PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 120, Number 1, January 1994

A REMARK ON THE DUNFORD-PETTIS PROPERTY IN $L_1(\mu, X)$

RAFFAELLA CILIA

(Communicated by Theodore W. Gamelin)

ABSTRACT. We prove that if X is an L_∞ space, then $L_1(\mu, X)$ has the Dunford-Pettis Property.

In this note we shall devote our attention to the research of those Banach spaces having the Dunford-Pettis property [3] such that $L_1(\mu, X)$ [3] verifies the same property. According to our knowledge, the results that are known so far are those due to Andrews [1] and Bourgain [2]. In particular, in Bourgain's paper it has been proved that $L_1(\mu, C(K))$ has the Dunford-Pettis property. Using this result, in the present note we show that if X is an L_∞ space [2], then $L_1(\mu, X)$ has the Dunford-Pettis property too. First of all we prove the following

Lemma. *The space* $L_1(\mu)\otimes_\pi X^{**} = L_1(\mu, X^{**})$ *is a closed subspace of the space* $(L_1(\mu)\otimes_\pi X)^{**} = L_1(\mu, X)^{**}$.

Proof. It is easy to see that $L_1(\mu)\otimes_\pi X^{**}$ is a subset of $(L_1(\mu)\otimes_\pi X)^{**}$ and that

$$\|T'\|_{(L_1(\mu)\otimes_\pi X)^{**}} \leq \|T\|_{L_1(\mu)\otimes_\pi X^{**}}.$$

So it is enough to assure the converse inequality. Let $\sum_{i=1}^n f_i \otimes x_i^{**}$, $f_i \in L_1(\mu)$, $x_i^{**} \in X^{**}$, $i = 1, \dots, n$, be one of the representations of T in the space $L_1(\mu)\otimes X^{**}$ that is dense in $L_1(\mu)\otimes_\pi X^{**}$. Since $L_1(\mu)$ is metrically accessible, given $\varepsilon > 0$, we can find a finite rank bounded linear operator v from $L_1(\mu)$ to $L_1(\mu)$ such that $\|v\| \leq 1$ and $\|v(f_i) - f_i\| \leq \varepsilon$ for all $i = 1, \dots, n$. Now we put $E = v(L_1(\mu))$ and $T_\varepsilon = \sum_{i=1}^n v(f_i)\otimes x_i^{**}$. Of course, $T_\varepsilon \in E\otimes_\pi X^{**} = E^{**}\otimes_\pi X^{**}$. Because E^* is finite dimensional and the projective norm is accessible [4], we have

$$E^{**}\otimes_\pi X^{**} = B^\pi(E^*, X^*) = (E^* \overset{\vee}{\otimes} X^*)^*,$$

where $\vee$ is the injective norm. On the other hand, we have [4]

$$(E\otimes_\pi X)^{**} = ((E\otimes_\pi X)^*)^* = B(E, X)^* = (E^* \overset{\vee}{\otimes} X^*)^*.$$

Received by the editors October 7, 1991 and, in revised form, April 28, 1992.
1991 *Mathematics Subject Classification*. Primary 46E40, 46M05.
Key words and phrases. Dunford-Pettis property, Bochner integrable functions, L_∞ space.
This work was partially supported by M.U.R.S.T. of Italy (40% 1990).

Then

$$\|T_\varepsilon\|_{E\otimes_\pi X^{**}} = \|T_\varepsilon\|_{(E\otimes_\pi X)^{**}} .$$

Since it is easy to prove that

$$\|T\|_{L_1(\mu)\otimes_\pi X^{**}} \leq \|T_\varepsilon\|_{E\otimes_\pi X^{**}} , \qquad \|T_\varepsilon\|_{(E\otimes_\pi X)^{**}} \leq \|T'\|_{(L_1(\mu)\otimes_\pi X)^{**}} ,$$

our lemma is proved.

Now we can prove

Theorem 1. *Let X be an L_∞ space. Then $L_1(\mu, X)$ has the Dunford-Pettis property.*

Proof. Let $T: L_1(\mu, X) \to Z$ be a weakly compact operator and $T^{**}: (L_1(\mu, X))^{**} \to Z$ be its second adjoint. Thanks to the previous lemma we can consider the restriction $\overline{T}$ of T^{**} to $L_1(\mu)\otimes_\pi X^{**}$. It is also weakly compact. Since X is an L_∞ space, X^{**} is complemented into some $C(K)$ [2]. So, there exists a projection $P': L_1(\mu, C(K)) \to L_1(\mu, X^{**})$. Let $T': L_1(\mu)\otimes_\pi C(K) \to Z$ be defined by $T' = \overline{T}\circ P'$. Obviously T' is weakly compact, so it is a Dunford-Pettis operator [2]. The restriction of T' to $L_1(\mu)\otimes_\pi X$ is just T, so T is Dunford-Pettis.

Remark. With similar techniques it is possible to prove

Theorem 2. *If X is an L_1 space, then $L_1(\mu, X)$ has the Dunford-Pettis property.*

References

1. K. T. Andrews, *Dunford-Pettis sets in the space of Bochner integrable functions*, Math. Ann. **241** (1979), 35–41.
2. J. Bourgain, *New classes of L^p-spaces*, Springer-Verlag, New York, 1981.
3. J. Diestel and J. J. Uhl, Jr., *Vector measures*, Math. Surveys Monographs, vol. 15, Amer. Math. Soc., Providence, RI, 1977.
4. J. Diestel, *Notes on the resume of Grothendieck*, preprint, 1988.

Department of Mathematics, University of Catania, 95125 Catania, Italy