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ON NUMERICALLY GORENSTEIN QUASI-SIMPLE ELLIPTIC SINGULARITIES WITH $\mathbb{C}^*$-ACTION

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ABSTRACT. Let (X, x) be a numerically Gorenstein elliptic singularity with $\mathbb{C}^*$-action and $\pi: (\widetilde{X}, A) \to (X, x)$ the minimal good resolution. Assume that the central curve of π is an elliptic curve. We will determine the configuration of the w.d. graph (weighted dual graph) of A and obtain a condition for (X, x) to be a maximally elliptic singularity in the sense of Yau.

1. INTRODUCTION

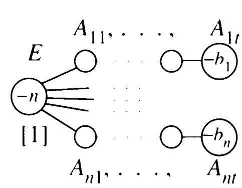
Let (X, x) be a normal surface singularity and $\pi: (\widetilde{X}, A) \to (X, x)$ a resolution, where $A = \pi^{-1}(x)$. For definitions and details of the following, please refer to [1] or [8]: Z (resp. K) the fundamental (resp. canonical) cycle on A, $p_g(X, x)$ (resp. $p_a(X, x)$) the geometric (resp. arithmetic) genus. If $p_a(X, x) = 1$, we call (X, x) an elliptic singularity. If K is an integral coefficient divisor, we say that (X, x) is numerically Gorenstein.

For any elliptic singularity, Yau [8] introduced the notion of the elliptic sequence. Also, for numerically Gorenstein elliptic singularities, he gave the definition of maximally elliptic singularities. Further, in [7], he found the condition for (X, x) to be maximally elliptic and gave the following definition.

Definition 1.1 [7, Definition 3.1]. Let π be the minimal good resolution. If the minimally elliptic cycle E is a nonsingular elliptic curve, we say that (X, x) is a quasi-simple elliptic singularity.

Theorem 1.2. *Let (X, x) be a quasi-simple elliptic singularity with star-shaped dual graph and $\pi: (\widetilde{X}, A) \to (X, x)$ the minimal good resolution. Assume that (X, x) is not a simple elliptic singularity.*

(i) *(X, x) is numerically Gorenstein if and only if the w.d. graph of A has the form*



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where $b_1, \ldots, b_n \geq 2$, $-E^2 =$ *the number of branches* $(= n)$, *the lengths of all branches are equal* $(= t \geq 1)$, *and* $\bigcirc = \textcircled{-2}$.

(ii) *Let* (X, x) *be a numerically Gorenstein singularity with* $\mathbb{C}^*$*-action. Then we have*:

(ii-a) (X, x) *is Gorenstein if and only if* $tR \sim t\sum_{i=1}^n P_i$;

(ii-b) (X, x) *is a maximally elliptic singularity if and only if* $R \sim \sum_{i=1}^n P_i$, *where* $\mathscr{O}(R)$ *is the conormal sheaf of* E *in* $\widetilde{X}$, $P_i = E \cap A_{i,1}$ $(i = 1, \ldots, n)$, *and "*$\sim$*" means two divisors are linearly equivalent on* E.

(iii) *Let* (X, x) *be a maximally elliptic singularity with* $\mathbb{C}^*$*-action whose w.d. graph is given as in* (i). *Then the* $\operatorname{emb\,dim}(X, x)$ $(=$ *embedding dimension of* $(X, x)) = \max(3, -Z^2)$, *where* $-Z^2 = \sum_{i=1}^n (b_i - 1)$.

Yau [7, Theorem 2.4] had obtained a fine condition for numerically Gorenstein elliptic singularities to be maximally elliptic. We can prove (ii-b) directly from his result. However, we give a different proof for it by using results about surface singularities with $\mathbb{C}^*$-action. For (iii), we also give a minimal generator system of the affine graded ring of (X, x) in terms of the elliptic functions on E. On the other hand, for every numerically Gorenstein elliptic singularity with $\mathbb{C}^*$-action whose central curve is $\mathbb{P}^1$, Tomari [4] had already proved that it is always a maximally elliptic singularity.

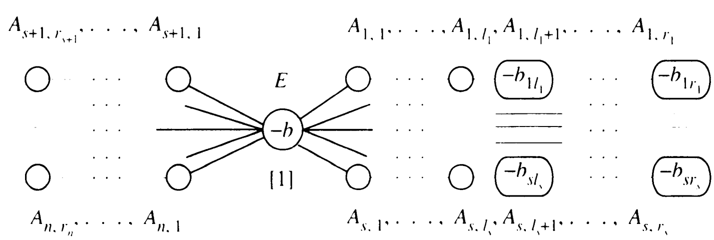
Before we prove Theorem 1.2, we prepare a fact. Let A be the exceptional set of a resolution of a normal surface singularity. For a cycle D on A, $\chi(D)$ is given by $-\frac{1}{2}(D^2 + K_{\widetilde{X}}D)$ (see [8, p. 272]), where $K_{\widetilde{X}}$ is the canonical divisor on $\widetilde{X}$. Let Z be the fundamental cycle on A. Assume $0 \leq D < Z$. Then, in the same way as in Laufer's computation sequence, we can choose a sequence of cycles $Z_0 = D$, $Z_1 = Z_0 + A_{i_1}, \ldots, Z_l = Z$ such that $Z_i A_{i+1} > 0$ for any i. Then, by the adjunction formula on algebraic curves, we have $\chi(Z_{i+1}) = \chi(Z_i + A_{i+1}) = \chi(Z_i) + \chi(A_i) - Z_i A_{i+1} = \chi(Z_i) + 1 - g_{i+1} - Z_i A_{i+1} \leq \chi(Z_i)$ for any i, where g_{i+1} is the genus of A_{i+1}. Hence we have

$$\chi(D) \geq \chi(Z_1) \geq \cdots \geq \chi(Z_i) \geq \cdots \geq \chi(Z). \tag{1.1}$$

2. Proof of Theorem 1.2

In this section, for a real number a, we put $[a] := \max\{n \in \mathbb{Z} | n \leq a\}$ and $\{a\} := \min\{n \in \mathbb{Z} | n \geq a\}$.

(i) Suppose that (X, x) is numerically Gorenstein. Let s be the number of the branches that contain an irreducible component whose intersection number is greater than two. Then the w.d. graph of A is given as



where $b_{1,l_1}, \ldots, b_{l,l_s} \geq 3$, $r_i > l_i \geq 0$ for $i = 1, \ldots, n$, and $l_i + 1 = r_i$ for $i = s+1, \ldots, n$. Let $D_1 = \sum_{i=1}^n \sum_{j=1}^{r_i} A_{i,j}$. Then $(E + D_1)A_{i,j} \leq 0$ for any i, j. By the adjunction formula we have $\chi(E) = 0$. Since there is a sequence from E to Z passing through $E + D_1$ as in (1.1) and (X, x) is an elliptic singularity, we have $0 = \chi(E) \geq \chi(E + D_1) \geq \chi(Z) = 0$ from (1.1). Then $\chi(E + D_1) = 0$, so $0 = \chi(E + E + D_1) = \chi(E) + \chi(E + D_1) - E(E + D_1) = -E(E + D_1)$. Then $Z = E + D_1$ is the fundamental cycle on A. Further $ZE = 0$ and $b = -E^2 = ED_1 = n$.

Now let $\{Z, Z_{B_1}, \dots, Z_{B_t} = E\}$ be the elliptic sequence for (X, x). From the proof of Theorem 3.7 in [8], we can see that (X_{B_i}, x) (= the contraction of B_i in $\widetilde{X}$) is a numerically Gorenstein elliptic singularity. If $l_1 = \cdots = l_n$ does not hold, the number of the branches of $(X_{B_{t-1}}, x)$ is less than $b = -E^2$. This contradicts the above fact, so $l_1 = \cdots = l_n$ $(:= l)$ and $t = l + 1$. It is easy to see that $Z_{B_1} = E + \sum_{i=1}^n \sum_{j=1}^l A_{i,j}$. Then, by Theorem 3.7 in [8] we have

$$-K = (t+1)E + \sum_{i=1}^{n} \left\{ \sum_{j=1}^{l} (l - j + 2) A_{i,j} + \sum_{j=l+1}^{r_i} A_{i,j} \right\}.$$

If $r_i \geq l + 2$ for some i, then $-KA_{i,t+1} = A_{i,t+1}^2 + 3$ by the above formula. On the other hand, $-KA_{i,t+1} = A_{i,t+1}^2 + 2$ by the adjunction formula. This is a contradiction, so $r_1 = \cdots = r_n = l + 1$.

Conversely, if the w.d. graph is given as in (i), then we can easily check that $-K$ is given as

$$-K = (t+1)E + \sum_{i=1}^{n} \sum_{j=1}^{t} (t - j + 1) A_{i,j}.$$

(ii-a) Since (X, x) has a $\mathbb{C}^*$-action, the w.d. graph of (X, x) is a star-shaped graph as in (i). From

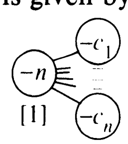
$$\underbrace{2 - \cfrac{1}{\ddots - \cfrac{1}{2 - \cfrac{1}{b_i}}}}_{t-1} = \frac{tb_i - t + 1}{(t-1)b_i - t + 2}$$

and Corollary 2.9 in [6], (X, x) is Gorenstein if and only if

$$\sum_{i=1}^{n} \frac{tb_i - t}{tb_i - t + 1} P_i \sim a \left(R - \sum_{i=1}^{n} \frac{(t-1)b_i - t + 2}{tb_i - t + 1} P_i \right)$$

for an integer a. Then $a = t$, so (X, x) is Gorenstein if and only if $tR \sim t\sum_{i=1}^n P_i$.

(ii-b) Suppose that (X, x) is a maximally elliptic singularity. Let $\{Z, Z_{B_1}, \dots, Z_{B_t} = E\}$ be the elliptic sequence of (X, x). From the definition of maximally elliptic singularities, (X_{B_i}, x) is a maximally elliptic singularity for any i and is Gorenstein from Theorem 3.11 in [8]. Let us consider $(X_{B_{t-1}}, x)$. Then the w.d. graph is given by



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where $c_i = 2$ (resp. $c_i = b_i$) if $t > 1$ (resp. $t = 1$) for any i. From (ii-a), $R \sim \sum_{i=1}^n P_i$.

Conversely we assume that $R \sim \sum_{i=1}^{n} P_i$. From Theorem 5.7 in [2], $p_g(X, x)$ is given by $\sum_{k=0}^{\infty} \dim_{\mathbb{C}} H^1(E, \mathscr{O}(D^{(k)}))$, where

$$D^{(k)} = k\sum_{i=1}^{n} P_i - \sum_{i=1}^{n} \left\{ \frac{k((t-1)b_i - t + 2)}{tb_i - t + 1} \right\} P_i .$$

Therefore, by using the Riemann-Roch Theorem for algebraic curves we can easily check that $p_g(X, x) = t + 1 =$ the length of the elliptic sequence of (X, x).

(iii) From Theorem 5.1 in [2], the affine graded ring of (X, x) is given by

$$R_X := \bigoplus_{k=0}^{\infty} H^0\left(E, \mathscr{O}\left(\sum_{i=1}^{n} \left[\frac{ke_i}{d_i}\right] \cdot P_i\right)\right) \cdot u^k ,$$

where $d_i = tb_i - t + 1$, $e_i = b_i - 1$, and $P_i = E \cap A_i$.

First we consider the case of $n = 1$. R_X is generated by $\{t, f_k u^{\{d_1 k/e_1\}} | k = 2, 3, \ldots\}$, where f_k is a meromorphic function on E which has a pole only at P_1 and the order is k. The semigroup $\mathbb{H}$ associated to P_1 is $\{2, 3, \ldots\}$. Let

$$\begin{aligned} I &= \left\{ k \in \mathbb{H} \;\middle|\; \left\{\frac{d_1 k}{e_1}\right\} < \left\{\frac{d_1 i}{e_1}\right\} + \left\{\frac{d_1(k-i)}{e_1}\right\} \text{ for any } i \in \mathbb{H} \text{ with } 0 < i < k \right\} \\ &= \left\{ k \in \mathbb{H} \;\middle|\; \left\{\frac{k}{b_1 - 1}\right\} < \left\{\frac{i}{b_1 - 1}\right\} + \left\{\frac{k-i}{b_1 - 1}\right\} \text{ for any } i \text{ with } 2 \le i < k \right\} \\ &= \{2, 3, \ldots, b_1 - 1\}. \end{aligned}$$

We can easily check that if $b_1 \ge 4$, $\{t, f_k u^{\{d_1 k/e_1\}} | k \in I\}$ is a minimal generator system of R_X. Then $\operatorname{emb\,dim}(X, x)$ is $b_1 - 1$. If $b_1 = 2$ or 3, then it is easy to check that $\operatorname{emb\,dim}(X, x)$ is 3.

Next we consider the case $n \ge 2$ and $\operatorname{emb\,dim}(X, x) \ge 4$. Let f_i be a meromorphic function on E which has poles only at two points P_1 and P_i and whose order is one $(i = 2, \ldots, n)$. Let $f_{i,k}$ be a meromorphic function on E which has a pole only at P_i and the order is k $(i = 1, \ldots, n$ and $k = 2, 3, \ldots)$. From a similar consideration as above we can see that the elements of the following set constitutes a minimal generator system of R_X:

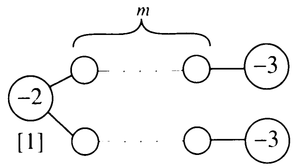
$$\{t, f_i u^{t+1} | 2 \le i \le n\} \cup \{f_{i,k} u^{\{d_i k/e_i\}} | 1 \le i \le n \text{ and } 2 \le k \le b_i - 1\}.$$

Hence $\operatorname{emb\,dim}(X, x) = \max(3, \sum_{i=1}^{n}(b_i - 1))$. □

Example 2.1. (i) Let $(Y, y) = \{(x, y, z) \in \mathbb{C}^3 | x^2 + y^6 + z^{6m+9} = 0\}$. Let G be the Veronese group of order 4 for (Y, y) (see [5]). Namely, G is a cyclic group generated by

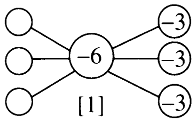
$$\begin{pmatrix} \zeta^{2m+1} & 0 & 0 \\ 0 & \zeta^{2m+3} & 0 \\ 0 & 0 & \zeta^2 \end{pmatrix},$$

where ζ is a primitive 4th root and $m = 0, 1, \ldots$. Let $(X, x) = (Y, y)/G$ (4th Veronese quotient). Then the w.d. graph associated to (X, x) is given by



and $p_g(X, x) = m+2$ (see [5]). Then (X, x) is a maximally elliptic singularity of emb dim $= 4$. Hence (X, x) is Gorenstein and codim $= 2$, so it is a complete intersection from Serre's result in [3].

(ii) Let $(Y, y) = \{(x, y, z) \in \mathbb{C}^3 | x^3 + y^6 + z^9 = 0\}$, and let (X, x) be the 7th Veronese quotient of (X, x). Then the w.d. graph associated to (X, x) is given by



and $p_g(X, x) = 2$. Then this is a maximally elliptic singularity of emb dim $= 9$.

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