A REMARK ON THE DUNFORD-PETTIS PROPERTY IN $L_1(\mu, X)$

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Abstract. We prove that if $X$ is an $L_\infty$ space, then $L_1(\mu, X)$ has the Dunford-Pettis Property.

In this note we shall devote our attention to the research of those Banach spaces having the Dunford-Pettis property [3] such that $L_1(\mu, X)$ [3] verifies the same property. According to our knowledge, the results that are known so far are those due to Andrews [1] and Bourgain [2]. In particular, in Bourgain's paper it has been proved that $L_1(\mu, C(K))$ has the Dunford-Pettis property. Using this result, in the present note we show that if $X$ is an $L_\infty$ space [2], then $L_1(\mu, X)$ has the Dunford-Pettis property too. First of all we prove the following

Lemma. The space $L_1(\mu) \otimes \pi X^{**} = L_1(\mu, X^{**})$ is a closed subspace of the space $(L_1(\mu) \otimes \pi X)^{**} = L_1(\mu, X)^{**}$.

Proof. It is easy to see that $L_1(\mu) \otimes \pi X^{**}$ is a subset of $(L_1(\mu) \otimes \pi X)^{**}$ and that

$$\|T'\|_{(L_1(\mu) \otimes \pi X)^{**}} \leq \|T\|_{L_1(\mu) \otimes \pi X^{**}}.$$ 

So it is enough to assure the converse inequality. Let $\sum_{i=1}^n f_i \otimes x_i^{**}, f_i \in L_1(\mu), x_i^{**} \in X^{**}, i = 1, \ldots, n$, be one of the representations of $T$ in the space $L_1(\mu) \otimes X^{**}$ that is dense in $L_1(\mu) \otimes X^{**}$. Since $L_1(\mu)$ is metrically accessible, given $\varepsilon > 0$, we can find a finite rank bounded linear operator $v$ from $L_1(\mu)$ to $L_1(\mu)$ such that $\|v\| \leq 1$ and $\|v(f_i) - f_i\| \leq \varepsilon$ for all $i = 1, \ldots, n$. Now we put $E = v(L_1(\mu))$ and $T_i = \sum_{i=1}^n v(f_i) \otimes x_i^{**}$. Of course, $T_i \in E \otimes \pi X^{**} = E^{**} \otimes \pi X^{**}$. Because $E^*$ is finite dimensional and the projective norm is accessible [4], we have

$$E^{**} \otimes \pi X^{**} = B^f(E^*, X^*) = (E^* \otimes X^*)^*,$$

where $\vee$ is the injective norm. On the other hand, we have [4]

$$(E \otimes \pi X)^* = ((E \otimes \pi X)^*)^* = B(E, X)^* = (E^* \vee X^*)^*.$$
Then
\[ \|T_\varepsilon\|_{E^\otimes_X} = \|T_\varepsilon\|_{(E^\otimes_X)^{\ast\ast}}. \]

Since it is easy to prove that
\[ \|T\|_{L_1(\mu)\otimes_X} \leq \|T_\varepsilon\|_{E^\otimes_X}, \quad \|T_\varepsilon\|_{(E^\otimes_X)^{\ast\ast}} \leq \|T'\|_{(L_1(\mu)\otimes_X)^{\ast\ast}}, \]
our lemma is proved.

Now we can prove

**Theorem 1.** Let \( X \) be an \( L_\infty \) space. Then \( L_1(\mu, X) \) has the Dunford-Pettis property.

**Proof.** Let \( T: L_1(\mu, X) \to Z \) be a weakly compact operator and \( T^{\ast\ast}: (L_1(\mu, X))^{\ast\ast} \to Z \) be its second adjoint. Thanks to the previous lemma we can consider the restriction \( \overline{T} \) of \( T^{\ast\ast} \) to \( L_1(\mu)\otimes_X X^{\ast\ast} \). It is also weakly compact. Since \( X \) is an \( L_\infty \) space, \( X^{\ast\ast} \) is complemented into some \( C(K) \) [2]. So, there exists a projection \( P': L_1(\mu, C(K)) \to L_1(\mu, X^{\ast\ast}) \). Let \( T': L_1(\mu)\otimes_K C(K) \to Z \) be defined by \( T' = \overline{T} \circ P' \). Obviously \( T' \) is weakly compact, so it is a Dunford-Pettis operator [2]. The restriction of \( T' \) to \( L_1(\mu)\otimes_X X \) is just \( T \), so \( T \) is Dunford-Pettis.

**Remark.** With similar techniques it is possible to prove

**Theorem 2.** If \( X \) is an \( L_1 \) space, then \( L_1(\mu, X) \) has the Dunford-Pettis property.

**References**