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DETERMINANT TYPE GENERALIZATIONS OF THE HEINZ-KATO THEOREM VIA THE FURUTA INEQUALITY

TAKAYUKI FURUTA

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Dedicated to Professor Osamu Takenouchi with respect and affection

ABSTRACT. A capital letter means a bounded linear operator on a complex Hilbert space H. By a nice application of the Furuta inequality, we give two kinds of determinant type generalizations (Theorems 1 and 2 in §1) of the famous and well-known Heinz-Kato theorem containing the terms T, $|T|$, and $|T^*|$.

0. INTRODUCTION

An operator T is said to be positive if $(Tx, x) \geq 0$ for all $x \in H$. We recall the following famous Löwner-Heinz theorem [5, 8]. If $A \geq B \geq 0$, then $A^\alpha \geq B^\alpha$ for each $\alpha \in [0, 1]$. There are a lot of proofs of this famous theorem, in particular, an elegant proof given in [9].

Also we recall the following famous Heinz-Kato theorem [5, 7]. If A and B are positive operators such that $\|Tx\| \leq \|Ax\|$ and $\|T^*y\| \leq \|By\|$ for all $x, y \in H$, then the following inequality holds: $|(Tx, y)| \leq \|A^\alpha x\| \, \|B^{1-\alpha}y\|$ for any $0 \leq \alpha \leq 1$.

We have the Furuta inequality [2] as some extension of this Löwner-Heinz theorem as follows: If $A \geq B \geq 0$, then for each $r \geq 0$

(i) $(B^r A^p B^r)^{1/q} \geq B^{(p+2r)/q}$ and

(ii) $A^{(p+2r)/q} \geq (A^r B^p A^r)^{1/q}$

hold for each p and q such that $p \geq 0$ and $q \geq 1$ with $(1+2r)q \geq p+2r$.

When we put $r = 0$ in (i) or (ii) in the Furuta inequality stated above, we have the famous Löwner-Heinz theorem. Alternative proofs of the Furuta inequality are given in [1, 3, 6], and an elementary proof is shown in [4].

In this paper, as an application of the Furuta inequality we shall show Theorems 1 and 2, which are two kinds of determinant type inequalities, and these two theorems yield Theorem 3, which is a generalization of the Heinz-Kato theorem. Also we shall show that any one of these generalizations is equivalent to the Furuta inequality.

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1. Determinant type generalizations of the Heinz-Kato theorem

Put $f=(1+2r)\alpha$ and $g=(1+2s)\beta$ for any $r\geq 0$, $s\geq 0$, and $\alpha, \beta\in[0,1]$. Let the $n\times n$ determinant $G_n^{(1)}(r,s,\alpha,\beta)$ be defined by the following formula for $h=(1+2r)\alpha+(1+2s)\beta-1\geq 0$ and $x_1, x_2, \dots, x_n\in H$:

$$G_n^{(1)}(r,s,\alpha,\beta) = \begin{vmatrix} (|T|^{2f}x_1, x_1) & (T|T|^h x_1, x_2) & (T|T|^h x_1, x_3) & \cdots & (T|T|^h x_1, x_n) \\ (|T|^h T^* x_2, x_1) & (|T^*|^{2g}x_2, x_2) & (|T^*|^{2g}x_2, x_3) & \cdots & (|T^*|^{2g}x_2, x_n) \\ (|T|^h T^* x_3, x_1) & (|T^*|^{2g}x_3, x_2) & (|T^*|^{2g}x_3, x_3) & \cdots & (|T^*|^{2g}x_3, x_n) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ (|T|^h T^* x_n, x_1) & (|T^*|^{2g}x_n, x_2) & (|T^*|^{2g}x_n, x_3) & \cdots & (|T^*|^{2g}x_n, x_n) \end{vmatrix}.$$

Theorem 1 (Type 1). *Let T be an operator on a Hilbert space H. If $A, B_2, B_3, \dots, B_n$ are positive operators such that $\|Tx\|\leq\|Ax\|$ and $\|T^*y\|\leq\|B_jy\|$ for all $x, y\in H$ and $j=2,3,\dots,n$, then, for each $r\geq 0$ and $s\geq 0$, the following inequality holds for all $x_1, x_2, \dots, x_n\in H$:*

$$\begin{aligned} &(|T|^{2(1+2r)\alpha}x_1, x_1)\prod_{j=2}^{n}(|T^*|^{2(1+2s)\beta}x_j, x_j) \\ &\leq G_n^{(1)}(r,s,\alpha,\beta)+((|T|^{2r}A^{2p}|T|^{2r})^{(1+2r)\alpha/(p+2r)}x_1, x_1) \\ &\quad\times\prod_{j=2}^{n}((|T^*|^{2s}B_j^{2q}|T^*|^{2s})^{(1+2s)\beta/(q+2s)}x_j, x_j) \end{aligned} \tag{1}$$

for any $p\geq 1$, $q\geq 1$, and $\alpha, \beta\in[0,1]$ such that $(1+2r)\alpha+(1+2s)\beta\geq 1$.

In the case $\alpha>0$ and $\beta>0$, the equality in (1) holds for some vectors $x_1, x_2, \dots, x_n\in H$ iff the following (a$_1$), (b$_1$), *and* (c$_1$) *hold together for some vectors $x_1, x_2, \dots, x_n\in H$:*

(a$_1$) $\{|T|^{2(1+2r)\alpha}x_1, |T|^{(1+2r)\alpha+(1+2s)\beta-1}T^*x_2, |T|^{(1+2r)\alpha+(1+2s)\beta-1}T^*x_3, \dots, |T|^{(1+2r)\alpha+(1+2s)\beta-1}T^*x_n\}$ *is a sequence of linearly dependent vectors,*

(b$_1$) $|T|^{2(1+2r)\alpha}x_1=(|T|^{2r}A^{2p}|T|^{2r})^{(1+2r)\alpha/(p+2r)}x_1$,

(c$_1$) $|T^*|^{2(1+2s)\beta}x_j=(|T^*|^{2s}B_J^{2q}|T^*|^{2S})^{(1+2s)\beta/(q+2s)}x_j$ *for* $j=2,3,\dots,n$.

We define f, g, and h the same as in the definition of $G_n^{(1)}(r,s,\alpha,\beta)$; that is, $f=(1+2r)\alpha$ and $g=(1+2s)\beta$ for any $r\geq 0$, $s\geq 0$, and $\alpha, \beta\in[0,1]$.

Let the $2n\times 2n$ determinant $G_{2n}^{(2)}(r,s,\alpha,\beta)$ be defined by the following formula for $h=(1+2r)\alpha+(1+2s)\beta-1\geq 0$ and $x_1, x_2, \dots, x_{2n-1}, x_{2n}\in H$:

$$G_{2n}^{(2)}(r,s,\alpha,\beta) = \begin{vmatrix} (|T|^{2f}x_1, x_1) & (T|T|^h x_1, x_2) & \cdots & (|T|^{2f}x_1, x_{2n-1}) & (T|T|^h x_1, x_{2n}) \\ (|T|^h T^* x_2, x_1) & (|T^*|^{2g}x_2, x_2) & \cdots & (|T|^h T^* x_2, x_{2n-1}) & (|T^*|^{2g}x_2, x_{2n}) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ (|T|^{2f}x_{2n-1}, x_1) & (T|T|^h x_{2n-1}, x_2) & \cdots & (|T|^{2f}x_{2n-1}, x_{2n-1}) & (T|T|^h x_{2n-1}, x_{2n}) \\ (|T|^h T^* x_{2n}, x_1) & (|T^*|^{2g}x_{2n}, x_2) & \cdots & (|T|^h T^* x_{2n}, x_{2n-1}) & (|T^*|^{2g}x_{2n}, x_{2n}) \end{vmatrix}.$$

Theorem 2 (Type 2). *Let T be an operator on a Hilbert space H. If $A_1, A_3, \dots, A_{2n-1}$ and $B_2, B_4, \dots, B_{2n}$ are positive operators such that $\|Tx\|\leq\|A_{2j-1}x\|$ and $\|T^*y\|\leq\|B_{2j}y\|$ for all $x, y\in H$ and $j=1,2,\dots,n$, then, for each $r\geq 0$ and $s\geq 0$, the following inequality holds for all $x_1, x_2, \dots, x_{2n-1}$,*

$x_{2n} \in H$:

$$\prod_{j=1}^{n}\{(|T|^{2(1+2r)\alpha}x_{2j-1}, x_{2j-1})(|T^*|^{2(1+2s)\beta}x_{2j}, x_{2j})\}$$

$$\text{(2)} \qquad \leq G_{2n}^{(2)}(r, s, \alpha, \beta) + \prod_{j=1}^{n}\{((|T|^{2r}A_{2j-1}^{2p}|T|^{2r})^{(1+2r)\alpha/(p+2r)}x_{2j-1}, x_{2j-1})$$

$$((|T^*|^{2s}B_j^{2q}|T^*|^{2s})^{(1+2s)\beta/(q+2s)}x_{2j}, x_{2j})\}$$

for any $p \geq 1$, $q \geq 1$, *and* $\alpha, \beta \in [0, 1]$ *such that* $(1+2r)\alpha + (1+2s)\beta \geq 1$.

In the case $\alpha > 0$ *and* $\beta > 0$, *the equality in* (2) *holds for some vectors* $x_1, x_2, \ldots, x_{2n-1}, x_{2n} \in H$ *iff the following* (a$_2$), (b$_2$), *and* (c$_2$) *hold together for some vectors* $x_1, x_2, \ldots, x_{2n-1}, x_{2n} \in H$:

(a$_2$)

$$\{|T|^{2(1+2r)\alpha}x_1, |T|^{(1+2r)\alpha+(1+2s)\beta-1}T^*x_2, |T|^{2(1+2r)\alpha}x_3,$$
$$|T|^{(1+2r)\alpha+(1+2s)\beta-1}T^*x_4, \ldots, |T|^{2(1+2r)\alpha}x_{2n-1}, |T|^{(1+2r)\alpha+(1+2s)\beta-1}T^*x_{2n}\}$$

is a sequence of linearly dependent vectors,

(b$_2$) $|T|^{2(1+2r)\alpha}x_{2j-1} = (|T|^{2r}A_{2j-1}^{2p}|T|^{2r})^{(1+2r)\alpha/(p+2r)}x_{2j-1}$ *for* $j = 1, 2, \ldots, n$,

(c$_2$) $|T^*|^{2(1+2s)\beta}x_{2j} = (|T^*|^{2s}B_{2j}^{2q}|T^*|^{2s})^{(1+2s)\beta/(q+2s)}x_{2j}$ *for* $j = 1, 2, \ldots, n$.

Theorem 3. *Let* T *be an operator on a Hilbert space* H. *If* A *and* B *are positive operators such that* $\|Tx\| \leq \|Ax\|$ *and* $\|T^*y\| \leq \|By\|$ *for all* $x, y \in H$, *then, for each* $r \geq 0$ *and* $s \geq 0$, *the following inequality holds for all* $x, y \in H$:

(3)

$$|(T|T|^{(1+2r)\alpha+(1+2s)\beta-1}x, y)|^2$$
$$\leq ((|T|^{2r}A^{2p}|T|^{2r})^{(1+2r)\alpha/(p+2r)}x, x)((|T^*|^{2s}B^{2q}|T^*|^{2s})^{(1+2s)\beta/(q+2s)}y, y)$$

for any $p \geq 1$, $q \geq 1$, *and* $\alpha, \beta \in [0, 1]$ *such that* $(1+2r)\alpha + (1+2s)\beta \geq 1$.

In the case $\alpha > 0$ *and* $\beta > 0$, *the equality in* (3) *holds for some vectors* x *and* $y \in H$ *iff the following* (a$_3$), (b$_3$), *and* (c$_3$) *hold together for some* x *and* $y \in H$:

(a$_3$) $|T|^{2(1+2r)\alpha}x$ *and* $|T|^{(1+2r)\alpha+(1+2s)\beta-1}T^*y$ *are linearly dependent,*

(b$_3$) $|T|^{2(1+2r)\alpha}x = (|T|^{2r}A^{2p}|T|^{2r})^{(1+2r)\alpha/(p+2r)}x$,

(c$_3$) $|T^*|^{2(1+2s)\beta}y = (|T^*|^{2s}B^{2q}|T^*|^{2s})^{(1+2s)\beta/(q+2s)}y$.

Remark 1. We remark that the condition $(1+2r)\alpha+(1+2s)\beta \geq 1$ in Theorems 1, 2, and 3 is unnecessary if T is a positive operator or invertible operator. This is easily seen in the proofs of their results.

Theorem 4. *Let* T *be an operator on a Hilbert space* H. *If* A *and* B *are positive operators such that* $\|Tx\| \leq \|Ax\|$ *and* $\|T^*y\| \leq \|By\|$ *for all* $x, y \in H$, *then the following inequality holds for all* $x, y \in H$:

$$\text{(4)} \qquad |(T|T|^{\alpha+\beta-1}x, y)| \leq \|A_x^{\alpha}\|\,\|B^{\beta}y\|$$

for any α *and* β *such that* $\alpha, \beta \in [0, 1]$ *and* $\alpha + \beta \geq 1$.

In the case $\alpha > 0$ *and* $\beta > 0$, *the equality in* (4) *holds for some* x *and* y *iff* $|T|^{2\alpha}x$ *and* $|T|^{\alpha+\beta-1}T^*y$ *are linearly dependent and* $|T|^{2\alpha}x = A^{2\alpha}x$ *and* $|T^*|^{2\beta}y = B^{2\beta}y$ *hold for some* x *and* y *together.*

Theorem A (Heinz-Kato). *Let* T *be an operator on a Hilbert space* H. *If* A *and* B *are positive operators such that* $\|Tx\| \le \|Ax\|$ *and* $\|T^*y\| \le \|By\|$ *for all* $x, y \in H$, *then the following inequality holds for all* $x, y \in H$:

$$|(Tx, y)| \le \|A^\alpha x\| \, \|B^{1-\alpha}y\| \quad \textit{for any } \alpha \in [0, 1].$$

2. Proofs of the results

At first we cite the following well-known folk lemma.

Lemma. *For any vectors* $x_1, x_2, \ldots, x_n \in H$, *let* G_n *be the determinant of the square matrix of order* n *defined by* $G_n = |((x_j, x_k))|$. *Then* $0 \le G_n$. *The equality holds if and only if* $x_1, x_2, \ldots, x_n$ *are linearly dependent.*

In order to give proofs of the results, we need the following inequalities which are equivalent to the Furuta inequality [2].

Theorem B. *If* $A \ge B \ge 0$, *then for each* $r \ge 0$

$$\text{(i)} \qquad (B^r A^p B^r)^{(1+2r)\theta/(p+2r)} \ge B^{(1+2r)\theta}$$

and

$$\text{(ii)} \qquad A^{(1+2r)\theta} \ge (A^r B^p A^r)^{(1+2r)\theta/(p+2r)}$$

hold for any $p \ge 1$ *and* $\theta \in [0, 1]$.

Proof of Theorem 1. Let $N(X)$ denote the kernel of an operator X. Let $T = U|T|$ be the polar decomposition of an operator T, where $|T| = (T^*T)^{1/2}$ and U is a partial isometry operator with $N(U) = N(|T|)$.

The case $g = (1+2s)\beta > 0$. In the Lemma we replace x_1 by $|T|^f x_1$ and x_k by $|T|^g U^* x_k$ for $k = 2, 3, \ldots, n$. Then we have

$$\begin{aligned}(5) \qquad (|T|^f x_1, |T|^g U^* x_k) &= (U|T|^{f+g} x_1, x_k) \\ &= (T|T|^{f+g-1} x_1, x_k) \quad \text{for } k = 2, 3, \ldots, n\end{aligned}$$

and

$$\begin{aligned}(6) \qquad (|T|^g U^* x_j, |T|^g U^* x_k) &= (U|T|^{2g} U^* x_j, x_k) \\ &= (|T^*|^{2g} x_j, x_k) \quad \text{for } j, k = 2, 3, \ldots, n,\end{aligned}$$

since $|T^*|^{2g} = U|T|^{2g}U^*$ holds for any positive number g in general.

Thus we can construct $G_n^{(1)}(r, s, \alpha, \beta)$ defined in §1. By the Lemma we have

$$(7) \qquad 0 \le G_n^{(1)}(r, s, \alpha, \beta).$$

The two hypotheses $\|Tx\| \le \|Ax\|$ and $\|T^*y\| \le \|B_j y\|$ for all $x, y \in H$ and $j = 2, 3, \ldots, n$ are equivalent to the following (8) and (9) respectively:

$$(8) \qquad |T|^2 \le A^2$$

and

$$(9) \qquad |T^*|^2 \le B_j^2 \quad \text{for } j = 2, 3, \ldots, n.$$

Applying Theorem B(i) to (8), for each $r \geq 0$ we have

$$(|T|^{2(1+2r)\alpha}x_1, x_1) \leq ((|T|^{2r}A^{2p}|T|^{2r})^{(1+2r)\alpha/(p+2r)}x_1, x_1) \tag{10}$$

for any $p \geq 1$ and $\alpha \in [0, 1]$.

Also applying Theorem B(i) to (9), for each $s \geq 0$ we have

$$(|T^*|^{2(1+2s)\beta}x_j, x_j) \leq ((|T^*|^{2s}B_j^{2q}|T^*|^{2s})^{(1+2s)\beta/(q+2s)}x_j, x_j) \tag{11}$$

for any $q \geq 1$, $\beta \in [0, 1]$, and $j = 2, \ldots, n$. Hence the following inequality (1) holds by (7), (10), and (11); that is, for each $r \geq 0$ and $s \geq 0$ and all $x_1, x_2, \ldots, x_n \in H$

$$\begin{aligned}&(|T|^{2(1+2r)\alpha}x_1, x_1)\prod_{j=2}^{n}(|T^*|^{2(1+2s)\beta}x_j, x_j)\\ &\leq G_n^{(1)}(r, s, \alpha, \beta) + ((|T|^{2r}A^{2p}|T|^{2r})^{(1+2r)\alpha/(p+2r)}x_1, x_1)\\ &\times\prod_{j=2}^{n}((|T^*|^{2s}B_j^{2q}|T^*|^{2s})^{(1+2s)\beta/(q+2s)}x_j, x_j)\end{aligned} \tag{1}$$

for any $p \geq 1$, $q \geq 1$, and $\alpha, \beta \in [0, 1]$ such that $(1+2r)\alpha + (1+2s)\beta \geq 1$.

The case $g = (1+2s)\beta = 0$. The result is obvious, so we omit its description.

We cite the following obvious results to scrutinize the equality case in (1):

(*) For a positive operator S, $(Sx, x) = 0$ holds for some vector x iff $Sx = 0$ holds for some vector x.

(**) For a positive operator S, $N(S^q) = N(S)$ holds for any positive number q.

In the case $f = (1+2r)\alpha > 0$ and $g = (1+2s)\beta > 0$, the equality in (7) holds iff

$$\{|T|^f x_1, |T|^g U^* x_2, |T|^g U^* x_3, \ldots, |T|^g U^* x_n\}$$

is a sequence of linearly dependent vectors. That is,

$$\{|T|^{2f}x_1, |T|^{f+g-1}|T|U^*x_2, |T|^{f+g-1}|T|U^*x_3, \ldots, |T|^{f+g-1}|T|U^*x_n\}$$

is a sequence of linearly dependent vectors by (**); namely,

$$\{|T|^{2f}x_1, |T|^{f+g-1}T^*x_2, |T|^{f+g-1}T^*x_3, \ldots, |T|^{f+g-1}T^*x_n\} \tag{12}$$

is a sequence of linearly dependent vectors. That is, (12) is (a_1) in Theorem 1.

The equality in (10) holds for some vector x_1 iff

$$(|T|^{2r}A^{2p}|T|^{2r})^{(1+2r)\alpha/(p+2r)}x_1 = |T|^{2(1+2r)\alpha}x_1 \tag{13}$$

holds for some vector x_1 by (*), and also the equality in (11) holds for some vectors x_j for $j = 1, 2, \ldots, n$, iff

$$(|T^*|^{2s}B_j^{2q}|T^*|^{2s})^{(1+2s)\beta/(q+2s)}x_j = |T^*|^{2(1+2s)\beta}x_j \tag{14}$$

holds for some vectors x_j for $j = 1, 2, \ldots, n$ by (*).

The case $\alpha > 0$ and $\beta > 0$ is equivalent to the case $(1+2r)\alpha > 0$ and $(1+2s)\beta > 0$. In this case, the equality in (1) holds iff the equalities in (7), (10), and (11) hold together; that is, the conditions (a_1), (b_1), and (c_1) in

Theorem 1 hold together by (12)–(14). Therefore, the proof of the equality in (1) is complete.

Hence, the proof of Theorem 1 is complete.

Proof of Theorem 2. *The case* $g = (1+2s)\beta > 0$. In the Lemma we replace x_{2j} by $|T|^g U^* x_{2j}$ for $j = 1, 2, \dots, n$ and also x_{2j-1} by $|T|^f x_{2j-1}$ for $j = 1, 2, \dots, n$. Then we have

$$(15)\qquad \begin{aligned}(|T|^f x_{2j-1}, |T|^g U^* x_{2k}) &= (U|T|^{f+g} x_{2j-1}, x_{2k}) \\ &= (T|T|^{f+g-1} x_{2j-1}, x_{2k}) \quad \text{for } j, k = 1, 2, \dots, n\end{aligned}$$

and

$$(16)\qquad \begin{aligned}(|T|^g U^* x_{2j}, |T|^g U^* x_{2k}) &= (U|T|^{2g} U^* x_{2j}, x_{2k}) \\ &= (|T^*|^{2g} x_{2j}, x_{2k}) \quad \text{for } j, k = 1, 2, \dots, n,\end{aligned}$$

since $|T^*|^{2g} = U|T|^{2g}U^*$ holds for any positive number g in general.

Thus we can construct $G_{2n}^{(2)}(r, s, \alpha, \beta)$ defined in §1. By the Lemma, we have

$$(17)\qquad 0 \le G_{2n}^{(2)}(r, s, \alpha, \beta).$$

The two hypotheses $\|Tx\| \le \|A_{2j-1}x\|$ and $\|T^*y\| \le \|B_{2j}y\|$ for all $x, y \in H$ and $j = 1, 2, \dots, n$ are equivalent respectively to

$$(18)\qquad |T|^2 \le A_{2j-1}^2 \quad \text{for } j = 1, 2, \dots, n$$

and

$$(19)\qquad |T^*|^2 \le B_{2j}^2 \quad \text{for } j = 1, 2, \dots, n.$$

Applying Theorem B(i) to (18), for each $r \ge 0$ we have

$$(20)\qquad (|T|^{2(1+2r)\alpha} x_{2j-1}, x_{2j-1}) \le ((|T|^{2r} A_{2j-1}^{2p} |T|^{2r})^{(1+2r)\alpha/(p+2r)} x_{2j-1}, x_{2j-1})$$

for any $p \ge 1$, $\alpha \in [0, 1]$, and $j = 1, 2, \dots, n$. Also applying Theorem B(i) to (19), for each $s \ge 0$ we have

$$(21)\qquad (|T^*|^{2(1+2s)\beta} x_{2j}, x_{2j}) \le ((|T^*|^{2s} B_j^{2q} |T^*|^{2s})^{(1+2s)\beta/(q+2s)} x_{2j}, x_{2j})$$

for any $q \ge 1$, $\beta \in [0, 1]$, and $j = 1, 2, \dots, n$. Hence the following inequality (2) holds by (17), (20), and (21); that is, for each $r \ge 0$, $s \ge 0$, and for all $x_1, x_2, \dots, x_{2n-1}, x_{2n} \in H$:

$$(2)\qquad \begin{aligned}&\prod_{j=1}^{n} \{(|T|^{2(1+2r)\alpha} x_{2j-1}, x_{2j-1})(|T^*|^{2(1+2s)\beta} x_{2j}, x_{2j})\} \\ &\le G_{2n}^{(2)}(r, s, \alpha, \beta) + \prod_{j=1}^{n} \{((|T|^{2r} A_{2j-1}^{2p} |T|^{2r})^{(1+2r)\alpha/(p+2r)} x_{2j-1}, x_{2j-1}) \\ &\qquad ((|T^*|^{2s} B_j^{2q} |T^*|^{2s})^{(1+2s)\beta/(q+2s)} x_{2j}, x_{2j})\}\end{aligned}$$

for any $p \ge 1$, $q \ge 1$, and $\alpha, \beta \in [0, 1]$ such that $(1+2r)\alpha + (1+2s)\beta \ge 1$.

The case $g = (1+2s)\beta = 0$. The result is obvious, so we omit its description.

Next we shall scrutinize the equality case in (2). In the case $f = (1+2r)\alpha > 0$ and $g = (1+2s)\beta > 0$, the equality in (17) holds iff

$$\{|T|^f x_1, |T|^g U^* x_2, |T|^f x_3, |T|^g U^* x_4, \dots, |T|^f x_{2n-1}, |T|^g U^* x_n\}$$

is a sequence of linearly dependent vectors by the Lemma. That is,

$$\{|T|^{2f}x_1, |T|^{f+g-1}|T|U^*x_2, |T|^{2f}x_3, |T|^{f+g-1}|T|U^*x_4, \dots, |T|^{2f}x_{2n-1}, |T|^{f+g-1}|T|U^*x_{2n}\}$$

is a sequence of linearly dependent vectors by $(**)$; namely,

$$\{|T|^{2f}x_1, |T|^{f+g-1}T^*x_2, |T|^{2f}x_3, |T|^{f+g-1}T^*x_4, \dots, |T|^{2f}x_{2n-1}, |T|^{f+g-1}T^*x_{2n}\} \tag{22}$$

is a sequence of linearly dependent vectors. That is, (22) is (a_2) in Theorem 2.

The equality in (20) holds for some vectors x_{2j-1} iff

$$(|T|^{2r}A^{2p}|T|^{2r})^{(1+2r)\alpha/(p+2r)}x_{2j-1} = |T|^{2(1+2r)\alpha}x_{2j-1} \tag{23}$$

holds for some vectors x_{2j-1} for $j = 1, 2, \dots, n$ by $(*)$, and also the equality in (21) holds for some vectors x_{2j} iff

$$(|T^*|^{2s}B_j^{2q}|T^*|^{2s})^{(1+2s)\beta/(q+2s)}x_{2j} = |T^*|^{2(1+2s)\beta}x_{2j} \tag{24}$$

holds for some vectors x_{2j} for $j = 1, 2, \dots, n$ by $(*)$.

The case $\alpha > 0$ and $\beta > 0$ is equivalent to the case $(1+2r)\alpha > 0$ and $(1+2s)\beta > 0$. In this case, the equality in (2) holds iff the equalities in (17), (20), and (21) hold together; that is, the conditions (a_2), (b_2), and (c_2) in Theorem 2 hold together by (22)–(24). Hence the proof of Theorem 2 is complete, together with the proof of the equality in (2).

Remark 2. *Equivalent conditions.* In the special case $\alpha = 0$, scrutiny of the equality in (1) is obvious. In the case $\beta = 0$, scrutiny of the equality in (1) is also obvious. In the case $\alpha > 0$ and $\beta > 0$ in Theorem 1, we shall show that the following two conditions (a_1) in Theorem 1 and (a_1') are equivalent:

(a_1) $\{|T|^{2(1+2r)\alpha}x_1, |T|^{(1+2r)\alpha+(1+2s)\beta-1}T^*x_2, |T|^{(1+2r)\alpha+(1+2s)\beta-1}T^*x_3, \dots, |T|^{(1+2r)\alpha+(1+2s)\beta-1}T^*x_n\}$ is a sequence of linearly dependent vectors.

(a_1') $\{T|T|^{(1+2r)\alpha+(1+2s)\beta-1}x_1, |T^*|^{2(1+2s)\beta}x_2, |T^*|^{2(1+2s)\beta}x_3, \dots, |T^*|^{2(1+2s)\beta}x_n\}$ is a sequence of linearly dependent vectors.

In fact the former condition (a_1) is equivalent to

$$\{|T|^f x_1, |T|^g U^*x_2, |T|^g U^*x_3, \dots, |T|^g U^*x_n\}$$

is a sequence of linearly dependent vectors, as is easily seen in the proof of the equality in (7), and this condition is equivalent to

$$\{U|T|^{f+g}x_1, U|T|^{2g}U^*x_2, U|T|^{2g}U^*x_3, \dots, U|T|^{2g}U^*x_n\}$$

is a sequence of linearly dependent vectors by $(**)$ and $N(|T|) = N(U)$. That is,

$$\{T|T|^{f+g-1}x_1, |T^*|^{2g}x_2, |T^*|^{2g}x_3, \dots, |T^*|^{2g}x_n\}$$

is a sequence of linearly dependent vectors; that is, (a_1') holds.

In the case $\alpha > 0$ and $\beta > 0$, we shall show that the following (a_2) and (a_2') are equivalent:

(a_2) $\{|T|^{2(1+2r)\alpha}x_1, |T|^{(1+2r)\alpha+(1+2s)\beta-1}T^*x_2, |T|^{2(1+2r)\alpha}x_3, |T|^{2(1+2r)\alpha+(1+2s)\beta-1}T^*x_4, \dots, |T|^{2(1+2r)\alpha}x_{2j-1}, |T|^{(1+2r)\alpha+(1+2s)\beta-1}T^*x_{2n}\}$ is a sequence of linearly dependent vectors.

(a_2') $\{T|T|^{(1+2r)\alpha+(1+2s)\beta-1}x_1, |T^*|^{2(1+2s)\beta}x_2, T|T|^{(1+2r)\alpha+(1+2s)\beta-1}x_3, |T^*|^{2(1+2s)\beta}x_4, \dots, T|T|^{(1+2r)\alpha+(1+2s)\beta-1}x_{2n-1}, |T^*|^{2(1+2s)\beta}x_{2n}\}$ is a sequence of linearly dependent vectors.

In fact the former condition (a_2) is equivalent to

$$\{|T|^f x_1, |T|^g U^* x_2, |T|^f x_3, |T|^g U^* x_4, \dots, |T|^f x_{2n-1}, |T|^g U^* x_{2n}\}$$

is a sequence of linear dependent vectors, as is easily seen in the proof of the equality in (17), and this condition is equivalent to

$$\{U|T|^{f+g}x_1, U|T|^{2g}U^*x_2, U|T|^{f+g}x_3, U|T|^{2g}U^*x_4, \dots, U|T|^{f+g}x_{2n-1}, U|T|^{2g}U^*x_{2n}\}$$

is a sequence of linearly dependent vectors by (**) and $N(|T|) = N(U)$. That is,

$$\{T|T|^{f+g-1}x_1, |T^*|^{2g}x_2, T|T|^{f+g-1}x_3, |T^*|^{2g}x_4, \dots, T|T|^{f+g-1}x_{2n-1}, |T^*|^{2g}x_{2n}\}$$

is a sequence of linearly dependent vectors; that is, (a_2') holds.

Proof of Theorem 3. In Theorem 1, we put $n = 2$. Then for each $r \geq 0$ and $s \geq 0$ the following inequality holds for all $x, y \in H$ by Theorem 1:

$$\begin{aligned}(|T|^{(1+2r)\alpha}x, x)(|T^*|^{2(1+2s)\beta}y, y) &\leq G_2^{(1)}(r, s, \alpha, \beta) + ((|T|^{2r}A^{2p}|T|^{2r})^{(1+2r)\alpha/(p+2r)}x, x)\\ &\quad\times((|T^*|^{2s}B^{2q}|T^*|^{2s})^{(1+2s)\beta/(q+2s)}y, y).\end{aligned}$$

That is,

$$\begin{aligned}&(|T|^{(1+2r)\alpha}x, x)(|T^*|^{2(1+2s)\beta}y, y)\\ &\quad\leq (|T|^{(1+2r)\alpha}x, x)(|T^*|^{2(1+2s)\beta}y, y) - |(T|T|^{2(1+2r)\alpha+2(1+2s)\beta-1}x, y)|^2\\ &\quad\quad+ ((|T|^{2r}A^{2p}|T|^{2r})^{(1+2r)\alpha/(p+2r)}x, x)((|T^*|^{2s}B^{2q}|T^*|^{2s})^{(1+2s)\beta/(q+2s)}y, y);\end{aligned}$$

namely,

$$\begin{aligned}|(T|T|^{(1+2r)\alpha+(1+2s)\beta-1}x, y)|^2 &\leq ((|T|^{2r}A^{2p}|T|^{2r})^{(1+2r)\alpha/(p+2r)}x, x)\\ &\quad\times((|T^*|^{2s}B^{2q}|T^*|^{2s})^{(1+2s)\beta/(q+2s)}y, y)\end{aligned} \tag{3}$$

for any $p \geq 1$, $q \geq 1$, and $\alpha, \beta \in [0, 1]$ such that $(1+2r)\alpha + (1+2s)\beta \geq 1$. Hence the proof of Theorem 3 is complete.

Also, Theorem 3 easily follows by Theorem 2, putting $n = 1$.

Proof of Theorem 4. Theorem 4 easily follows by Theorem 3, putting $r = s = 0$.

Proof of Theorem A. Theorem A easily follows by Theorem 4, putting $\alpha+\beta = 1$.

3. The equivalence relation between Theorems 1, 2, and B

Theorems 1 and 2 are proved by Theorem B, which is an extension of the Löwner-Heinz theorem. In this section, conversely we shall show that Theorem B can be derived from Theorems 1 or 2 as follows:

Theorem 1 $\Rightarrow$ Theorem B. In (1) of Theorem 1, we put $n = 2$, $T = B$, $\alpha = \beta$, $r = s$, and also we put $x_1 = x_2 = x$. Then the hypothesis $\|Tx\| \leq \|Ax\|$ is equivalent to $B^2 \leq A^2$. Theorem 1 and Remark 1 ensure the following inequality for each $r \geq 0$ and $\alpha \in [0, 1]$:

$$|(B^{2(1+2r)\alpha}x, x)|^2 \leq G_2^{(1)}(r, r, \alpha, \alpha) + ((B^{2r}A^{2p}B^{2r})^{(1+2r)\alpha/(p+2r)}x, x)(B^{2(1+2r)\alpha}x, x)$$

for any $p \geq 1$. In this case

$$G_2^{(1)}(r, r, \alpha, \alpha) = |(B^{2(1+2r)\alpha}x, x)|^2 - |(B^{2(1+2r)\alpha}x, x)|^2 = 0,$$

so we have for each $r \geq 0$ and $\alpha \in [0, 1]$

$$(25) \qquad |(B^{2(1+2r)\alpha}x, x)| \leq ((B^{2r}A^{2p}B^{2r})^{(1+2r)\alpha/(p+2r)}x, x) \quad \text{for any } p \geq 1.$$

Then by (25) we have for each $r \geq 0$

$$(26) \qquad B^{2(1+2r)\alpha} \leq (B^{2r}A^{2p}B^{2r})^{(1+2r)\alpha/(p+2r)} \quad \text{for any } p \geq 1$$

holds under the hypothesis $B^2 \leq A^2$, and the inequality (26) is equivalent to Theorem B(i), which is also equivalent to Theorem B(ii).

Theorem 2 $\Rightarrow$ Theorem B. Also putting $n = 1$ in Theorem 2, we can show that Theorem B can be derived from Theorem 2 by the same way as in the proof that Theorem 1 $\Rightarrow$ Theorem B.

Hence Theorems 1, 2, and B are mutually equivalent.

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Department of Applied Mathematics, Faculty of Science, Science University of Tokyo, 1-3 Kagurazaka, Shinjuku-ku, Tokyo, 162 Japan