x+\frac{1}{4})$ and $g(x)+g(x+\frac{1}{2})$, for any function g in $C^\infty(\mathbb{R}/\mathbb{Z})$ supported $(\bmod 1)$ on an interval of length $\frac{1}{4}$ but $\not\equiv 0$, will have the same f_r's but will not be translates.

If f is entire with period 1, then f_r determines φ_r, since integrating products of Taylor series gives

$$\varphi_r(\mathbf{y}) = \sum_{\mathbf{n}} f_r(\mathbf{n})\frac{\mathbf{y}^{\mathbf{n}}}{\mathbf{n}!}$$

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with $\mathbf{y}^{\mathbf{n}} = y_1^{n_1} \cdots y_r^{n_r}$ and $\mathbf{n}! = n_1! \cdots n_r!$. If f is only real-analytic on $\mathbb{R}/\mathbb{Z}$, i.e., analytic with period 1 in a strip neighborhood of $\mathbb{R}$, then f_r still determines φ_r. This may be shown by integrating products of iterated Taylor series

$$f(x+y) = \sum_{\nu_1} \cdots \sum_{\nu_s} f^{(n)}(x) \frac{\eta_1^{\nu_1} \cdots \eta_s^{\nu_s}}{\nu_1! \cdots \nu_s!}$$

with $n = \nu_1 + \cdots + \nu_s$ and $y = \eta_1 + \cdots + \eta_s$ with small η_i. We omit the details since the result is contained in Theorem 3 as the case $m_n = n!$.

For a positive sequence $\mathbf{m} = (m_0, m_1, \dots)$ and a real interval I, the Denjoy-Carleman class $C_{\mathbf{m}}(I)$ consists of the functions f in $C^\infty(I)$ for which there are positive constants a, b depending on f such that

$$(1_{\mathrm{D}}) \qquad \sup|f^{(n)}| \le ab^n m_n \qquad (n \in \mathbb{N})$$

or, equivalently,

$$(1_{\mathrm{I}}) \qquad |f_r(\mathbf{n})| \le a^r b^{|\mathbf{n}|} m_{\mathbf{n}} \qquad (\mathbf{n} \in \mathbb{N}^r;\ r = 1, 2, 3, \dots)$$

with $|\mathbf{n}| = n_1 + \cdots + n_r$ and $m_{\mathbf{n}} = m_{n_1} \cdots m_{n_r}$. From (1_{D}) to (1_{I}) is immediate. The opposite direction follows from

$$\sup|f^{(n)}| = \lim \|f^{(n)}\|_r \qquad (r \to \infty)$$

and

$$\|f^{(n)}\|_r^r = f_r(n, \dots, n) \qquad (r \text{ even}).$$

Theorem 2 (Denjoy-Carleman [2, 3, 5]). *The derivative sequence $f^{(n)}$ at any point in I is a complete set of invariants for $C_{\mathbf{m}}(I)$ if and only if*

$$(2) \qquad \sum_{n=0}^{\infty} \left(\inf_{k \ge n} m_k^{1/k} \right)^{-1} = \infty.$$

The same holds with I replaced by $\mathbb{R}/\mathbb{Z}$, $C_{\mathbf{m}}(\mathbb{R}/\mathbb{Z})$ being defined similarly.

Theorem 3. *The integrals $f_r(\mathbf{n})$ for $\mathbf{n}$ in $\mathbb{N}^r$ $(r = 1, 2, 3, \dots)$ are a complete set of invariants on real $C_{\mathbf{m}}(\mathbb{R}/\mathbb{Z})$ up to translation if and only if* (2) *holds. If* (2) *holds and f in real $C_{\mathbf{m}}(\mathbb{R}/\mathbb{Z})$ has no vanishing Fourier coefficient, then the f_r with $r = 1, 2, 3$, which amounts to*

$$\int_0^1 f, \quad \int_0^1 (f^{(n)})^2, \quad \int_0^1 (f^{(n)})^2 f^{(k)} \quad \textit{for } n, k \textit{ in } \mathbb{N},\ k \le n,$$

suffice to determine f up to translation.

For n a positive integer and $\mathbf{m} = (m_0, m_1, \dots)$, put $\mathbf{m}^{(n)} = (m_n, m_{n+1}, \dots)$. From (1_{D}) it follows that $f \to f^{(n)}$ maps $C_{\mathbf{m}}(I)$ onto $C_{\mathbf{m}^{(n)}}(I)$, with a, b replaced by ab^n, b. Condition (2) is preserved by $\mathbf{m} \to \mathbf{m}^{(n)}$: For this we make use of the fact, due to Carleman [2, p. 105] and used by him for the same purpose, that for series with positive terms,

$$(3) \qquad \sum_{n=2}^{\infty} c_n^{1-1/n} \le \sum_{n=2}^{\infty} c_n + 2 \left(\sum_{n=2}^{\infty} c_n \right)^{1/2}.$$

For the convenience of the reader we repeat Carleman's proof. For each $\lambda > 1$, the terms with $c_n^{-1/n} \le \lambda$ contribute $\le \lambda s$ to the series on the left, s being the series on the right. For the remaining terms, $c_n^{1-1/n} \leq \lambda^{-(n-1)}$, giving a contribution $\leq 1/(\lambda - 1)$. Choosing $\lambda = 1 + s^{-1/2}$ gives the total bound $s + 2s^{1/2}$. For the preservation of (2), it suffices to show that

$$(4) \qquad \sum_n \Big(\inf_{k \geq n} m_k^{1/k}\Big)^{-1} \text{ converges if } \sum_n \Big(\inf_{k \geq n} m_{k+1}^{1/k}\Big)^{-1} \text{ converges}.$$

For this we may assume all $m_n \geq 1$. This implies

$$\Big(\inf_{k \geq n+1} m_k^{1/k}\Big)^{-1} \leq \Big(\Big(\inf_{k \geq n} m_{k+1}^{1/k}\Big)^{-1}\Big)^{1-1/n},$$

which with (3) gives (4).

Proof of Theorem 3. Let $\mathbf{m}$ satisfy (2). For f in $C_{\mathbf{m}}(\mathbb{R}/\mathbb{Z})$, each $D^{\mathbf{n}}\varphi_r$ with $D^{\mathbf{n}} = D^{n_1} \cdots D^{n_r}$ belongs to $C_{\mathbf{m}}(n_j)$ in the jth variable, the other variables being held fixed, with a, b replaced by $a^r b^{|\mathbf{n}|} m_{\mathbf{n}}/m_{n_j}$, b. This fact and Theorem 2 show that f_r, i.e.,

$$f_r(\mathbf{n}) = D^{(\mathbf{n})}\varphi_r(0, \ldots, 0) \quad \text{for all } \mathbf{n},$$

determines

$$D^{(\mathbf{n})}\varphi_r(y_1, 0, \ldots, 0) \quad \text{for all } \mathbf{n}, y_1.$$

Taking r steps in this way leads to

$$D^{(\mathbf{n})}\varphi_r(y_1, \ldots, y_r) \quad \text{for all } \mathbf{n}, \mathbf{y}$$

and, in particular, to φ_r. The first statement in Theorem 1 now shows that the f_r determine f up to translation. The relation

$$f_r(n_1 + 1, n_2, \ldots, n_r) + \cdots + f_r(n_1, \ldots, n_{r-1}, n_r + 1) = 0 \qquad (\mathbf{n} \in \mathbb{N}^r)$$

shows that for $r = 1, 2, 3$ the f_r are determined by

$$f_1(0), \ f_2(n, n), \ f_3(n, n, k) \quad \text{for } n, k \in \mathbb{N}, \ k \leq n.$$

If $\mathbf{m}$ fails (2), then, as Mandelbrojt has shown [4, 5], there are real g in $C_{\mathbf{m}}(\mathbb{R}/\mathbb{Z})$ with arbitrarily small support (mod 1) but $\not\equiv 0$. The functions $g(x) + g(x + \frac{1}{4})$ and $g(x) + g(x + \frac{1}{2})$, for such g supported in an interval of length $\frac{1}{4}$, belong to $C_{\mathbf{m}}(\mathbb{R}/\mathbb{Z})$ and, as before, have the same f_r's but are not translates.

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