PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 120, Number 1, January 1994

ON SINGULAR H-CLOSED EXTENSIONS

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(Communicated by Richard Tall)

ABSTRACT. Introducing the notion of singular H-closed extensions of a locally H-closed space, we obtain a necessary and sufficient condition for a semiregular H-closed extension to be singular.

1. INTRODUCTION

By a space we mean a Hausdorff space, and a map will mean a continuous map. Letters X and Y are used for spaces.

The concept of singular set of a mapping introduced by Cain [2] to study the relationship of compact mappings with other mappings is found useful in the study of compactifications of a locally compact space [5, 7, 8, 10]. This concept which arose earlier in [16] led to the notion of singular compactifications. These have been further studied in [3–5, 9, 17]. In fact, the simple observation by Cain, Chandler, and Faulkner [5] that for a map f from a locally compact space X to a locally compact space Y, the remainder induced by f (see [6]) and the singular set of f are the same has led to a fruitful combination of the independent studies of both these concepts.

As is remarked in the treatise by Porter and Woods [15], that in many ways compact spaces are to Tychonoff spaces as H-closed spaces are to Hausdorff spaces, it is desirable to undertake the above-mentioned study in the context of H-closed extensions. In this spirit we introduce the concept of H-singular sets in §2 and use this to construct H-closed extensions of locally H-closed spaces in §3. Also, the notion of singular H-closed extensions is introduced in this section. We obtain a necessary and sufficient condition for a semiregular H-closed extension of a locally H-closed space to be a singular H-closed extension in §4.

A space is said to be *H-closed* if it is closed in every Hausdorff space in which it can be embedded as a subspace, and a *locally H-closed* (LHC) space is a space in which each point has an H-closed neighborhood. An *H-closed extension* of a space X is an H-closed space Y in which X is densely embedded; $Y - X$ is called the *remainder* of X in Y. An open cover $\mathscr{U}$ of a space X is said to

Received by the editors July 6, 1991.
1991 *Mathematics Subject Classification*. Primary 54D35, 54D40.
Key words and phrases. H-closed spaces, H-closed extensions, locally H-closed spaces, singular H-closed extensions, p-map, p-cover, remainder, retract.

be a *p-cover* if there is a finite subfamily of $\mathscr{U}$ whose union is dense in X. A map f from a space X to a space Y is said to be *p-map* if for each p-cover $\mathscr{U}$ of Y, $f^{-1}(\mathscr{U}) = \{f^{-1}(U) \mid U \in \mathscr{U}\}$ is a p-cover of X. A space is called *semiregular* if the family of regular open sets of the space forms a base for its topology. An extensive account of H-closed spaces and H-closed extensions can be found in [13–15]. Also, we refer to [1, 11] for other related studies.

For $A \subset X$, the closure of A in X will be denoted by $\mathrm{Cl}_X A$ or simply by $\mathrm{Cl}\, A$ if it is clear from the context. The family of H-closed sets of X is denoted by $\mathscr{H}_X$.

The real number system is denoted by R and the closed interval $[0, 1]$ by I. The symbols Q and N denote the sets of rational numbers and natural numbers respectively. Unless stated otherwise, R carries the usual topology and I, Q, and N are treated as its subspaces.

Let τ be the topology on X and $A \subset X$ be such that $A \notin \tau$. Then X equipped with the topology generated by $\tau \cup \tau_1$, where $\tau_1 = \{\varnothing, A, X\}$ is called the *simple extension* of X by A [18].

2. H-singular sets

2.1. **Definition.** Let f be a map from X to Y. Then the *H-singular set* $S_h(f)$ of f is the set $\{y \in Y \mid$ for every open set U of Y containing y, $f^{-1}(U)$ is not contained in an H-closed set of $X\}$.

Note that

$$\begin{aligned} S_h(f) &= \{y \in Y \mid \text{for every open set } U \text{ of } Y \text{ containing } y, \\ &\qquad \mathrm{Cl}\, f^{-1}(U) \text{ is not contained in an } H\text{-closed set of } X\} \\ &= \{y \in Y \mid \text{for every open set } U \text{ of } Y \text{ containing } y, \\ &\qquad \mathrm{Cl}\, f^{-1}(U) \text{ is not } H\text{-closed}\}. \end{aligned}$$

2.2. **Proposition.** *For a map f from X to Y,*

$$S_h(f) = \bigcap_{H \in \mathscr{H}_X} \mathrm{Cl}\, f(X - H).$$

Proof. The proof is straightforward.

2.3. **Proposition.** *Let f be a map from a space X to the real line R. If f has the extension Hf to the H-closed extension HX of X and if $HX - X$ is H-closed, then $Hf(HX - X) = S_h(f)$.*

Proof. Noting that, for $H \in \mathscr{H}_X$, $HX - X \subset \mathrm{Cl}_{HX}(X - H)$ and hence $Hf(HX - X) \subset Hf(\mathrm{Cl}_{HX}(X - H)) \subset \mathrm{Cl}\, Hf(X - H) = \mathrm{Cl}\, f(X - H)$, we obtain $Hf(HX - X) \subset S_h(f)$ in view of Proposition 2.2. Since $Hf(HX - X)$ is closed, to complete the proof it suffices to show that $Hf(HX - X)$ is dense in $S_h(f)$. If otherwise, then there exists a $p \in S_h(f)$ and an open set U in R containing p such that $\mathrm{Cl}\, U \cap Hf(HX - X) = \varnothing$, so that

$$(Hf)^{-1}(\mathrm{Cl}\, U) \subset (Hf)^{-1}(R - Hf(HX - X)) \subset X,$$

which gives that $\mathrm{Cl}_X f^{-1}(U) = \mathrm{Cl}_{HX}(Hf)^{-1}(U)$ is H-closed in X. This contradicts that $p \in S_h(f)$.

3. Singular H-closed extensions

Let f be a p-map from an LHC space X to an H-closed space Y such that $f(X)$ is dense in Y. Topologize the disjoint union $X \cup Y$ of X and Y as follows: Each open set of X is open in $X \cup Y$ and, for y in Y, the family $\{U \cup (f^{-1}(U) - H) \mid U$ is open in Y, $y \in U$ and $H \in \mathscr{H}_X\}$ forms a neighborhood base. With this topology $X \cup Y$ is an H-closed space. That it is a Hausdorff space can be verified by using local H-closedness of X. To prove that $X \cup Y$ is H-closed, let $\vartheta = \{V_a \mid a \in \mathscr{A}$, $\mathscr{A}$ is an index set$\}$ be an open cover of $X \cup Y$ by basic open sets. Then there exists a subset $\mathscr{B}$ of $\mathscr{A}$ such that, for b in $\mathscr{B}$, $V_b = U_b \cup (f^{-1}(U_b) - H_b)$, where U_b is open in Y, $H_b \in \mathscr{H}_X$, and $\{U_b \mid b \in \mathscr{B}\}$ is a p-cover of Y. Thus $\{f^{-1}(U_b) \mid b \in \mathscr{B}\}$ is a p-cover of X. Therefore, there exists a finite set $\mathscr{C}$ of $\mathscr{B}$ such that $\bigcup_{c \in \mathscr{C}} \mathrm{Cl}_X f^{-1}(U_c) = X$. Since $\bigcup_{c \in \mathscr{C}} (\mathrm{Cl}_X f^{-1}(U_c) - H_c)$ is contained in $\bigcup_{c \in \mathscr{C}} \mathrm{Cl}_X (f^{-1}(U_c) - H_c)$, $\bigcup_{c \in \mathscr{C}} \mathrm{Cl}_{X \cup Y} (f^{-1}(U_c) - H_c)$ contains $X - \bigcup_{c \in \mathscr{C}} H_c$. As $\bigcup_{c \in \mathscr{C}} H_c$ is H-closed in $X \cup Y$, there exists a finite set $\mathscr{D}$ contained in $\mathscr{A}$ such that $\bigcup_{c \in \mathscr{C}} H_c \subset \bigcup_{d \in \mathscr{D}} \mathrm{Cl}_{X \cup Y} V_d$. Further, the p-cover $\{U_b \mid b \in \mathscr{B}\}$ of Y provides a finite set $\mathscr{E}$ in $\mathscr{B}$ such that $Y \subset \bigcup_{e \in \mathscr{E}} \mathrm{Cl}_{X \cup Y} V_e$. Now, it follows that the union of members of ϑ corresponding to the finite set $\mathscr{C} \cup \mathscr{D} \cup \mathscr{E}$ of $\mathscr{A}$ is dense in $X \cup Y$.

Noting that $\mathrm{Cl}_{X \cup Y} X = X \cup S_h(f)$, we find that $X \cup S_h(f)$ is an H-closed extension of X, and thus we have established the following theorem.

3.1. **Theorem.** *If f is a p-map from a locally H-closed space X to an H-closed space Y such that $f(X)$ is dense in Y, then X has an H-closed extension with $S_h(f)$ as the remainder.*

3.2. *Remark.* Let f be a map from a locally compact space X to a compact space Y such that $f(X)$ is dense in Y. Then the *singular set* $S(f)$ of f is the set $\{y \in Y \mid$ for every open set U of Y containing y, $\mathrm{Cl}\, f^{-1}(U)$ is not compact$\}$. In this case $S_h(f) = S(f)$ and $X \cup S_h(f)$ coincides with the singular compactification $X \cup S(f)$ of X. It may be noted that, since Y is compact, the map f is a p-map.

The following example shows that if f is not a p-map, then the method described may not give an H-closed extension.

3.3. **Example.** Denote by I_d the closed interval $[0, 1]$ of R with the discrete topology, and let I_Q be the simple extension of I by $I \cap Q$. Then I_Q is an H-closed space. Consider the identity map f from I_d to I_Q. It is clear that $S_h(f) = I_Q$ and f is not a p-map. The space $I_d \cup I_Q$ fails to be H-closed because the open cover $\{(1/(n+1), 1/n) \cup f^{-1}(1/(n+1), 1/n) \mid n \in N\} \cup \{(I \cap Q) \cup f^{-1}(I \cap Q)\}$ is not a p-cover.

3.4. **Proposition.** *Let f be a p-map from a locally H-closed space X to an H-closed space Y such that $S_h(f) = Y$. Then X has an H-closed extension with Y as the remainder.*

Proof. The proof follows immediately from Theorem 3.1.

3.5. **Definition.** The H-closed extension $X \cup S_h(f)$ of a locally H-closed space X, where f is a p-map from X to an H-closed space Y such that $S_h(f) = Y$, is called a *singular H-closed extension of X*.

We give below some examples of singular H closed extensions.

3.6. **Example.** The constant map from a non-H-closed LHC space X to the one-point space provides a singular H-closed extension of X with remainder as the one-point space. Obviously for an LHC space, a singular H-closed extension with one-point space as the remainder is unique.

3.7. **Example.** Consider a map f from N to I_Q which is injective and satisfies $f(N) = I \cap Q$. The map f is a p-map with $S_h(f) = I_Q$. This provides a singular H-closed extension of N with I_Q as the remainder.

3.8. **Example.** Let D be the discrete space of cardinality c and $I_{Q'}$ be the simple extension of I by $I \cap (R - Q)$. Define f from D to $I_{Q'}$ which is injective and satisfies $f(D) = I \cap (R - Q)$. The map f is a p-map with $S_h(f) = I_{Q'}$. This gives a singular H-closed extension of D with $I_{Q'}$ as the remainder.

4. Singular H-closed extensions and retracts

Guglielmi [12] obtained that a compactification αX of a locally compact space X is singular iff $\alpha X - X$ is a retract of αX. An analogue of this result may be expected in the case of a singular H-closed extension of an LHC space. In this section we obtain that this is true under the condition when the extension is semiregular. We provide an example to show the need for this condition.

4.1. **Theorem.** *A semiregular H-closed extension HX of a locally H-closed space X is singular iff $HX - X$ is a retract of HX.*

Proof. If HX is a singular H-closed extension with respect to a map $f\colon X \to HX - X$, then the map $r\colon HX \to HX - X$ defined by $r(x) = x$, if $x \in HX - X$ and $r(x) = f(x)$, if $x \in X$, is a retraction. Conversely, suppose that $r\colon HX \to HX - X$ is a retraction and f is the restriction of r to X. Because the inclusion of X into HX and r are p-maps, f is also a p-map. To show that $S_h(f) = HX - X$, let, for $p \in HX - X$, U be an open set of $HX - X$ containing p. Since X is dense in HX and $r^{-1}(U)$ is an open neighbourhood of p, p is necessarily in $\mathrm{Cl}_{HX}(X \cap r^{-1}(U)) = \mathrm{Cl}_{HX}(f^{-1}(U))$. But p is not in X and therefore not in $\mathrm{Cl}_X(f^{-1}(U))$, implying that $\mathrm{Cl}_X(f^{-1}(U)) \neq \mathrm{Cl}_{HX}(f^{-1}(U))$, and thus $\mathrm{Cl}_X f^{-1}(U)$ is not H-closed. Therefore $p \in S_h(f)$. Since for an open set U of $HX - X$ and $H \in \mathscr{H}_X$, $h^{-1}(U \cup (f^{-1}(U) - H)) = r^{-1}(U) - H$, where $h\colon HX \to X \cup S_h(f)$ is the identity map, h is continuous. Further, since HX is semiregular, h is closed as well. Hence HX is a singular H-closed extension of X.

The following example shows that an H-closed extension HX of an LHC space X which is not semiregular is not necessarily a singular H-closed extension even if $HX - X$ is a retract of HX.

4.2. **Example.** Consider the subspace

$$X = \left\{ \left(\frac{1}{n}, \frac{1}{m} \right) \mid n, m \in N \right\} \cup \left\{ \left(\frac{1}{n}, 0 \right) \mid n \in N \right\}$$

of the Euclidean plane R^2. Let $Y = X \cup \{p\}$, where p is not in X. Equip Y with a topology such that $U \subset Y$ is open provided that $U \cap X$ is open in X and if $p \in U$, then there exists a $k \in N$ such that $\{(\frac{1}{n}, \frac{1}{m}) \mid n \geq k$

and $m \in N\} \subset U$. Then Y is an H-closed extension of X which is not semiregular [15, 7.3(d)]. Obviously, $Y - X = \{p\}$ is a retract of Y. In view of Remark 3.2, the singular H-closed extension of X with remainder as the one-point space is one-point compactification of X. Since Y and the one-point compactification of X are not homeomorphic [15, 7.3(d)], Y is not a singular H-closed extension of X.

4.3. *Remark.* From the proof of Theorem 4.1, it is clear that if HX is a singular H-closed extension of an LHC space X, then $HX - X$ is a retract of HX. However, a singular H-closed extension need not be semiregular: The fact that a space which is not semiregular cannot have a semiregular H-closed extension provides examples. Specifically, the one-point singular H-closed extension of the simple extension R_Q of R by Q is not semiregular.

Acknowledgments

The author expresses her gratefulness to Dr. K. K. Azad for his continued encouragement and valuable suggestions during the preparation of this manuscript.

The author thanks the referee for giving some significant suggestions, particularly for providing a simpler argument for a statement in the proof of Theorem 4.1.

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