PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 120, Number 1, January 1994

EXISTENCE OF PERIODIC SOLUTIONS FOR NONLINEAR EVOLUTION EQUATIONS IN HILBERT SPACES

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(Communicated by Barbara Lee Keyfitz)

ABSTRACT. In this paper, we consider the existence and multiplicity of periodic solutions of the problem $u' + Au \ni g(t, u)$ where A is a subdifferential of a convex function defined in a Hilbert space H and $g: R \times H \to H$ is a Carathéodory function periodic with respect to the first variable.

1. INTRODUCTION

Let $(H, \|\cdot\|)$ be a Hilbert space with an inner product $\langle\cdot, \cdot\rangle$ and $A \subset H \times H$ be an m-accretive operator given by the form $A = \partial\varphi$, where $\partial\varphi$ is the subdifferential of a lower semicontinuous proper convex function $\varphi: D(\varphi) \subset H \to R$, with domain $D(\varphi)$ dense in H. In the present paper, we consider the existence of T-periodic solutions for nonlinear evolution equations of the form

$$\text{(P)} \qquad \frac{du}{dt} + Au \ni g(t, u), \qquad t \in R,$$

where $g: R \times H \to H$ is a Carathéodory function.

We state our assumptions imposed on A and g:

(H_1) For some $\lambda > 0$, $J_\lambda = (I + \lambda^{-1}A)^{-1}$ is a compact mapping on H;

(H_2) $g: R \times H \to H$ is a Carathéodory mapping (i.e., for each $v \in H$ the mapping $t \to g(t, v)$ is measurable, and for each $t \in R$ the mapping $v \to g(t, v)$ is continuous) and satisfies that for some $M_1, M_2 > 0$

$$\|g(t, v)\| \le M_1\|v\| + M_2 \quad \text{for all } t \in R \text{ and } v \in H.$$

We now state our main result.

Theorem. *Suppose that (H_1) and (H_2) hold. Assume further that g is T-periodic with respect to the first variable and satisfies that there exist positive constants a and b such that*

$$(*) \qquad \langle z - g(t, v), v\rangle \ge a\|v\|^2 - b \quad \text{for all } v \in D(A) \text{ and } z \in Av.$$

Then the problem (P) *has at least one T-periodic mild solution.*

Received by the editors November 26, 1991 and, in revised form, March 2, 1992 and April 28, 1992.

1991 *Mathematics Subject Classification.* Primary 34G20, 34C25; Secondary 35K55.

Key words and phrases. Accretive operator, periodic solution, nonlinear evolution equations.

Existence of T-periodic solutions for (P) has been investigated by many authors (cf. [5–7]) under the assumption that A is m-accretive and g is Lipschitz continuous with respect to the second variable. In the case that g is merely continuous with respect to the second variable, the existence of periodic solutions of (P) was studied in [2, 8, 9]. Becker [2] considered the case that g is continuous and $A\colon D(A) \to H$ is a closed densely defined linear operator. Recently, Vrabie [9] extended Becker's result to fully nonlinear cases. In [9] it is assumed that H is a real Banach space and $A\colon D(A) \to 2^H$ is an m-accretive operator such that $\overline{D(A)}$ is convex, A generates a compact semigroup,

(c_1) there exists $a > 0$ such that $A - aI$ is m-accretive,

(c_2) g is a T-periodic Carathéodory mapping and satisfies

$$\lim_{r\to\infty} (1/r) \sup\{\|g(t, v)\| : t \in R,\ v \in \overline{D(A)},\ \|v\| \le r\} = m < a.$$

For additional references for periodic solutions of the problem (P), the reader is referred to Vrabie [9]. In the introduction of [9], the reader can find detailed explanations of the difficulty caused by the lack of Lipschitz continuity for g and of the results established so far. Our approach is quite different from that employed in [9]. Though our method requires that A is a subdifferential of a functional, we do not need that $A - aI$ is m-accretive for some $a > 0$ and then our results can be applied to a wide class of elliptic operators. The assumption $(*)$ is a unilateral condition and then we do not need the opposite side restriction. (See §3.) We assume that the reader is familiar with the theory of nonlinear evolution equations. (See Barbu [1] and Brezis [4] for the basic concepts and results of nonlinear evolution equations.)

2. Proof of Theorem

We denote by $L^2(0, T; H)$ the space of functions $v\colon [0, T] \to H$ such that $\int_0^T \|v\|^2\, dt < \infty$. The norm and the inner product of $L^2(0, T; H)$ are denoted by $\|\cdot\|_T$ and $\langle\langle\cdot, \cdot\rangle\rangle$, respectively. We identify the functions in $L^2(0, T; H)$ with T-periodic functions. For each positive integer m, we denote by $W^{m,2}(0, T; H)$ the space of functions $v\colon [0, T] \to H$ such that $v^{(i)} \in L^2(0, T; H)$ for $0 \le i \le m$, where $v^{(i)}$ denotes the ith derivative in the sense of distribution. For a reflexive Banach space E, a mapping $T\colon E \to E^*$ is said to be pseudomonotone if $u_n \to u$ weakly in E and $\limsup_{n\to\infty}\langle Tu_n, u_n - u\rangle \le 0$ imply that $Tu_n \to Tu$ weakly in E^* and

$$\langle Tu, u - v\rangle \le \liminf_{n\to\infty}\langle Tu_n, u_n - v\rangle \quad \text{for all } v \in E.$$

We put $A_\lambda = \lambda(I - J_\lambda)$ for $\lambda > 0$. It is known that J_λ is nonexpansive (i.e., $\|J_\lambda x - J_\lambda y\| \le \|x - y\|$ for $x, y \in H$). It is also known that $A_\lambda x \in AJ_\lambda x$ and $J_\lambda x = J_\mu(J_\lambda x + \mu^{-1}A_\lambda x)$ for $\lambda, \mu > 0$ and $x \in H$ (cf. [4]). It follows from this equality that (H_1) implies that J_λ is compact for all $\lambda > 0$. For each $n \ge 1$, we put $A_n = \partial\varphi_n$, where φ_n is a functional defined by $\varphi_n(x) = \inf_{u\in H}\{n\|x - u\|^2/2 + \varphi(u)\}$. Since φ is a proper convex lower semicontinuous function, the induced functional defined by $\tilde\varphi(u)(t) = \varphi(u(t))$ for $u \in L^2(0, T; H)$ is proper convex lower semicontinuous and the domain $D(\tilde\varphi)$ is dense in $L^2(0, T; H)$.

Therefore, $\widetilde{A} = \partial\tilde{\varphi}$ is maximal monotone in $L^2(0, T; H)$. For simplicity, we write A and φ instead $\widetilde{A}$ and $\tilde{\varphi}$, respectively. In the following, we assume that (H_1) and (H_2) hold.

To find a solution of the problem (P), we consider the approximate equations

$$(\mathrm{P}_n) \qquad \begin{cases} -\dfrac{1}{n}\dfrac{d^2u}{dt^2} + \dfrac{du}{dt} + A_n u = g(t, J_n u), \\ u(T+t) = u(t), \qquad t \in R. \end{cases}$$

Definition 1. A strong solution of the problem (P_n) is a T-periodic function $u\colon R \to H$ whose restriction to $[0, T]$ belongs to $W^{2,2}(0, T; H)$ and which satisfies (P_n) a.e. for $t \in R$.

Definition 2. A generalized solution of (P_n), or a solution of $(\widetilde{\mathrm{P}}_n)$, is a T-periodic function $u\colon R \to H$ whose restriction to $[0, T]$ belongs to $W^{1,2}(0, T; H)$ and satisfies

$$(\widetilde{\mathrm{P}}_n) \qquad \left\langle\left\langle \frac{1}{n}\frac{du}{dt}, \frac{dv}{dt} \right\rangle\right\rangle + \left\langle\left\langle \frac{du}{dt} + A_n u - g(t, J_n u), v \right\rangle\right\rangle = 0$$

for all $v \in W$ verifying $v(0) = v(T)$.

In the following, we denote by W the space defined by

$W = \{v\,;\, v\colon R \to H,\ v(t+T) = v(t) \text{ for } t \in R, \text{ and } v|_{[0,T]} \in W^{2,2}(0, T; H)\}$

endowed with the norm $\|\cdot\|_{1,T}$ of $W^{1,2}(0, T; H)$. That is,

$$\|v\|_{1,T}^2 = \|v_t\|_T^2 + \|v\|_T^2 \quad \text{for } v \in W.$$

Here we define an operator $T_n\colon W \to W^*$ by

$$\langle\langle T_n u, v\rangle\rangle = \left\langle\left\langle \frac{1}{n}\frac{du}{dt}, \frac{dv}{dt} \right\rangle\right\rangle + \left\langle\left\langle \frac{du}{dt} + A_n u - g(t, J_n u), v \right\rangle\right\rangle \quad \text{for } u, v \in W.$$

Then it is obvious that if $u \in W$ satisfies $T_n u = 0$, then u is a solution of the problem $(\widetilde{\mathrm{P}}_n)$. It also follows from the definitions above that each solution $u \in W$ of $(\widetilde{\mathrm{P}}_n)$ is a solution of (P_n) if $u|_{[0,T]} \in W^{2,2}(0, T; H)$.

Remark 1. The idea of considering approximating equations (P_n) is suggested by the observation that for each $n \geq 1$ the critical points of the functional F_n defined by

$$F_n(u) = \int_0^T \left(e^{-nt}\left(\frac{1}{2n}\|u_t\|^2 + \varphi(u)\right)\right) dt - \int_0^T e^{-nt} \int_\Omega \int^{u(t,x)} g(\tau)\, d\tau\, dx\, dt$$

are solutions of the problem (P_n) provided that each critical point is contained in $W^{2,2}(0, T; H)$. The existence of the critical points of F_n follows easily from the fact that $F_n(v) \to \infty$, as $\|v\|_{1,T} \to \infty$ under the assumption $(*)$.

Remark 2. We may assume without any loss of generality that φ attains its minimum at 0. In fact, if φ attains its minimum at $u_0 \neq 0$, we put $\widetilde{A}(v) = A(v + u_0)$ and $\tilde{g}(v) = g(v + u_0)$. Then if $\tilde{u}$ is a solution of the problem (P) with A and g replaced by $\widetilde{A}$ and $\tilde{g}$, we can see that $u = \tilde{u} + u_0$ is a solution of (P). In the following, we assume that 0 is the minimal point of φ. Then, since J_n is nonexpansive and $J_n 0 = 0$ for $n \geq 1$, we have that $\|J_n v\| \leq \|v\|$ and $\|A_n v\| \leq 2n\|v\|$ for $n \geq 1$ and $v \in H$.

Lemma 1. *For each* $n \geq 1$, $T_n: W \to W^*$ *is a pseudomonotone operator.*

Proof. Fix $n \geq 1$. Let $\{u_i\} \subset W$ be a sequence such that $u_i \to u \in W$ weakly in W and

$$\limsup_{i\to\infty} \langle\langle T_n u_i, u_i - u\rangle\rangle \leq 0. \tag{2.1}$$

Since $\{\|u_i\|_{1,T}\}$ is bounded we may assume that $u_i \to u \in L^2(0, T; H)$ weakly in $L^2(0, T; H)$ and $A_n u_i \to z$ weakly in $L^2(0, T; H)$. It also follows that

$$\sup\{\|u_i(t)\|: i \geq 1, 0 \leq t \leq T\} < \infty. \tag{2.2}$$

Since J_n is compact, we have that $\{J_n(u_i(t)): i \geq 1, 0 \leq t \leq T\}$ is relatively compact in H. Since J_n is nonexpansive, it follows from the boundedness of $\{\|u_i\|_{1,T}\}$ that $\{(J_n(u_i))_t: t \geq 1\}$ is bounded in $L^2(0, T; H)$. Thus we obtain that $\{J_n(u_i)\}$ is relatively compact in $L^2(0, T; H)$. Then we may assume by extracting subsequences that $J_n(u_i)$ converges to $v \in L^2(0, T; H)$ strongly in $L^2(0, T; H)$. Since $u_i \to u$ weakly in $L^2(0, T; H)$, we have that $v = J_n u$ (cf. [4, Proposition 2.5]). Then it follows that $g(J_n(u_i)) \to g(J_n(u))$ strongly in $L^2(0, T; H)$. Therefore, we have

$$\limsup_{i\to\infty} \int_0^T \left(\frac{1}{n}\langle u_{it}, u_{it} - u_t\rangle - \langle u_{it}, u\rangle + \langle A_n u_i, u_i - u\rangle\right) dt \leq 0.$$

Noting that

$$\|u_t\|_T \leq \liminf_{i\to\infty} \|u_{it}\|_T$$

and

$$\lim_{n\to\infty} \langle\langle u_{it}, u\rangle\rangle = \lim_{n\to\infty} \langle\langle u_i, u_t\rangle\rangle = 0,$$

we find that $\limsup_{i\to\infty}\langle\langle A_n u_i, u_i - u\rangle\rangle \leq 0$. Then, since A_n is maximal monotone, we have from [1, Chapter II, Lemma 1.3] that $\lim_{i\to\infty}\langle\langle A_n u_i, u_i\rangle\rangle = \langle\langle A_n u, u\rangle\rangle$ and $A_n u_i$ converges weakly to $A_n u$. Then we obtain that $T_n u_i$ converges weakly to $T_n u$ and then $\langle\langle T_n u, u - v\rangle\rangle \leq \liminf_{i\to\infty}\langle\langle T_n u_i, u_i - v\rangle\rangle$ for all $v \in W$. This completes the proof. □

Lemma 2. *There exists* $n_0 \geq 1$ *such that for each* $n \geq n_0$ *the mapping* T_n *satisfies*

$$\lim_{\|v\|_{1,T}\to\infty} \langle\langle T_n v, v\rangle\rangle = \infty. \tag{2.3}$$

Proof. From the definition of A_n, we have that

$$\langle A_n v, v\rangle = n\|v - J_n v\|^2 + \langle A_n v, J_n v\rangle \quad \text{for } v \in H.$$

Then we have by $(*)$ that for each $n \geq 1$ and $v \in H$

$$\begin{aligned} &\langle A_n v - g(t, J_n v), v\rangle \\ &\quad \geq \langle A_n v - g(t, J_n v), J_n v\rangle + n\|v - J_n v\|^2 \\ &\qquad - (M_1\|J_n v\| + M_2)\|v - J_n v\| \\ &\quad \geq a\|J_n v\|^2 + n\|v - J_n v\|^2 - (M_1\|J_n v\| + M_2)\|v - J_n v\| - b. \end{aligned} \tag{2.4}$$

Let $m \geq 1$ satisfy

$$c = a\left(\frac{m-1}{m}\right)^2 - \frac{M_1}{m} > 0. \tag{2.5}$$

If $v \in H$ satisfies $\|v - J_n v\| \leq (1/m)\|v\|$, then, recalling that $\|J_n v\| \leq \|v\|$, we have by (2.4) and (2.5) that

$$\begin{aligned}\langle A_n v - g(t, J_n v), v\rangle &\geq a\left(\frac{m-1}{m}\right)^2 \|v\|^2 - \frac{M_1}{m}\|v\|^2 - \frac{M_2}{m}\|v\| - b \\ &= c\|v\|^2 - \frac{M-2}{m}\|v\| - b.\end{aligned} \tag{2.6}$$

On the other hand, for $v \in H$ satisfying $\|v - J_n v\| > (1/m)\|v\|$, we find by (2.4) that

$$\langle A_n v - g(t, J_n v), v\rangle \geq \left(\frac{n}{m^2} - 2M_1\right)\|v\|^2 - 2M_2\|v\| - b. \tag{2.7}$$

Then combining (2.6) and (2.7), we find that there exist $n_0 \geq 1$, $C \geq 0$, and $\rho > 0$ such that for each $n \geq n_0$

$$\langle A_n v - g(t, J_n v), v\rangle \geq \rho\|v\|^2 - C \quad \text{for all } t \in R \text{ and } v \in H. \tag{2.8}$$

Now let $n \geq n_0$. Then for each $v \in W$ we have by (2.8) and the definition of T_n that

$$\langle\langle T_n v, v\rangle\rangle \geq \int_0^T \left(\frac{1}{n}\|v_t\|^2 + \rho\|v\|^2 - C\right) dt.$$

This implies that $\lim_{\|v\|_{1,T}\to\infty}\langle\langle T_n v, v\rangle\rangle = \infty$. □

Lemma 3. *For each $n \geq n_0$, there exists a strong solution $u_n \in W^{2,2}(0, T; H)$ of the problem (P_n) satisfying*

$$\left\|\frac{1}{n}u_{ntt}\right\|_T^2 \leq 4\|u_{nt}\|_T^2 + 2M_1^2\|u_n\|_T^2 + 2TM_2^2. \tag{2.9}$$

Proof. Let $n \geq n_0$. Then we have by Lemma 2 that T_n is coercive in W. Then, since T_n is pseudomonotone, we have that there exists $u_n \in W$ such that $T_n u_n = 0$ (cf. [3]). That is, u_n is a solution of $(\widetilde{\mathrm{P}}_n)$. We show that $u_n \in W^{2,2}(0, T; H)$. Let $\{u_n^i\} \subset W^{2,2}(0, T; H) \cap W$ be an approximating sequence such that u_n^i converges to u_n strongly in $W^{1,2}(0, T; H)$. Then from the definition of T_n we have that for each $v \in W$

$$\left\langle\left\langle -\frac{1}{n}u_{ntt}^i + u_{nt}^i + A_n u_n^i - g(t, J_n u_n^i), v\right\rangle\right\rangle \to 0 \quad \text{as } i \to \infty.$$

This implies that $(1/n)u_{ntt}^i$ converges to $u_{nt} + A_n u_n - g(t, J_n u_n)$ weakly in W^*. Since $u_{nt} + A_n u_n - g(t, J_n u_n) \in L^2(0, T; H)$, we obtain that $u_{ntt} \in L^2(0, T; H)$. That is, $u_n \in W^{2,2}(0, T; H)$ and

$$-\frac{1}{n}u_{ntt} + u_{nt} + A_n u_n - g(t, J_n u_n) = 0. \tag{2.10}$$

We multiply (2.10) by u_{ntt} and integrate over $[0, T]$. Then, noting that A_n is Lipschitz continuous with Lipschitz constant $2n$, we find

$$\begin{aligned}\int_0^T \frac{1}{n}\|u_{ntt}\|^2\, dt &\leq \int_0^T (\|u_{nt}\|\,\|(A_n u_n)_t\| + \|u_{ntt}\|(M_1\|J_n u_n\| + M_2))\, dt \\ &\leq \int_0^T (2n\|u_{nt}\|^2 + \|u_{ntt}\|(M_1\|u_n\| + M_2))\, dt.\end{aligned}$$

Then we find

$$\int_0^T \left\| \frac{1}{n} u_{ntt} \right\|^2 dt \leq \int_0^T \left(2\|u_{nt}\|^2 + \left\| \frac{1}{n} u_{ntt} \right\| (M_1 \|u_n\| + M_2) \right) dt$$

$$\leq 2\|u_{nt}\|_T^2 + \frac{1}{2} \left\| \frac{1}{n} u_{ntt} \right\|_T^2 + M_1^2 \|u_n\|_T^2 + TM_2^2.$$

Then we can see that (2.9) follows. □

Proof of Theorem. By Lemma 3, we have that for each $n \geq n_0$ there exists a strong solution u_n of the problem (P_n). That is, u_n is T-periodic and satisfies

$$-\frac{1}{n} \frac{d^2 u_n}{dt^2} + \frac{du_n}{dt} + A_n u_n = g(t, J_n u_n) \quad \text{a.e. } t \in R. \tag{2.11}$$

We multiply (2.11) by u_n and integrate over $[0, T]$. Then we have by (2.8)

$$\int_0^T \left(\frac{1}{n} \|u_{nt}\|^2 + \rho \|u_n\|^2 \right) dt \leq TC. \tag{2.12}$$

Then we have that $\{\|u_n\|_T\}$ is bounded and that $\{\|g(t, J_n u_n)\|_T\}$ is bounded. We next multiply (2.11) by u_{nt} and integrate over $[0, T]$. Then, noting that

$$\int_0^T \langle A_n u_n, u_{nt} \rangle \, dt = \varphi_n(u_n(T)) - \varphi_n(u_n(0)) = 0,$$

we find

$$\|u_{nt}\|_T^2 \leq \|u_{nt}\|_T \|g(t, J_n u_n)\|_T \leq \|u_{nt}\|_T (M_1 \|u_n\|_T + M_2). \tag{2.13}$$

This implies that $\{\|u_{nt}\|_T\}$ is bounded. It then follows that

$$\sup_{n \geq n_0} \|(J_n u_n)_t\|_T < \infty. \tag{2.14}$$

We also have by (2.14) that

$$\sup\{\|J_n u_n(t)\| : n \geq n_0, \ 0 \leq t \leq T\} < \infty. \tag{2.15}$$

On the other hand, recalling that (2.9) holds for all $n \geq n_0$, we have that $\{\|u_{ntt}/n\|\}$ is bounded. Then it follows from (2.11) that

$$\sup_{n \geq n_0} \int_0^T \|A_n u_n\|^2 \, dt < \infty. \tag{2.16}$$

We now show that $\{J_n u_n\}$ is relatively compact in $L^2(0, T; H)$. Let $\varepsilon > 0$. Then, by (2.14) and (2.15), there exists an integer $m_0 > 0$ such that

$$\|J_n u_n(t) - J_n u_n(s)\|^2 < \varepsilon/6T \quad \text{for all } n \geq n_0 \text{ and } |t - s| < 2T/m_0. \tag{2.17}$$

On the other hand, we have that there exists $D > 0$ such that

$$\inf\{\|A_n u_n(\tau)\| : t \leq \tau \leq t + T/m_0\} < D \quad \text{for all } n \geq n_0 \text{ and } 0 \leq t \leq T - T/m_0.$$

We now choose $\{t_{m,n} : n_0 \leq n, \ 1 \leq m \leq m_0\} \subset [0, T]$ such that

$$T(m-1)/m_0 \leq t_{m,n} \leq Tm/m_0 \quad \text{and} \quad \|A_n u_n(t_{m,n})\| \leq D$$

for $n \geq n_0$ and $1 \leq m \leq m_0$. Here we fix $n_1 \geq 1$. Then, since

$$J_n(u_n(t_{m,n})) = J_{n_1}(J_n u_n(t_{m,n}) + A_n u_n(t_{m,n})/n_1) \quad \text{for } n \geq 1,$$

we find by (H_1) that $\{J_n(u_n(t_{m,n}))\}$ is relatively compact. Then we may assume by extracting subsequences that $\{J_n(u_n(t_{m,n})) : n \geq 1\}$ is a convergent sequence for all $1 \leq m \leq m_0$. Then we have from (2.17) and the observation above that $\|J_n(u_n) - J_{n'}(u_{n'})\|_T^2 < \varepsilon$ for n, n' sufficiently large. Since $\varepsilon > 0$ is arbitrary, we have that $\{J_n(u_n)\}$ is relatively compact in $L^2(0, T; H)$.

Therefore, we may assume that there exists a T-periodic function $u \in W$ and

$$J_n u_n \to u \quad \text{strongly in } L^2(0, T; H). \tag{2.18}$$

Also we may assume that $A_n u_n \to z \in L^2(0, T; H)$ weakly in $L^2(0, T; H)$ by the boundedness of $\{\|A_n u_n\|_T\}$. On the other hand, we have

$$\int_0^T \|u_n - J_n u_n\|^2 \, dt = \frac{1}{n^2} \int_0^T \|A_n u_n\|^2 \, dt \to 0 \quad \text{as } n \to \infty. \tag{2.19}$$

Then we find by (2.18) and (2.19) that

$$\lim_{m, n \to \infty} \int_0^T \langle A_n u_n - A_m u_m, u_n - u_m \rangle \, dt = 0.$$

Since A is maximal monotone in $L^2(0, T; H)$, we have by [1, Chapter II, Proposition 1.1] that

$$A_n u_n \to z \in Au \quad \text{weakly in } L^2(0, T; H). \tag{2.20}$$

We also have that $\frac{1}{n} u_{ntt} \to 0$ weakly in $W^{1,2}(0, T; H)$, because $\frac{1}{n} \int_0^T |u_{nt}|^2 \, dt \to 0$ as $n \to \infty$. Therefore, by (2.18), (2.19), and (2.20), we have that u is a T-periodic solution of the problem (P). □

3. Example

Let $\Omega \subseteq R^N$ be a bounded domain with a smooth boundary $\partial\Omega$ and $\psi : R \to R$ be a convex function. We put

$$Au = \sum_{i=1}^n D_i a_i(Du) \quad \text{for } u \in D(A) \tag{3.1}$$

where $a_i(p) = \partial\psi(p)/\partial p_i$ for $p \in R^N$ and $D(A) = \{u \in H_0^1(\Omega) : Au \in L^2(\Omega)\}$. We assume that a_i is continuously differentiable and satisfies

$$\sum_{i=1}^n a_i(p) p_i \geq \omega |p|^2 - C_0 \quad \text{and} \quad \sum_{i=1}^n |a_i(p)| \leq C_1 |p| + C_2$$

for all $p \in R^N$, where C_0, C_1, and C_2 are positive constants. Then A is a subdifferential of the functional $u \to \int_\Omega \psi(u) \, dx$ and satisfies the assumption (H_1) (cf. Vrabie [10]) with $H = L^2(\omega)$. Let $g : R \times R \to R$ be a Carathéodory mapping satisfying $g(t, x)x < \lambda_1(\omega - \varepsilon)|x|^2$ for some $\varepsilon > 0$, where λ_1 is the first eigenvalue of $-\Delta$ under the Dirichlet boundary condition. Then, since

$$\langle Au - g(t, u), u \rangle \geq \omega \|\nabla u\|^2 - C_0 - \lambda_1(\omega - \varepsilon)\|u\|^2 \geq \lambda_1 \varepsilon \|u\|^2 - C_0$$

for $u \in H_0^1(\omega)$, the condition $(*)$ is satisfied. We lastly see that the condition $(*)$ is strictly weaker than (c_1) and (c_2). Let $\psi(\tau) = \tau^2$ for $\tau \in R$ and

$g(t, x) = \lambda_1(\sin t - 2)\tau^3/(1+\tau^2)$ for $(t, \tau) \in R \times R$. Let A be the operator defined by (3.1) (i.e., $A = -\Delta$). Then (H_1), (H_2), and $(*)$ are satisfied. The operator A satisfies (c_1) for $a < \lambda_1$. But one can see that condition (c_2) is not satisfied. Let $\psi(\tau) = \tau^4/(1+\tau^2)$ for $\tau \in R$ and g be as above. In this case, (H_1), (H_2), and $(*)$ are satisfied, but A does not satisfy (c_1) for any $a > 0$.

Acknowledgment

I am grateful to the referee for a careful reading of the paper and giving valuable suggestions.

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