PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 120, Number 1, January 1994

EXISTENCE OF PERIODIC SOLUTIONS FOR NONLINEAR EVOLUTION EQUATIONS IN HILBERT SPACES

NORIMICHI HIRANO

(Communicated by Barbara Lee Keyfitz)

ABSTRACT. In this paper, we consider the existence and multiplicity of periodic solutions of the problem $u' + Au \ni g(t, u)$ where A is a subdifferential of a convex function defined in a Hilbert space H and $g: R \times H \to H$ is a Carathéodory function periodic with respect to the first variable.

1. INTRODUCTION

Let $(H, \|\cdot\|)$ be a Hilbert space with an inner product $\langle\cdot, \cdot\rangle$ and $A \subset H \times H$ be an m-accretive operator given by the form $A = \partial\varphi$, where $\partial\varphi$ is the subdifferential of a lower semicontinuous proper convex function $\varphi: D(\varphi) \subset H \to R$, with domain $D(\varphi)$ dense in H. In the present paper, we consider the existence of T-periodic solutions for nonlinear evolution equations of the form

$$\text{(P)} \qquad \frac{du}{dt} + Au \ni g(t, u), \qquad t \in R,$$

where $g: R \times H \to H$ is a Carathéodory function.

We state our assumptions imposed on A and g:

(H_1) For some $\lambda > 0$, $J_\lambda = (I + \lambda^{-1}A)^{-1}$ is a compact mapping on H;

(H_2) $g: R \times H \to H$ is a Carathéodory mapping (i.e., for each $v \in H$ the mapping $t \to g(t, v)$ is measurable, and for each $t \in R$ the mapping $v \to g(t, v)$ is continuous) and satisfies that for some $M_1, M_2 > 0$

$$\|g(t, v)\| \le M_1\|v\| + M_2 \quad \text{for all } t \in R \text{ and } v \in H.$$

We now state our main result.

Theorem. *Suppose that (H_1) and (H_2) hold. Assume further that g is T-periodic with respect to the first variable and satisfies that there exist positive constants a and b such that*

$$(*) \qquad \langle z - g(t, v), v\rangle \ge a\|v\|^2 - b \quad \textit{for all } v \in D(A) \textit{ and } z \in Av.$$

Then the problem (P) *has at least one T-periodic mild solution.*

Received by the editors November 26, 1991 and, in revised form, March 2, 1992 and April 28, 1992.

1991 *Mathematics Subject Classification*. Primary 34G20, 34C25; Secondary 35K55.

Key words and phrases. Accretive operator, periodic solution, nonlinear evolution equations.

Existence of T-periodic solutions for (P) has been investigated by many authors (cf. [5–7]) under the assumption that A is m-accretive and g is Lipschitz continuous with respect to the second variable. In the case that g is merely continuous with respect to the second variable, the existence of periodic solutions of (P) was studied in [2, 8, 9]. Becker [2] considered the case that g is continuous and $A: D(A) \to H$ is a closed densely defined linear operator. Recently, Vrabie [9] extended Becker's result to fully nonlinear cases. In [9] it is assumed that H is a real Banach space and $A: D(A) \to 2^H$ is an m-accretive operator such that $\overline{D(A)}$ is convex, A generates a compact semigroup,

(c_1) there exists $a > 0$ such that $A - aI$ is m-accretive,

(c_2) g is a T-periodic Carathéodory mapping and satisfies

$$\lim_{r\to\infty} (1/r) \sup\{\|g(t, v)\| : t \in R,\ v \in \overline{D(A)},\ \|v\| \le r\} = m < a.$$

For additional references for periodic solutions of the problem (P), the reader is referred to Vrabie [9]. In the introduction of [9], the reader can find detailed explanations of the difficulty caused by the lack of Lipschitz continuity for g and of the results established so far. Our approach is quite different from that employed in [9]. Though our method requires that A is a subdifferential of a functional, we do not need that $A - aI$ is m-accretive for some $a > 0$ and then our results can be applied to a wide class of elliptic operators. The assumption $(*)$ is a unilateral condition and then we do not need the opposite side restriction. (See §3.) We assume that the reader is familiar with the theory of nonlinear evolution equations. (See Barbu [1] and Brezis [4] for the basic concepts and results of nonlinear evolution equations.)

2. Proof of Theorem

We denote by $L^2(0, T; H)$ the space of functions $v: [0, T] \to H$ such that $\int_0^T \|v\|^2\, dt < \infty$. The norm and the inner product of $L^2(0, T; H)$ are denoted by $\|\cdot\|_T$ and $\langle\langle \cdot, \cdot\rangle\rangle$, respectively. We identify the functions in $L^2(0, T; H)$ with T-periodic functions. For each positive integer m, we denote by $W^{m,2}(0, T; H)$ the space of functions $v: [0, T] \to H$ such that $v^{(i)} \in L^2(0, T; H)$ for $0 \le i \le m$, where $v^{(i)}$ denotes the ith derivative in the sense of distribution. For a reflexive Banach space E, a mapping $T: E \to E^*$ is said to be pseudomonotone if $u_n \to u$ weakly in E and $\limsup_{n\to\infty} \langle Tu_n, u_n - u\rangle \le 0$ imply that $Tu_n \to Tu$ weakly in E^* and

$$\langle Tu, u - v\rangle \le \liminf_{n\to\infty} \langle Tu_n, u_n - v\rangle \quad \text{for all } v \in E.$$

We put $A_\lambda = \lambda(I - J_\lambda)$ for $\lambda > 0$. It is known that J_λ is nonexpansive (i.e., $\|J_\lambda x - J_\lambda y\| \le \|x - y\|$ for $x, y \in H$). It is also known that $A_\lambda x \in AJ_\lambda x$ and $J_\lambda x = J_\mu(J_\lambda x + \mu^{-1} A_\lambda x)$ for $\lambda, \mu > 0$ and $x \in H$ (cf. [4]). It follows from this equality that (H_1) implies that J_λ is compact for all $\lambda > 0$. For each $n \ge 1$, we put $A_n = \partial\varphi_n$, where φ_n is a functional defined by $\varphi_n(x) = \inf_{u\in H}\{n\|x - u\|^2/2 + \varphi(u)\}$. Since φ is a proper convex lower semicontinuous function, the induced functional defined by $\tilde\varphi(u)(t) = \varphi(u(t))$ for $u \in L^2(0, T; H)$ is proper convex lower semicontinuous and the domain $D(\tilde\varphi)$ is dense in $L^2(0, T; H)$.

Therefore, $\widetilde{A} = \partial\tilde{\varphi}$ is maximal monotone in $L^2(0, T; H)$. For simplicity, we write A and φ instead $\widetilde{A}$ and $\tilde{\varphi}$, respectively. In the following, we assume that (H_1) and (H_2) hold.

To find a solution of the problem (P), we consider the approximate equations

$$(\mathrm{P}_n) \qquad \begin{cases} -\dfrac{1}{n}\dfrac{d^2u}{dt^2} + \dfrac{du}{dt} + A_n u = g(t, J_n u), \\ u(T+t) = u(t), \qquad t \in R. \end{cases}$$

Definition 1. A strong solution of the problem (P_n) is a T-periodic function $u: R \to H$ whose restriction to $[0, T]$ belongs to $W^{2,2}(0, T; H)$ and which satisfies (P_n) a.e. for $t \in R$.

Definition 2. A generalized solution of (P_n), or a solution of $(\widetilde{\mathrm{P}}_n)$, is a T-periodic function $u: R \to H$ whose restriction to $[0, T]$ belongs to $W^{1,2}(0, T; H)$ and satisfies

$$(\widetilde{\mathrm{P}}_n) \qquad \left\langle\!\left\langle \frac{1}{n}\frac{du}{dt}, \frac{dv}{dt} \right\rangle\!\right\rangle + \left\langle\!\left\langle \frac{du}{dt} + A_n u - g(t, J_n u), v \right\rangle\!\right\rangle = 0$$
$$\text{for all } v \in W \text{ verifying } v(0) = v(T).$$

In the following, we denote by W the space defined by
$W = \{v; v: R \to H, \ v(t+T) = v(t) \text{ for } t \in R, \ \text{and } v|_{[0,T]} \in W^{2,2}(0, T; H)\}$
endowed with the norm $\|\cdot\|_{1,T}$ of $W^{1,2}(0, T; H)$. That is,

$$\|v\|_{1,T}^2 = \|v_t\|_T^2 + \|v\|_T^2 \quad \text{for } v \in W.$$

Here we define an operator $T_n: W \to W^*$ by

$$\langle\langle T_n u, v \rangle\rangle = \left\langle\!\left\langle \frac{1}{n}\frac{du}{dt}, \frac{dv}{dt} \right\rangle\!\right\rangle + \left\langle\!\left\langle \frac{du}{dt} + A_n u - g(t, J_n u), v \right\rangle\!\right\rangle \quad \text{for } u, v \in W.$$

Then it is obvious that if $u \in W$ satisfies $T_n u = 0$, then u is a solution of the problem $(\widetilde{\mathrm{P}}_n)$. It also follows from the definitions above that each solution $u \in W$ of $(\widetilde{\mathrm{P}}_n)$ is a solution of (P_n) if $u|_{[0,T]} \in W^{2,2}(0, T; H)$.

Remark 1. The idea of considering approximating equations (P_n) is suggested by the observation that for each $n \geq 1$ the critical points of the functional F_n defined by

$$F_n(u) = \int_0^T \left(e^{-nt}\left(\frac{1}{2n}\|u_t\|^2 + \varphi(u)\right)\right) dt - \int_0^T e^{-nt} \int_\Omega \int^{u(t,x)} g(\tau)\, d\tau\, dx\, dt$$

are solutions of the problem (P_n) provided that each critical point is contained in $W^{2,2}(0, T; H)$. The existence of the critical points of F_n follows easily from the fact that $F_n(v) \to \infty$, as $\|v\|_{1,T} \to \infty$ under the assumption $(*)$.

Remark 2. We may assume without any loss of generality that φ attains its minimum at 0. In fact, if φ attains its minimum at $u_0 \neq 0$, we put $\widetilde{A}(v) = A(v + u_0)$ and $\tilde{g}(v) = g(v + u_0)$. Then if $\tilde{u}$ is a solution of the problem (P) with A and g replaced by $\widetilde{A}$ and $\tilde{g}$, we can see that $u = \tilde{u} + u_0$ is a solution of (P). In the following, we assume that 0 is the minimal point of φ. Then, since J_n is nonexpansive and $J_n 0 = 0$ for $n \geq 1$, we have that $\|J_n v\| \leq \|v\|$ and $\|A_n v\| \leq 2n\|v\|$ for $n \geq 1$ and $v \in H$.

Lemma 1. *For each $n \geq 1$, $T_n \colon W \to W^*$ is a pseudomonotone operator.*

Proof. Fix $n \geq 1$. Let $\{u_i\} \subset W$ be a sequence such that $u_i \to u \in W$ weakly in W and

$$\limsup_{i\to\infty} \langle\langle T_n u_i, u_i - u\rangle\rangle \leq 0. \tag{2.1}$$

Since $\{\|u_i\|_{1,T}\}$ is bounded we may assume that $u_i \to u \in L^2(0, T; H)$ weakly in $L^2(0, T; H)$ and $A_n u_i \to z$ weakly in $L^2(0, T; H)$. It also follows that

$$\sup\{\|u_i(t)\| \colon i \geq 1, 0 \leq t \leq T\} < \infty. \tag{2.2}$$

Since J_n is compact, we have that $\{J_n(u_i(t)) \colon i \geq 1, 0 \leq t \leq T\}$ is relatively compact in H. Since J_n is nonexpansive, it follows from the boundedness of $\{\|u_i\|_{1,T}\}$ that $\{(J_n(u_i))_t \colon t \geq 1\}$ is bounded in $L^2(0, T; H)$. Thus we obtain that $\{J_n(u_i)\}$ is relatively compact in $L^2(0, T; H)$. Then we may assume by extracting subsequences that $J_n(u_i)$ converges to $v \in L^2(0, T; H)$ strongly in $L^2(0, T; H)$. Since $u_i \to u$ weakly in $L^2(0, T; H)$, we have that $v = J_n u$ (cf. [4, Proposition 2.5]). Then it follows that $g(J_n(u_i)) \to g(J_n(u))$ strongly in $L^2(0, T; H)$. Therefore, we have

$$\limsup_{i\to\infty} \int_0^T \left(\frac{1}{n}\langle u_{it}, u_{it} - u_t\rangle - \langle u_{it}, u\rangle + \langle A_n u_i, u_i - u\rangle\right) dt \leq 0.$$

Noting that

$$\|u_t\|_T \leq \liminf_{i\to\infty} \|u_{it}\|_T$$

and

$$\lim_{n\to\infty} \langle\langle u_{it}, u\rangle\rangle = \lim_{n\to\infty} \langle\langle u_i, u_t\rangle\rangle = 0,$$

we find that $\limsup_{i\to\infty}\langle\langle A_n u_i, u_i - u\rangle\rangle \leq 0$. Then, since A_n is maximal monotone, we have from [1, Chapter II, Lemma 1.3] that $\lim_{i\to\infty}\langle\langle A_n u_i, u_i\rangle\rangle = \langle\langle A_n u, u\rangle\rangle$ and $A_n u_i$ converges weakly to $A_n u$. Then we obtain that $T_n u_i$ converges weakly to $T_n u$ and then $\langle\langle T_n u, u - v\rangle\rangle \leq \liminf_{i\to\infty}\langle\langle T_n u_i, u_i - v\rangle\rangle$ for all $v \in W$. This completes the proof. □

Lemma 2. *There exists $n_0 \geq 1$ such that for each $n \geq n_0$ the mapping T_n satisfies*

$$\lim_{\|v\|_{1,T}\to\infty} \langle\langle T_n v, v\rangle\rangle = \infty. \tag{2.3}$$

Proof. From the definition of A_n, we have that

$$\langle A_n v, v\rangle = n\|v - J_n v\|^2 + \langle A_n v, J_n v\rangle \quad \text{for } v \in H.$$

Then we have by $(*)$ that for each $n \geq 1$ and $v \in H$

$$\begin{aligned}
&\langle A_n v - g(t, J_n v), v\rangle \\
&\quad\geq \langle A_n v - g(t, J_n v), J_n v\rangle + n\|v - J_n v\|^2 \\
&\qquad - (M_1\|J_n v\| + M_2)\|v - J_n v\| \\
&\quad\geq a\|J_n v\|^2 + n\|v - J_n v\|^2 - (M_1\|J_n v\| + M_2)\|v - J_n v\| - b.
\end{aligned} \tag{2.4}$$

Let $m \geq 1$ satisfy

$$c = a\left(\frac{m-1}{m}\right)^2 - \frac{M_1}{m} > 0. \tag{2.5}$$

If $v \in H$ satisfies $\|v - J_n v\| \le (1/m)\|v\|$, then, recalling that $\|J_n v\| \le \|v\|$, we have by (2.4) and (2.5) that

$$\begin{aligned}\langle A_n v - g(t, J_n v), v\rangle &\ge a\left(\frac{m-1}{m}\right)^2 \|v\|^2 - \frac{M_1}{m}\|v\|^2 - \frac{M_2}{m}\|v\| - b \\ &= c\|v\|^2 - \frac{M-2}{m}\|v\| - b.\end{aligned} \tag{2.6}$$

On the other hand, for $v \in H$ satisfying $\|v - J_n v\| > (1/m)\|v\|$, we find by (2.4) that

$$\langle A_n v - g(t, J_n v), v\rangle \ge \left(\frac{n}{m^2} - 2M_1\right)\|v\|^2 - 2M_2\|v\| - b. \tag{2.7}$$

Then combining (2.6) and (2.7), we find that there exist $n_0 \ge 1$, $C \ge 0$, and $\rho > 0$ such that for each $n \ge n_0$

$$\langle A_n v - g(t, J_n v), v\rangle \ge \rho\|v\|^2 - C \quad \text{for all } t \in R \text{ and } v \in H. \tag{2.8}$$

Now let $n \ge n_0$. Then for each $v \in W$ we have by (2.8) and the definition of T_n that

$$\langle\langle T_n v, v\rangle\rangle \ge \int_0^T \left(\frac{1}{n}\|v_t\|^2 + \rho\|v\|^2 - C\right) dt.$$

This implies that $\lim_{\|v\|_{1,T}\to\infty}\langle\langle T_n v, v\rangle\rangle = \infty$. □

Lemma 3. *For each $n \ge n_0$, there exists a strong solution $u_n \in W^{2,2}(0, T; H)$ of the problem* (P_n) *satisfying*

$$\left\|\frac{1}{n}u_{ntt}\right\|_T^2 \le 4\|u_{nt}\|_T^2 + 2M_1^2\|u_n\|_T^2 + 2TM_2^2. \tag{2.9}$$

Proof. Let $n \ge n_0$. Then we have by Lemma 2 that T_n is coercive in W. Then, since T_n is pseudomonotone, we have that there exists $u_n \in W$ such that $T_n u_n = 0$ (cf. [3]). That is, u_n is a solution of $(\widetilde{\mathrm{P}}_n)$. We show that $u_n \in W^{2,2}(0, T; H)$. Let $\{u_n^i\} \subset W^{2,2}(0, T; H) \cap W$ be an approximating sequence such that u_n^i converges to u_n strongly in $W^{1,2}(0, T; H)$. Then from the definition of T_n we have that for each $v \in W$

$$\left\langle\left\langle -\frac{1}{n}u_{ntt}^i + u_{nt}^i + A_n u_n^i - g(t, J_n u_n^i), v\right\rangle\right\rangle \to 0 \quad \text{as } i \to \infty.$$

This implies that $(1/n)u_{ntt}^i$ converges to $u_{nt} + A_n u_n - g(t, J_n u_n)$ weakly in W^*. Since $u_{nt} + A_n u_n - g(t, J_n u_n) \in L^2(0, T; H)$, we obtain that $u_{ntt} \in L^2(0, T; H)$. That is, $u_n \in W^{2,2}(0, T; H)$ and

$$-\frac{1}{n}u_{ntt} + u_{nt} + A_n u_n - g(t, J_n u_n) = 0. \tag{2.10}$$

We multiply (2.10) by u_{ntt} and integrate over $[0, T]$. Then, noting that A_n is Lipschitz continuous with Lipschitz constant $2n$, we find

$$\begin{aligned}\int_0^T \frac{1}{n}\|u_{ntt}\|^2\,dt &\le \int_0^T (\|u_{nt}\|\,\|(A_n u_n)_t\| + \|u_{ntt}\|(M_1\|J_n u_n\| + M_2))\,dt \\ &\le \int_0^T (2n\|u_{nt}\|^2 + \|u_{ntt}\|(M_1\|u_n\| + M_2))\,dt.\end{aligned}$$

Then we find

$$\int_0^T \left\|\frac{1}{n}u_{ntt}\right\|^2 dt \leq \int_0^T \left(2\|u_{nt}\|^2 + \left\|\frac{1}{n}u_{ntt}\right\|(M_1\|u_n\| + M_2)\right) dt$$
$$\leq 2\|u_{nt}\|_T^2 + \frac{1}{2}\left\|\frac{1}{n}u_{ntt}\right\|_T^2 + M_1^2\|u_n\|_T^2 + TM_2^2.$$

Then we can see that (2.9) follows. □

Proof of Theorem. By Lemma 3, we have that for each $n \geq n_0$ there exists a strong solution u_n of the problem (P_n). That is, u_n is T-periodic and satisfies

$$-\frac{1}{n}\frac{d^2u_n}{dt^2} + \frac{du_n}{dt} + A_nu_n = g(t, J_nu_n) \quad \text{a.e. } t \in R. \tag{2.11}$$

We multiply (2.11) by u_n and integrate over $[0, T]$. Then we have by (2.8)

$$\int_0^T \left(\frac{1}{n}\|u_{nt}\|^2 + \rho\|u_n\|^2\right) dt \leq TC. \tag{2.12}$$

Then we have that $\{\|u_n\|_T\}$ is bounded and that $\{\|g(t, J_nu_n)\|_T\}$ is bounded. We next multiply (2.11) by u_{nt} and integrate over $[0, T]$. Then, noting that

$$\int_0^T \langle A_nu_n, u_{nt}\rangle\, dt = \varphi_n(u_n(T)) - \varphi_n(u_n(0)) = 0,$$

we find

$$\|u_{nt}\|_T^2 \leq \|u_{nt}\|_T\|g(t, J_nu_n)\|_T \leq \|u_{nt}\|_T(M_1\|u_n\|_T + M_2). \tag{2.13}$$

This implies that $\{\|u_{nt}\|_T\}$ is bounded. It then follows that

$$\sup_{n\geq n_0} \|(J_nu_n)_t\|_T < \infty. \tag{2.14}$$

We also have by (2.14) that

$$\sup\{\|J_nu_n(t)\|: n \geq n_0,\ 0 \leq t \leq T\} < \infty. \tag{2.15}$$

On the other hand, recalling that (2.9) holds for all $n \geq n_0$, we have that $\{\|u_{ntt}/n\|\}$ is bounded. Then it follows from (2.11) that

$$\sup_{n\geq n_0} \int_0^T \|A_nu_n\|^2\, dt < \infty. \tag{2.16}$$

We now show that $\{J_nu_n\}$ is relatively compact in $L^2(0, T; H)$. Let $\varepsilon > 0$. Then, by (2.14) and (2.15), there exists an integer $m_0 > 0$ such that

$$\|J_nu_n(t) - J_nu_n(s)\|^2 < \varepsilon/6T \quad \text{for all } n \geq n_0 \text{ and } |t - s| < 2T/m_0. \tag{2.17}$$

On the other hand, we have that there exists $D > 0$ such that

$$\inf\{\|A_nu_n(\tau)\|: t \leq \tau \leq t + T/m_0\} < D \quad \text{for all } n \geq n_0 \text{ and } 0 \leq t \leq T - T/m_0.$$

We now choose $\{t_{m,n}: n_0 \leq n,\ 1 \leq m \leq m_0\} \subset [0, T]$ such that

$$T(m-1)/m_0 \leq t_{m,n} \leq Tm/m_0 \quad \text{and} \quad \|A_nu_n(t_{m,n})\| \leq D$$

for $n \geq n_0$ and $1 \leq m \leq m_0$. Here we fix $n_1 \geq 1$. Then, since

$$J_n(u_n(t_{m,n})) = J_{n_1}(J_nu_n(t_{m,n}) + A_nu_n(t_{m,n})/n_1) \quad \text{for } n \geq 1,$$

we find by (H_1) that $\{J_n(u_n(t_{m,n}))\}$ is relatively compact. Then we may assume by extracting subsequences that $\{J_n(u_n(t_{m,n})) \colon n \geq 1\}$ is a convergent sequence for all $1 \leq m \leq m_0$. Then we have from (2.17) and the observation above that $\|J_n(u_n) - J_{n'}(u_{n'})\|_T^2 < \varepsilon$ for n, n' sufficiently large. Since $\varepsilon > 0$ is arbitrary, we have that $\{J_n(u_n)\}$ is relatively compact in $L^2(0, T; H)$.

Therefore, we may assume that there exists a T-periodic function $u \in W$ and

$$J_n u_n \to u \quad \text{strongly in } L^2(0, T; H). \tag{2.18}$$

Also we may assume that $A_n u_n \to z \in L^2(0, T; H)$ weakly in $L^2(0, T; H)$ by the boundedness of $\{\|A_n u_n\|_T\}$. On the other hand, we have

$$\int_0^T \|u_n - J_n u_n\|^2 \, dt = \frac{1}{n^2} \int_0^T \|A_n u_n\|^2 \, dt \to 0 \quad \text{as } n \to \infty. \tag{2.19}$$

Then we find by (2.18) and (2.19) that

$$\lim_{m,n\to\infty} \int_0^T \langle A_n u_n - A_m u_m, u_n - u_m \rangle \, dt = 0.$$

Since A is maximal monotone in $L^2(0, T; H)$, we have by [1, Chapter II, Proposition 1.1] that

$$A_n u_n \to z \in Au \quad \text{weakly in } L^2(0, T; H). \tag{2.20}$$

We also have that $\frac{1}{n} u_{ntt} \to 0$ weakly in $W^{1,2}(0, T; H)$, because $\frac{1}{n}\int_0^T |u_{nt}|^2 \, dt \to 0$ as $n \to \infty$. Therefore, by (2.18), (2.19), and (2.20), we have that u is a T-periodic solution of the problem (P). □

3. Example

Let $\Omega \subsetneq R^N$ be a bounded domain with a smooth boundary $\partial\Omega$ and $\psi \colon R \to R$ be a convex function. We put

$$Au = \sum_{i=1}^n D_i a_i(Du) \quad \text{for } u \in D(A) \tag{3.1}$$

where $a_i(p) = \partial\psi(p)/\partial p_i$ for $p \in R^N$ and $D(A) = \{u \in H_0^1(\Omega) \colon Au \in L^2(\Omega)\}$. We assume that a_i is continuously differentiable and satisfies

$$\sum_{i=1}^n a_i(p) p_i \geq \omega |p|^2 - C_0 \quad \text{and} \quad \sum_{i=1}^n |a_i(p)| \leq C_1 |p| + C_2$$

for all $p \in R^N$, where C_0, C_1, and C_2 are positive constants. Then A is a subdifferential of the functional $u \to \int_\Omega \psi(u) \, dx$ and satisfies the assumption (H_1) (cf. Vrabie [10]) with $H = L^2(\omega)$. Let $g \colon R \times R \to R$ be a Carathéodory mapping satisfying $g(t, x)x < \lambda_1(\omega - \varepsilon)|x|^2$ for some $\varepsilon > 0$, where λ_1 is the first eigenvalue of $-\Delta$ under the Dirichlet boundary condition. Then, since

$$\langle Au - g(t, u), u \rangle \geq \omega \|\nabla u\|^2 - C_0 - \lambda_1(\omega - \varepsilon)\|u\|^2 \geq \lambda_1 \varepsilon \|u\|^2 - C_0$$

for $u \in H_0^1(\omega)$, the condition $(*)$ is satisfied. We lastly see that the condition $(*)$ is strictly weaker than (c_1) and (c_2). Let $\psi(\tau) = \tau^2$ for $\tau \in R$ and

$g(t, x) = \lambda_1(\sin t - 2)\tau^3/(1 + \tau^2)$ for $(t, \tau) \in R \times R$. Let A be the operator defined by (3.1) (i.e., $A = -\Delta$). Then (H_1), (H_2), and $(*)$ are satisfied. The operator A satisfies (c_1) for $a < \lambda_1$. But one can see that condition (c_2) is not satisfied. Let $\psi(\tau) = \tau^4/(1 + \tau^2)$ for $\tau \in R$ and g be as above. In this case, (H_1), (H_2), and $(*)$ are satisfied, but A does not satisfy (c_1) for any $a > 0$.

ACKNOWLEDGMENT

I am grateful to the referee for a careful reading of the paper and giving valuable suggestions.

REFERENCES

1. V. Barbu, *Nonlinear semigroups and differential equations in Banach spaces*, Editura Academiei, 1976.
2. R. I. Becker, *Periodic solutions of semilinear equations of evolutions of compact type*, J. Math. Anal. Appl. **82** (1981), 33–48.
3. H. Brézis, *Equations et inequations nonlineaires dans des espace vectoriels en dualite*, Ann. Inst. Fourier (Grenoble) **18** (1968), 115–175.
4. ——, *Operateurs maximaux monotones*, North-Holland, Amsterdam, 1973.
5. F. E. Browder, *Existence of periodic solutions for nonlinear equations of evolution*, Proc. Nat. Acad. Sci. U.S.A. **53** (1965), 1100–1103.
6. J. H. Deimling, *Periodic solutions of differential equations in Banach spaces*, Manuscripta Math. **24** (1978), 31–44.
7. N. Hirano, *Existence of multiple periodic solutions for a semilinear evolution equations*, Proc. Amer. Math. Soc. **106** (1989), 107–114.
8. J. Prüss, *Periodic solutions of semilinear evolution equations*, Nonlinear Anal. **3** (1979), 221–235.
9. I. Vrabie, *Periodic solutions for nonlinear evolution equations in a Banach space*, Proc. Amer. Math. Soc. **109** (1990), 653–661.
10. ——, *Nonlinear version of Pazy's local existence theorem*, Israel J. Math. **32** (1979), 221–235.

DEPARTMENT OF MATHEMATICS, FACULTY OF ENGINEERING, YOKOHAMA NATIONAL UNIVERSITY, 156 TOKIWADAI, HODOGAYAKU, YOKOHAMA 240, JAPAN

E-mail address: `hirano@math.sci.ynu.ac.jp`