ON NUMERICALLY GORENSTEIN QUASI-SIMPLE
ELLiptic SINGULARITIES WITH C*-ACTION

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Abstract. Let \((X, x)\) be a numerically Gorenstein elliptic singularity with
\(C^\ast\)-action and \(\pi: (\tilde{X}, A) \to (X, x)\) the minimal good resolution. Assume that
the central curve of \(\pi\) is an elliptic curve. We will determine the configuration
of the w.d. graph (weighted dual graph) of \(A\) and obtain a condition for \((X, x)\)
to be a maximally elliptic singularity in the sense of Yau.

1. Introduction

Let \((X, x)\) be a normal surface singularity and \(\pi: (\tilde{X}, A) \to (X, x)\) a
resolution, where \(A = \pi^{-1}(x)\). For definitions and details of the following,
please refer to [1] or [8]: \(Z\) (resp. \(K\)) the fundamental (resp. canonical) cy-
cle on \(A\), \(p_g(X, x)\) (resp. \(p_a(X, x)\)) the geometric (resp. arithmetic) genus.
If \(p_a(X, x) = 1\), we call \((X, x)\) an elliptic singularity. If \(K\) is an integral
coefficient divisor, we say that \((X, x)\) is numerically Gorenstein.

For any elliptic singularity, Yau [8] introduced the notion of the elliptic se-
quence. Also, for numerically Gorenstein elliptic singularities, he gave the defi-
nition of maximally elliptic singularities. Further, in [7], he found the condition
for \((X, x)\) to be maximally elliptic and gave the following definition.

Definition 1.1 [7, Definition 3.1]. Let \(\pi\) be the minimal good resolution. If the
minimally elliptic cycle \(E\) is a nonsingular elliptic curve, we say that \((X, x)\)
is a quasi-simple elliptic singularity.

Theorem 1.2. Let \((X, x)\) be a quasi-simple elliptic singularity with star-shaped
dual graph and \(\pi: (\tilde{X}, A) \to (X, x)\) the minimal good resolution. Assume that
\((X, x)\) is not a simple elliptic singularity.

\(\text{(i) } (X, x) \text{ is numerically Gorenstein if and only if the w.d. graph of } A \text{ has}
\text{the form}

\[\begin{array}{ccc}
   E & \cdots & A_{1r} \\
   \cdots & \cdots & \cdots \\
   A_{n1} & \cdots & A_{nt}
\end{array}\]
where $b_1, \ldots, b_n \geq 2$, $-E^2 = \text{the number of branches (}= n)$, \text{the lengths of all branches are equal (}= t \geq 1)$, and $\bigcirc = \bigcirc$.

(ii) Let $(X, x)$ be a numerically Gorenstein singularity with $\mathbb{C}^*$-action. Then we have:

(ii-a) $(X, x)$ is Gorenstein if and only if $tR \sim t \sum_{i=1}^n P_i$;

(ii-b) $(X, x)$ is a maximally elliptic singularity if and only if $R \sim \sum_{i=1}^n P_i$, where $\mathcal{O}(R)$ is the conormal sheaf of $E$ in $\tilde{X}$, $P_i = E \cap A_{i,1}$ ($i = 1, \ldots, n$), and $\sim$ means two divisors are linearly equivalent on $E$.

(iii) Let $(X, x)$ be a maximally elliptic singularity with $\mathbb{C}^*$-action whose w.d. graph is given as in (i). Then the embdim$(X, x)$ ( = embedding dimension of $(X, x)$) $= \max(3, -Z^2)$, where $-Z^2 = \sum_{i=1}^n (b_i - 1)$.

Yau [7, Theorem 2.4] had obtained a fine condition for numerically Gorenstein elliptic singularities to be maximally elliptic. We can prove (ii-b) directly from his result. However, we give a different proof for it by using results about surface singularities with $\mathbb{C}^*$-action. For (iii), we also give a minimal generator system of the affine graded ring of $(X, x)$ in terms of the elliptic functions on $E$. On the other hand, for every numerically Gorenstein elliptic singularity with $\mathbb{C}^*$-action whose central curve is $\mathbb{P}^1$, Tomari [4] had already proved that it is always a maximally elliptic singularity.

Before we prove Theorem 1.2, we prepare a fact. Let $A$ be the exceptional set of a resolution of a normal surface singularity. For a cycle $D$ on $A$, $\chi(D)$ is given by $-\frac{1}{2}(D^2 + K_A D)$ (see [8, p. 272]), where $K_A$ is the canonical divisor on $\tilde{X}$. Let $Z$ be the fundamental cycle on $A$. Assume $0 \leq D < Z$. Then, in the same way as in Laufer’s computation sequence, we can choose a sequence of cycles $Z_0 = D$, $Z_1 = Z_0 + A_{i_1}, \ldots, Z_t = Z$ such that $Z_i A_{i+1} > 0$ for any $i$. Then, by the adjunction formula on algebraic curves, we have $\chi(Z_{i+1}) = \chi(Z_i) + \chi(A_i) - Z_i A_{i+1} = \chi(Z_i) + 1 - g_{i+1} - Z_i A_{i+1} \leq \chi(Z_i)$ for any $i$, where $g_{i+1}$ is the genus of $A_{i+1}$. Hence we have

$\chi(D) \geq \chi(Z_1) \geq \cdots \geq \chi(Z_i) \geq \cdots \geq \chi(Z)$.

2. Proof of Theorem 1.2

In this section, for a real number $a$, we put $[a] := \max\{n \in \mathbb{Z}|n \leq a\}$ and \(\{a\} := \min\{n \in \mathbb{Z}|n \geq a\}\).

(i) Suppose that $(X, x)$ is numerically Gorenstein. Let $s$ be the number of the branches that contain an irreducible component whose intersection number is greater than two. Then the w.d. graph of $A$ is given as

\[
\begin{array}{c}
\begin{array}{c}
A_{s+1, r_1, \cdots, A_{r_1, 1}} \quad \cdots \quad A_{1, 1, \cdots, A_{1, r_1}}
\end{array}
\end{array}
\]

where $b_{1, i_1}, \ldots, b_{1, i_1} \geq 3$, $r_i > l_i > 0$ for $i = 1, \ldots, n$, and $l_i + 1 = r_i$ for $i = s + 1, \ldots, n$. Let $D_1 = \sum_{i=1}^n \sum_{j=1}^n A_{i,j}$. Then $(E + D_1) A_{i,j} \leq 0$ for any $i, j$. By the adjunction formula we have $\chi(E) = 0$. Since there is a sequence from $E$ to $Z$ passing through $E + D_1$ as in (1.1) and $(X, x)$ is an elliptic
singularity, we have $0 = \chi(E) \geq \chi(E + D_1) \geq \chi(Z) = 0$ from (1.1). Then
\[ \chi(E + D_1) = 0, \] so $0 = \chi(E + E + D_1) = \chi(E) + \chi(E + D_1) - E(E + D_1) = -E(E + D_1). \] Then $Z = E + D_1$ is the fundamental cycle on $A$. Further $ZE = 0$ and $b = -E^2 = ED_1 = n$.

Now let $\{Z, Z_{B_1}, \ldots, Z_{B_n} = E\}$ be the elliptic sequence for $(X, x)$. From the proof of Theorem 3.7 in [8], we can see that $(X_{B_i}, x)$ (the contraction of $B_i$ in $\tilde{X}$) is a numerically Gorenstein elliptic singularity. If $l_1 = \cdots = l_n$ does not hold, the number of the branches of $(X_{B_{l-1}}, x)$ is less than $b = -E^2$. This contradicts the above fact, so $l_1 = \cdots = l_n (:= l)$ and $t = l + 1$. It is easy to see that $Z_{B_i} = E + \sum_{i=1}^n \sum_{j=1}^{l_i} A_{i,j}$. Then, by Theorem 3.7 in [8] we have

\[ -K = (t+1)E + \sum_{i=1}^n \left\{ \sum_{j=1}^{l_i} (l - j + 2)A_{i,j} + \sum_{j=l_i+1}^{r_i} A_{i,j} \right\}. \]

If $r_i \geq l + 2$ for some $i$, then $-KA_{i,t+1} = A_{i,t+1}^2 + 3$ by the above formula. On the other hand, $-KA_{i,t+1} = A_{i,t+1}^2 + 2$ by the adjunction formula. This is a contradiction, so $r_1 = \cdots = r_n = t + 1$.

Conversely, if the w.d. graph is given as in (i), then we can easily check that $-K$ is given as

\[ -K = (t+1)E + \sum_{i=1}^n \sum_{j=1}^t (t - j + 1)A_{i,j}. \]

(ii-a) Since $(X, x)$ has a $\mathbb{C}^*$-action, the w.d. graph of $(X, x)$ is a star-shaped graph as in (i). From
\[
\begin{align*}
2 - \frac{1}{t} & = \frac{tb_i - t + 1}{(t-1)b_i - t + 2} \\
(2) & = \frac{1}{2 - \frac{1}{b_i}}
\end{align*}
\]
and Corollary 2.9 in [6], $(X, x)$ is Gorenstein if and only if

\[ \sum_{i=1}^n \frac{tb_i - t}{tb_i - t + 1} P_i \sim a \left( R - \sum_{i=1}^n \frac{(t-1)b_i - t + 2}{tb_i - t + 1} P_i \right) \]

for an integer $a$. Then $a = t$, so $(X, x)$ is Gorenstein if and only if $tR \sim t \sum_{i=1}^n P_i$.

(ii-b) Suppose that $(X, x)$ is a maximally elliptic singularity. Let $\{Z, Z_{B_1}, \ldots, Z_{B_n} = E\}$ be the elliptic sequence of $(X, x)$. From the definition of maximally elliptic singularities, $(X_{B_i}, x)$ is a maximally elliptic singularity for any $i$ and is Gorenstein from Theorem 3.11 in [8]. Let us consider $(X_{B_{l-1}}, x)$. Then the w.d. graph is given by

\[ \begin{array}{c}
\circ \\
\cdots \\
-\frac{n}{t_i} \\
\cdots \\
\circ \\
\circ \end{array} \]

where $c_i = 2$ (resp. $c_i = b_i$) if $t > 1$ (resp. $t = 1$) for any $i$. From (ii-a), $R \sim \sum_{i=1}^n P_i$. 


Conversely we assume that \( R \sim \sum_{i=1}^{n} P_i \). From Theorem 5.7 in [2], \( p_g(X, x) \) is given by \( \sum_{k=0}^{\infty} \dim C H^1(E, \Theta(D^{(k)})) \), where
\[
D^{(k)} = k \sum_{i=1}^{n} P_i - \sum_{i=1}^{n} \left\{ k((t-1)b_i - t+2) \right\} P_i.
\]

Therefore, by using the Riemann-Roch Theorem for algebraic curves we can easily check that \( p_g(X, x) = t + 1 = \) the length of the elliptic sequence of \((X, x)\).

(iii) From Theorem 5.1 in [2], the affine graded ring of \((X, x)\) is given by
\[
R_X := \bigoplus_{k=0}^{\infty} H^0 \left( E, \Theta \left( \sum_{i=1}^{n} \left\lfloor \frac{ke_i}{d_i} \right\rfloor P_i \right) \right) \cdot u^k,
\]
where \( d_i = tb_i - t + 1 \), \( e_i = b_i - 1 \), and \( P_i = E \cap A_i \).

First we consider the case of \( n = 1 \). \( R_X \) is generated by \( \{ t, f_k u^{(d_i/e_i)} | k = 2, 3, \ldots \} \), where \( f_k \) is a meromorphic function on \( E \) which has a pole only at \( P_1 \) and the order is \( k \). The semigroup \( \mathbb{H} \) associated to \( P_1 \) is \{2, 3, \ldots\}. Let
\[
I = \left\{ k \in \mathbb{H} \left| \frac{d_i k}{e_i} < \frac{d_i i}{e_i} \right. \right\} \text{ for any } i \in \mathbb{H} \text{ with } 0 < i < k \right\}
\]
\[
= \left\{ k \in \mathbb{H} \left| \frac{k}{b_i - 1} < \frac{i}{b_i - 1} \right. \right\} \text{ for any } i \text{ with } 2 \leq i < k \right\}
\]
\[
= \{ 2, 3, \ldots, 6, -1 \}.
\]
We can easily check that if \( b_1 \geq 4 \), \( \{ t, f_k u^{(d_i/e_i)} | k \in I \} \) is a minimal generator system of \( R_X \). Then \( \text{emb dim}(X, x) \) is \( b_1 - 1 \). If \( b_1 = 2 \) or \( 3 \), then it is easy to check that \( \text{emb dim}(X, x) \) is \( 3 \).

Next we consider the case \( n \geq 2 \) and \( \text{emb dim}(X, x) \geq 4 \). Let \( f_i \) be a meromorphic function on \( E \) which has poles only at two points \( P_1 \) and \( P_i \) and whose order is one \( (i = 2, \ldots, n) \). Let \( f_{i,k} \) be a meromorphic function on \( E \) which has a pole only at \( P_i \) and the order is \( k \) \( (i = 1, \ldots, n \text{ and } k = 2, 3, \ldots) \). From a similar consideration as above we can see that the elements of the following set constitutes a minimal generator system of \( R_X \):
\[
\{ t, f_i u^{i+1} | 2 \leq i \leq n \} \cup \{ f_{i,k} u^{(d_i/e_i)} | 1 \leq i \leq n \text{ and } 2 \leq k \leq b_i - 1 \}.
\]
Hence \( \text{emb dim}(X, x) = \max(3, \sum_{i=1}^{n} (b_i - 1)) \). \( \square \)

**Example 2.1.** (i) Let \( (Y, y) = \{(x, y, z) \in \mathbb{C}^3 | x^2 + y^6 + z^{6m+9} = 0 \} \). Let \( G \) be the Veronese group of order 4 for \((Y, y)\) (see [5]). Namely, \( G \) is a cyclic group generated by
\[
\begin{pmatrix}
\zeta^{2m+1} & 0 & 0 \\
0 & \zeta^{2m+3} & 0 \\
0 & 0 & \zeta^2
\end{pmatrix},
\]
where \( \zeta \) is a primitive 4th root and \( m = 0, 1, \ldots \). Let \( (X, x) = (Y, y)/G \) (4th Veronese quotient). Then the w.d. graph associated to \((X, x)\) is given by

```
\begin{tikzpicture}
  \node (1) at (0,0) {1};
  \node (2) at (1,0) {2};
  \node (3) at (2,0) {3};

  \draw (1) -- (2);
  \draw (1) -- (3);
  \draw (2) -- (3);
\end{tikzpicture}
```
and $p_g(X, x) = m+2$ (see [5]). Then $(X, x)$ is a maximally elliptic singularity of $\text{emb dim} = 4$. Hence $(X, x)$ is Gorenstein and $\text{codim} = 2$, so it is a complete intersection from Serre’s result in [3].

(ii) Let $(Y, y) = \{(x, y, z) \in \mathbb{C}^3|x^3 + y^6 + z^9 = 0\}$, and let $(X, x)$ be the 7th Veronese quotient of $(X, x)$. Then the w.d. graph associated to $(X, x)$ is given by

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-6

[1]

-3
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and $p_g(X, x) = 2$. Then this is a maximally elliptic singularity of $\text{emb dim} = 9$.

References


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