

## PARABOLICITY OF A CLASS OF HIGHER-ORDER ABSTRACT DIFFERENTIAL EQUATIONS

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**ABSTRACT.** Let  $E$  be a complex Banach space,  $c_i \in \mathbb{C}$  ( $1 \leq i \leq n-1$ ), and  $A$  be a nonnegative operator in  $E$ . We discuss the parabolicity of the higher-order abstract differential equations

$$(*) \quad u^{(n)}(t) + \sum_{i=1}^{n-1} c_i A^{k_i} u^{(n-i)}(t) + Au(t) = 0$$

and some perturbation cases of  $(*)$ . A sufficient and necessary condition for  $(*)$  to be parabolic is obtained, provided  $k_1 > k_2 - k_1 > \dots > 1 - k_{n-1} > 0$ ,  $c_i \neq 0$  ( $1 \leq i \leq n-1$ ). For  $A$  strictly nonnegative (Definition 1.3),  $n = 3$ ,  $c_1, c_2 \geq 0$ , a sharp criterion is given.

### 1. INTRODUCTION AND PRELIMINARIES

Throughout this paper,  $E$  will be a complex Banach space and  $n \in \mathbb{N}$  (the set of natural numbers). For  $\theta \in (0, \pi/2]$  and  $\omega \in \mathbb{R}$  (the set of real numbers), write

$$\begin{aligned} \Sigma(\theta, \omega) &= \{z \in \mathbb{C}: z \neq \omega, |\arg(z - \omega)| < \frac{\pi}{2} + \theta\}, \\ \Sigma_\theta &= \{z \in \mathbb{C}: z \neq 0, |\arg z| < \theta\}. \end{aligned}$$

**Definition 1.1.** Suppose  $A_1, \dots, A_n$  are closed linear operators in  $E$  and  $\theta \in (0, \pi/2]$ . We say  $[A_1, \dots, A_n] \in \mathcal{A}_n(\theta)$ , if for each  $\theta' \in (0, \theta)$  there exist  $C_{\theta'}, \omega_{\theta'} > 0$  such that

$$(1.1) \quad \left\| \lambda^{n-i} A_i \left( \lambda^n + \sum_{i=1}^n \lambda^{n-i} A_i \right)^{-1} \right\| \leq C_{\theta'},$$

whenever  $\lambda \in \Sigma(\theta', \omega_{\theta'})$ ,  $1 \leq i \leq n$ .

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Write  $\mathcal{A}_n = \bigcup_{\theta \in (0, \pi/2)} \mathcal{A}_n(\theta)$ . When  $[A_1, \dots, A_n] \in \mathcal{A}_n$ , we also say the abstract differential equation

$$(1.2) \quad u^{(n)}(t) + \sum_{i=1}^n A_i u^{(n-i)}(t) = 0$$

is parabolic.

Clearly, when  $[A_1, \dots, A_n] \in \mathcal{A}_n(\theta)$  ( $\theta \in (0, \pi/2)$ ), (1.1) also holds for  $i = 0$  ( $A_0 = I$ , the identity operator);  $[A_1] \in \mathcal{A}_1(\theta)$  for some  $\theta \in (0, \pi/2]$  iff  $-A_1$  is the generator of an exponentially bounded holomorphic semigroup.

We note that parabolicity of equation (1.2) is ‘comparable’ with existence of an analytic exponentially bounded semigroup for the corresponding first-order system

$$(1.3) \quad v'(t) + G_n v(t) = 0,$$

in a proper  $B$ -space, where  $v = (u_0, u_1, \dots, u_{n-1}) = (u, u', \dots, u^{(n-1)})$  and

$$(1.4) \quad G_n = \begin{pmatrix} 0 & -I & 0 & \cdots & 0 \\ 0 & 0 & -I & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & -I \\ A_n & A_{n-1} & A_{n-2} & \cdots & A_1 \end{pmatrix}.$$

It is known that the parabolicity of (1.2) ensures the existence and uniqueness of the solution of the Cauchy problem for (1.2) (see, e.g., [10, 11]). But under what condition is equation (1.2) parabolic? For second-order equations, this problem has been studied by many authors (see, e.g., [1–3, 5–9] and references therein); here (1.2) amounts to

$$(1.5) \quad u''(t) + A_1 u'(t) + A_2 u(t) = 0,$$

and (1.3) to

$$(1.6) \quad G_2 = \begin{pmatrix} 0 & -I \\ A_2 & A_1 \end{pmatrix}.$$

In [1] Chen and Russell posed two conjectures in their study of linear elastic systems with structure damping, which state, qualitatively, that  $-G_2$  (in a suitable product space) generates an exponentially bounded analytic semigroup in the case where  $A_1, A_2$  are positive and selfadjoint operators in a Hilbert space and the dissipation operator  $A_1$  is ‘comparable’ with the  $\frac{1}{2}$ th power of the elastic operator  $A_2$ . Huang [6, 7] and, independently, Chen and Triggiani [2] proved these two conjectures; furthermore, they [3, 8, 9] discussed the general case where  $A_1$  is ‘comparable’ with the  $\alpha$ th power of  $A_2$  over the entire range  $0 \leq \alpha \leq 1$  of the parameter  $\alpha$ . Recently, in the framework of Banach spaces, Favini and Obrecht [10] studied sufficient and necessary conditions ensuring equation (1.5) with  $A_1 = \rho A_2^\alpha$  ( $\rho \in \mathbb{C}, 0 < \alpha < 1$ ) parabolic.

This paper aims at investigating the parabolicity of (1.2) for any  $n$ , but under the special condition  $A_n = A > 0, A_i = c_i A^{k_i}, c_i \in \mathbb{C} (1 \leq i \leq n - 1)$ , that is,

$$(1.7) \quad u^{(n)}(t) + \sum_{i=1}^{n-1} c_i A^{k_i} u^{(n-i)}(t) + Au(t) = 0.$$

First (in §2), assuming  $c_i \neq 0$  ( $1 \leq i \leq n - 1$ ),  $k_1 > k_2 - k_1 > \dots > 1 - k_{n-1} > 0$ , we obtain a sufficient and necessary condition for (1.7) to be parabolic. Furthermore, some perturbation theorems are presented. Following this (in §3), we specialize to the case where  $A$  is strictly nonnegative,  $n = 3$ ,  $c_1, c_2 \geq 0$ , and give a complete and clear answer for the problem of whether (1.7) is parabolic.

**Definition 1.2.** Suppose  $S$  is an arbitrary linear operator in  $E$ .  $S$  is nonnegative, if for each  $\lambda > 0$ ,  $\lambda \in \rho(-S)$  and

$$\sup\{\|\lambda(\lambda + S)^{-1}\| : \lambda > 0\} < +\infty.$$

It can be shown (cf. [4, Lemma 6.4.1]) that, if  $S$  is a nonnegative operator in  $E$ , then there exists  $\theta \in (0, \pi/2]$  such that  $\lambda \in \rho(-S)$  for each  $\lambda \in \Sigma_\theta$  with  $\{\|\lambda(\lambda + S)^{-1}\| : \lambda \in \Sigma_\theta\}$  bounded.

Let  $S$  be a nonnegative operator in  $E$ . Set as in [5]

$$\begin{aligned} \theta_\infty^+(S) &= \inf\{\theta \in (-\pi, \pi) : \text{there exist } C, \omega > 0 \text{ such that,} \\ &\quad \text{for each } \lambda \text{ with } |\lambda| \geq \omega \text{ and } \theta \leq \arg \lambda \leq \pi, \\ &\quad \lambda \in \rho(S) \text{ and } \|\lambda(\lambda - S)^{-1}\| \leq C\}, \\ \theta_\infty^-(S) &= \sup\{\theta \in (-\pi, \pi) : \text{there exist } C, \omega > 0 \text{ such that,} \\ &\quad \text{for each } \lambda \text{ with } |\lambda| \geq \omega \text{ and } -\pi \leq \arg \lambda \leq \theta, \\ &\quad \lambda \in \rho(S) \text{ and } \|\lambda(\lambda - S)^{-1}\| \leq C\}. \end{aligned}$$

Obviously,  $\theta_\infty^+(S) \geq \theta_\infty^-(S)$ ;  $[S] \in \mathcal{A}_1(\theta)$  ( $\theta \in (0, \pi/2]$ ) iff  $\theta_\infty^+(S) \leq \pi/2 - \theta$  and  $\theta_\infty^-(S) \geq -\pi/2 + \theta$ . It is not difficult to verify that, for  $c \in \mathbb{C}$ ,  $cS$  is nonnegative iff either

- (i)  $\arg c < -\pi - \theta_\infty^+(S)$ , or
- (ii)  $-\pi - \theta_\infty^-(S) < \arg c < \pi - \theta_\infty^+(S)$ , or
- (iii)  $\arg c > \pi - \theta_\infty^-(S)$ ,

and if  $cS$  is nonnegative, we have

$$(1.8) \quad \theta_\infty^\pm(cS) = \begin{cases} \arg c + \theta_\infty^\pm(S) + 2\pi & \text{if } \arg c < -\pi - \theta_\infty^+(S), \\ \arg c + \theta_\infty^\pm(S) & \text{if } -\pi - \theta_\infty^-(S) < \arg c < \pi - \theta_\infty^+(S), \\ \arg c + \theta_\infty^\pm(S) - 2\pi & \text{if } \arg c > \pi - \theta_\infty^-(S). \end{cases}$$

Finally, for each  $0 < \alpha < 1$ , as pointed out in the proof of [5, Lemma 3.3],

$$(1.9) \quad \theta_\infty^\pm(S^\alpha) = \alpha\theta_\infty^\pm(S).$$

**Definition 1.3.** We say that  $S$  is strictly nonnegative if  $\theta_\infty^\pm(S) = 0$ .

## 2. RESULTS FOR ARBITRARY ORDER

Throughout this section,  $A$  will be a densely defined and nonnegative operator in  $E$ ,  $c_i \in \mathbb{C}$  ( $1 \leq i \leq n - 1$ ), and

$$P_0(\lambda) = \lambda^n + \sum_{i=1}^{n-1} c_i A^{k_i} \lambda^{n-i} + A.$$

First, we state the well-known Moment inequality:

Let  $0 \leq \alpha < \beta < \varepsilon \leq 1$ . Then there exists a constant  $C = C(\alpha, \beta, \varepsilon)$  such that

$$(2.1) \quad \|A^\beta u\| \leq C \|A^\varepsilon u\|^{(\beta-\alpha)/(\varepsilon-\alpha)} \|A^\alpha u\|^{(\varepsilon-\beta)/(\varepsilon-\alpha)} \quad (u \in D(A^\varepsilon)).$$

**Theorem 2.1.** Let  $k_1 > k_2 - k_1 > \dots > k_{n-1} - k_{n-2} > 1 - k_{n-1} > 0$ ,  $c_i \neq 0$  for each  $1 \leq i \leq n - 1$ . Then  $[c_1 A^{k_1}, \dots, c_{n-1} A^{k_{n-1}}, A] \in \mathcal{A}_n(\theta)$  ( $\theta \in (0, \pi/2]$ ) iff, for each  $1 \leq i \leq n$ ,  $[c_{i-1}^{-1} c_i A^{k_i - k_{i-1}}] \in \mathcal{A}_1(\theta)$ , where  $c_0 = c_n = 1$ .

*Proof. Sufficiency.* Set  $c_{i-1}^{-1} c_i = \tilde{c}_i$ ,  $k_i - k_{i-1} = t_i$  ( $1 \leq i \leq n$ ),

$$P_1(\lambda) = \prod_{i=1}^n (\lambda + \tilde{c}_i A^{t_i}),$$

$$Q(\lambda) = \sum_{m=1}^{n-1} \sum_{(i_1, \dots, i_m) \in I_m} \tilde{c}_{i_1} \dots \tilde{c}_{i_m} A^{t_{i_1} + \dots + t_{i_m}} \lambda^{n-m},$$

where, for each  $1 \leq m \leq n - 1$ ,

$$I_m = \{(i_1, \dots, i_m) : 1 \leq i_1 < \dots < i_m \leq n, (i_1, \dots, i_m) = (1, \dots, m)\}.$$

Then

$$t_1 > t_2 > \dots > t_n, \quad k_m = \sum_{i=1}^m t_i \quad (1 \leq m \leq n - 1).$$

By hypothesis, for each  $\theta' \in (0, \theta)$ , there exist  $C_{\theta'}, \omega_{\theta'} > 0$  such that

$$(2.2) \quad \|\lambda^{n-m} A^{t_{i_1} + \dots + t_{i_m}} P_1^{-1}(\lambda)\| \leq C_{\theta'}, \quad \|\lambda^n P_1^{-1}(\lambda)\| \leq C_{\theta'}$$

whenever  $\lambda \in \sum(\theta', \omega_{\theta'})$ ,  $1 \leq i_1 < \dots < i_m \leq n$ ,  $1 \leq m \leq n$ . This together with (2.1) yields that, for each  $\theta' \in (0, \theta)$ , there exist  $C, C_{\theta'}, \omega_{\theta'} > 0$  such that, for  $\lambda \in \sum(\theta', \omega_{\theta'})$ ,  $(i_1, \dots, i_m) \in I_m$ ,  $1 \leq m \leq n - 1$ ,

$$\begin{aligned} & \|\lambda^{n-m} A^{t_{i_1} + \dots + t_{i_m}} P_1^{-1}(\lambda)\| \\ & \leq C |\lambda|^{n-m} \|A^{k_m} P_1^{-1}(\lambda)\|^{(t_{i_1} + \dots + t_{i_m})k_m^{-1}} \|P_1^{-1}(\lambda)\|^{1 - (t_{i_1} + \dots + t_{i_m})k_m^{-1}} \\ & \leq C (C_{\theta'} |\lambda|)^{[(t_{i_1} + \dots + t_{i_m})k_m^{-1} - 1]m}, \end{aligned}$$

which approaches 0 as  $|\lambda| \rightarrow \infty$ .

Therefore, for each  $\theta' \in (0, \theta)$ , there is  $\tilde{\omega}_{\theta'} > \omega_{\theta'}$  such that for  $\lambda \in \sum(\theta', \tilde{\omega}_{\theta'})$

$$(2.3) \quad \|Q(\lambda) P_1^{-1}(\lambda)\| < \frac{1}{2}.$$

Thus using (2.2) again we obtain that, for each  $\theta' \in (0, \theta)$ ,  $\lambda \in \sum(\theta', \tilde{\omega}_{\theta'})$ ,  $1 \leq m \leq n$ ,

$$\|c_m \lambda^{n-m} A^{k_m} P_0^{-1}(\lambda)\| = \|c_m \lambda^{n-m} A^{k_m} P_1^{-1}(\lambda) [I - Q(\lambda) P_1^{-1}(\lambda)]^{-1}\| \leq 2c_m C_{\theta'}.$$

In conclusion,  $[c_1 A^{k_1}, \dots, c_{n-1} A^{k_{n-1}}, A] \in \mathcal{A}_n(\theta)$ .

*Necessity.* Making use of (2.1) as in the proof of sufficiency, we obtain that, for each  $\theta' \in (0, \theta)$ , there exists  $\omega_{\theta'} > 0$  such that, for  $\lambda \in \sum(\theta', \omega_{\theta'})$ ,  $(i_1, \dots, i_m) \in I_m$ ,  $1 \leq m \leq n$ ,

$$(2.4) \quad \|\lambda^{n-m} A^{t_{i_1} + \dots + t_{i_m}} P_0^{-1}(\lambda)\| \leq C_{\theta'} |\lambda|^{[(t_{i_1} + \dots + t_{i_m})k_m^{-1} - 1]m},$$

and therefore there exist  $\tilde{\omega}_{\theta'} > \omega_{\theta'}$ ,  $M_{\theta'} > 0$  such that, for  $\lambda \in \Sigma(\theta', \tilde{\omega}_{\theta'})$ ,  $1 \leq m \leq n$ ,

$$\begin{cases} \|Q(\lambda)P_0^{-1}(\lambda)\| < \frac{1}{2}, \\ \|\lambda(\lambda + \tilde{c}_m A^{l_m})^{-1}P_1(\lambda)P_0^{-1}(\lambda)\| \leq M_{\theta'}. \end{cases}$$

Accordingly, for each  $\theta' \in (0, \theta)$ ,  $\lambda \in \Sigma(\theta', \omega_{\theta'})$ ,  $1 \leq m \leq n$ ,

$$\|\lambda(\lambda + \tilde{c}_m A^{l_m})^{-1}\| = \|\lambda(\lambda + \tilde{c}_m A^{l_m})^{-1}P_1(\lambda)P_0^{-1}(\lambda)[I + Q(\lambda)P_0^{-1}(\lambda)]^{-1}\| \leq 2M_{\theta'}.$$

This ends the proof. Q.E.D.

**Corollary 2.2.** Let  $\theta_{\infty}^{\pm}(A) = 0$ ,  $c_i > 0$  ( $1 \leq i \leq n - 1$ ), and  $k_1 > k_2 - k_1 > \dots > k_{n-1} - k_{n-2} > 1 - k_{n-1} > 0$ . Then  $[c_1 A^{k_1}, \dots, c_{n-1} A^{k_{n-1}}, A] \in \mathcal{A}_n(\pi/2)$ .

The following are some perturbation cases.

**Theorem 2.3.** Let  $B_1, \dots, B_{n-1}$  be closed linear operators in  $E$  satisfying that, for each  $1 \leq m \leq n - 1$ , there is  $l_m$  with  $k_{m-1} < l_m < \frac{1}{2}(k_{m-1} + k_{m+1})$  such that  $D(B_m) \supset D(A^{l_m})$ . Then if  $[c_1 A^{k_1}, \dots, c_{n-1} A^{k_{n-1}}, A] \in \mathcal{A}_n(\theta)$  ( $\theta \in (0, \pi/2]$ ), so does  $[c_1 A^{k_1} + B_1, \dots, c_{n-1} A^{k_{n-1}} + B_{n-1}, A]$ .

*Proof.* By hypothesis, there is  $C > 0$  such that, for each  $1 \leq m \leq n - 1$ ,  $u \in D(A^{l_m})$ ,

$$\|B_m u\| \leq C\|x\| + C\|A^{l_m} u\|.$$

So using (2.1) yields that, for each  $\theta' \in (0, \theta)$ , there exist  $C_{\theta'}$ ,  $\omega_{\theta'} > 0$  such that, for each  $1 \leq m \leq n - 1$ ,  $\lambda \in \Sigma(\theta', \omega_{\theta'})$ ,

$$\begin{aligned} \|\lambda^{n-m} B_m P_0^{-1}(\lambda)\| &\leq C|\lambda|^{n-m} \|P_0^{-1}(\lambda)\| + C|\lambda|^{n-m} \|A^{l_m} P_0^{-1}(\lambda)\| \\ &\leq C C_{\theta'} |\lambda|^{-m} + C|\lambda|^{n-m} \|A^{k_{m-1}} P_0^{-1}(\lambda)\|^{\tau} \|A^{k_{m+1}} P_0^{-1}(\lambda)\|^{1-\tau} \\ &\leq C C_{\theta'} (|\lambda|^{-m} + |\lambda|^{n-m} |\lambda|^{(m-n-1)\tau} |\lambda|^{(m-n+1)(1-\tau)}) \\ &= C C_{\theta'} (|\lambda|^{-m} + |\lambda|^{1-2\tau}) \end{aligned}$$

which approaches 0 as  $|\lambda| \rightarrow \infty$ , where  $\tau = (l_m - k_{m-1})(k_{m+1} - k_{m-1})^{-1} < 1$ . Consequently, for each  $\theta' \in (0, \theta)$  there is  $\tilde{\omega}_{\theta'} > \omega_{\theta'}$  such that, for  $\lambda \in \Sigma(\theta', \tilde{\omega}_{\theta'})$ ,

$$\left\| \sum_{m=1}^{n-1} \lambda^{n-m} B_m P_0^{-1}(\lambda) \right\| < \frac{1}{2}.$$

This leads to the result as claimed. Q.E.D.

**Corollary 2.4.** Let  $0 < k_1 < \dots < k_{n-1} < 1$  and  $k_j < \frac{1}{2}(k_{j-1} + k_{j+1})$  for some  $1 \leq j \leq n - 1$ . Then  $[c_1 A^{k_1}, \dots, c_{n-1} A^{k_{n-1}}, A] \in \mathcal{A}_n(\theta)$  ( $\theta \in (0, \pi/2]$ ) implies  $[c_1 A^{k_1}, \dots, c_{j-1} A^{k_{j-1}}, 0, c_{j+1} A^{k_{j+1}}, \dots, c_{n-1} A^{k_{n-1}}, A] \in \mathcal{A}_n(\theta)$ .

**Theorem 2.5.** Let  $\tilde{c}_i \in \mathbb{C}$ ,  $0 < l_i < i/n$  for each  $1 \leq i \leq n - 1$ ,  $\theta \in (0, \pi/2]$ . Then  $[c_1 A^{k_1}, \dots, c_{n-1} A^{k_{n-1}}, A] \in \mathcal{A}_n(\theta)$  iff  $[c_1 A^{k_1} + \tilde{c}_1 A^{l_1}, \dots, c_{n-1} A^{k_{n-1}} + \tilde{c}_{n-1} A^{l_{n-1}}, A] \in \mathcal{A}_n(\theta)$ .

This theorem is an immediate consequence of the following (general) result by taking  $i_m = n$ ,  $\varepsilon_m = l_m$  ( $1 \leq m \leq n - 1$ ), and using (2.1).

**Theorem 2.6.** Assume  $[A_1, \dots, A_n] \in \mathcal{A}_n(\theta)$  for some  $\theta \in (0, \pi/2]$ , and  $B_m, A_m + B_m$  are closed and densely defined linear operators in  $E$  ( $1 \leq m \leq n$ ). If, for each  $1 \leq m \leq n$ , there exist  $i_m, \varepsilon_m$  with  $1 \leq i_m \leq n, 0 < \varepsilon_m \leq 1$  such that  $D(B_m) \supset D(A_{i_m})$  and, for each  $u \in D(A_{i_m})$ ,

$$\|B_m u\| \leq C\|u\| + C\|A_{i_m} u\|^{\varepsilon_m} \|u\|^{1-\varepsilon_m} \text{ for some } C > 0,$$

then, for each  $\theta' \in (0, \theta)$ , there is  $\omega_{\theta'} > 0$  such that, for  $1 \leq m \leq n$ ,

$$(2.5) \quad \sup_{\lambda \in \Sigma(\theta', \omega_{\theta'})} \|\lambda^{n-m} B_m P^{-1}(\lambda)\| < \begin{cases} \frac{1}{2} & \text{if } i_m \varepsilon_m < m, \\ +\infty & \text{if } i_m \varepsilon_m = m, \end{cases}$$

where  $P(\lambda) = \lambda^n + \sum_{i=1}^n \lambda^{n-i} A_i$ . Furthermore, when  $i_m \varepsilon_m < m, [A_1 + B_1, \dots, A_n + B_n] \in \mathcal{A}_n(\theta)$ .

*Proof.* Observing that, for each  $\theta' \in (0, \theta)$ , there exist  $C_{\theta'}, \omega_{\theta'} > 0$  such that, for  $\lambda \in \Sigma(\theta', \omega_{\theta'})$ ,  $1 \leq m \leq n$ ,

$$\begin{aligned} \|\lambda^{n-m} B_m P^{-1}(\lambda)\| &\leq C|\lambda|^{n-m} \|P^{-1}(\lambda)\| + C|\lambda|^{n-m} \|A_{i_m} P^{-1}(\lambda)\|^{\varepsilon_m} \|P^{-1}(\lambda)\|^{1-\varepsilon_m} \\ &\leq C C_{\theta'} (|\lambda|^{-m} + |\lambda|^{n-m} |\lambda|^{(i_m-n)\varepsilon_m} |\lambda|^{-n(1-\varepsilon_m)}) \\ &= C C_{\theta'} (|\lambda|^{-m} + |\lambda|^{i_m \varepsilon_m - m}), \end{aligned}$$

we obtain (2.5). The remaining part follows from the plain equality

$$\left( P(\lambda) + \sum_{m=1}^n \lambda^{n-m} B_m \right)^{-1} = P^{-1}(\lambda) \left[ I + \sum_{m=1}^n \lambda^{n-m} B_m P^{-1}(\lambda) \right]^{-1}. \quad \text{Q.E.D.}$$

### 3. THE CASE OF $n = 3$

Throughout this section,  $A$  will be densely defined, unbounded, and strictly nonnegative. First we state several basic facts.

*Basic Facts.* For  $0 < \beta \leq 1, a > 0, \operatorname{Re} c > 0$ , we have:

- (i)  $\theta_{\infty}^{\pm}(aA^{\beta}) = 0$ ;
- (ii)  $[cA^{\beta}] \in \mathcal{A}_1, [-cA^{\beta}] \notin \mathcal{A}_1$ ;
- (iii) for  $b \in \mathbb{R}, [bA^{\beta/2}, aA^{\beta}] \in \mathcal{A}_2$  iff  $b > 0$ ;
- (iv) (see [5, Theorem 3.7]) let  $\frac{1}{2} < \beta < 1$ ; then  $[cA^{\beta}, A] \in \mathcal{A}_2$  iff

$$\begin{cases} \arg c > -\pi/2 + \max\{(1-\beta)\theta_{\infty}^+(A), -\beta\theta_{\infty}^-(A)\}, \\ \arg c < \pi/2 - \max\{\beta\theta_{\infty}^+(A), -(1-\beta)\theta_{\infty}^-(A)\}. \end{cases}$$

**Theorem 3.1.** Let  $a_1, a_2 > 0$  and  $0 < k_1, k_2 < 1$ . Then

$$[a_1 A^{k_1}, 0, A], [0, a_2 A^{k_2}, A], [0, 0, A] \notin \mathcal{A}_3.$$

*Proof.* Observe that, for each  $y_1 \geq 0$ , the function

$$y(x) = x^{-1} + x(y_1 - x)$$

is continuous in  $(0, +\infty)$ , and  $y \rightarrow +\infty$  as  $x \rightarrow 0^+$ ,  $y \rightarrow -\infty$  as  $x \rightarrow +\infty$ . Hence, for each  $y_1, y_2 \geq 0$ , there exists  $x_1 > 0$  such that

$$y_2 = x_1^{-1} + x_1(y_1 - x_1).$$

Set  $x_2 = y_1 - x_1$ . If  $x_2 > 0$ , i.e.,  $y_1 > x_1$ , then  $y_2 > x^{-1}$ ; therefore,  $y_1 y_2 > y_1 x_1^{-1} > 1$ . If  $x_2 \leq 0$ , i.e.,  $y_1 \leq x_1$ , then  $y_2 \leq x_1^{-1}$ ; therefore,  $y_1 y_2 > y_1 x_1^{-1} > 1$ . In other words,

$$x_2 > 0 \text{ if } y_1 y_2 > 1; \quad x_2 \leq 0 \text{ if } y_1 y_2 \leq 1.$$

So from the equality

$$\lambda^3 + y_1 A^{1/3} \lambda^2 + y_2 A^{2/3} \lambda + A = (\lambda + x_1 A^{1/3})(\lambda^2 + x_2 A^{1/3} \lambda + x_1^{-1} A^{2/3}),$$

we see by Basic Facts (ii) and (iii)

$$(3.1) \quad [y_1 A^{1/3}, y_2 A^{2/3}, A] \in \mathcal{A}_3 \text{ if } y_1 y_2 > 1.$$

But

$$(3.2) \quad [y_1 A^{1/3}, y_2 A^{2/3}, A] \notin \mathcal{A}_3 \text{ if } y_1 y_2 \leq 1.$$

In fact, if  $[y_1 A^{1/3}, y_2 A^{2/3}, A] \in \mathcal{A}_3$  ( $y_1 y_2 \leq 1$ ), then by virtue of (2.5) we have that there are  $C, \omega > 0, \theta \in (0, \pi/2]$  such that, for  $\lambda \in \Sigma(\theta, \omega)$ ,  $i = 1, 2, 3$ ,

$$\|\lambda^{3-i} A^{i/3} (\lambda^3 + y_1 A^{1/3} \lambda^2 + y_2 A^{2/3} \lambda + A)^{-1}\| \leq C.$$

According to this, the equality

$$(\lambda^2 + x_2 A^{1/3} \lambda + x_1^{-1} A^{2/3})^{-1} = (\lambda + x_1 A^{1/3})(\lambda^3 + y_1 A^{1/3} \lambda^2 + y_2 A^{2/3} \lambda + A)^{-1}$$

shows  $[x_2 A^{1/3}, x_1^{-1} A^{2/3}] \in \mathcal{A}_2$ , which contradicts Basic Fact (iii). So (3.2) holds. (3.2) indicates  $[a_1 A^{1/3}, 0, A], [0, a_2 A^{2/3}, A], [0, 0, A] \notin \mathcal{A}_3$ . Since  $[0, 0, A] \notin \mathcal{A}_3$ , using Theorem 2.5 yields that  $[a_1 A^{k_1}, 0, A], [0, a_2 A^{k_2}, A] \notin \mathcal{A}_3$  if  $k_1 < \frac{1}{3}, k_2 < \frac{2}{3}$ . Finally, we have that, for each  $a > 0, 0 < \beta < 1$ ,

$$[a A^\beta, a^{-1} A^{1-\beta}, A] \notin \mathcal{A}_3.$$

Indeed, if not, then the equality

$$(\lambda^2 + a^{-1} A^{1-\beta})^{-1} = (\lambda + a A^\beta)(\lambda^3 + a A^\beta \lambda^2 + a^{-1} A^{1-\beta} \lambda + A)^{-1}$$

yields  $[0, a^{-1} A^{1-\beta}] \in \mathcal{A}_2$ , which contradicts Basic Facts (iii). Thus, we conclude by Theorem 2.5 again that

$$[a A^\beta, 0, A] \notin \mathcal{A}_3 \text{ if } \beta > \frac{1}{3}, \\ [0, a^{-1} A^{1-\beta}, A] \notin \mathcal{A}_3 \text{ if } \beta < \frac{1}{3}.$$

The proof is then complete. Q.E.D.

**Theorem 3.2.** Let  $a_1, a_2 > 0$  and  $0 < k_1 < k_2 < 1$ . Then  $[a_1 A^{k_1}, a_2 A^{k_2}, A] \in \mathcal{A}_3$  iff either

- (i)  $k_1 > \frac{1}{3}, \frac{1}{2}(1 + k_1) \leq k_2 \leq 2k_1$ , or
- (ii)  $k_1 = \frac{1}{3}, k_2 = \frac{2}{3}, a_1 a_2 > 1$ .

*Proof.* Observing

$$\lambda^3 + (a_2 a_1^{-1} A^{(1-k_1)/2} + a_1 A^{k_1}) \lambda^2 + (a_1^{-1} A^{1-k_1} + a_2 A^{(1+k_1)/2}) \lambda + A \\ = (\lambda + a_1 A^{k_1})(\lambda^2 + a_2 a_1^{-1} A^{(1-k_1)/2} \lambda + a_1^{-1} A^{1-k_1}),$$

we obtain

$$[a_2 a_1^{-1} A^{(1-k_1)/2} + a_1 A^{k_1}, a_1^{-1} A^{1-k_1} + a_2 A^{(1+k_1)/2}, A] \in \mathcal{A}_3.$$

Thus appealing to Theorem 2.5 gives

$$(3.3) \quad [a_1 A^{k_1}, a_2 A^{(1+k_1)/2}, A] \in \mathcal{A}_3 \quad (k_1 > \frac{1}{3}).$$

Next, let  $\frac{1}{3} < k_1 < \frac{1}{2}$ . Set  $\tau = k_1(1 - k_1)^{-1}$ ;

$$b_1 = \begin{cases} \frac{1}{2}[a_1 + (a_1^2 - 4a_2)^{1/2}] & \text{if } a_1^2 \geq 4a_2, \\ re^{i\theta} & \text{if } a_1^2 < 4a_2; \end{cases}$$

$$b_2 = \begin{cases} \frac{1}{2}[a_1 - (a_1^2 - 4a_2)^{1/2}] & \text{if } a_1^2 \geq 4a_2, \\ re^{-i\theta} & \text{if } a_1^2 < 4a_2, \end{cases}$$

where  $\theta = \arccos(\frac{1}{2}a_1 a_2^{-1/2})$ ,  $r = a_2^{1/2}$ ,

$$B = r^{-1}e^{-i\theta}A^{1-k_2}.$$

Then  $\theta_\infty^\pm(B) = -\theta$ ,  $\frac{1}{2} < \tau < 1$ ,  $b_1 + b_2 = a_1$ , and  $b_1 b_2 = a_2$ . Therefore, if  $a_1^2 < 4a_2$ ,

$$\max\{(1 - \tau)\theta_\infty^+(B), -\tau\theta_\infty^-(B)\} = \theta\tau,$$

$$\max\{\tau\theta_\infty^+(B), -(1 - \tau)\theta_\infty^-(B)\} = \theta(1 - \tau),$$

which implies by Basic Facts (iv) that

$$[r^{\tau+1}e^{(\tau-1)\theta i}B^\tau, B] \in \mathcal{A}_3.$$

Consequently, using

$$\begin{aligned} &\lambda^3 + a_1 A^{k_1} \lambda^2 + (a_2 A^{2k_1} + b_1^{-1} A^{1-k_1}) \lambda + A \\ &= (\lambda + b_1 A^{k_1})(\lambda^2 + b_2 A^{k_1} \lambda + b_1^{-1} A^{1-k_1}), \\ &\lambda^2 + b_2 A^{k_1} \lambda + b_1^{-1} A^{1-k_1} = \lambda^2 + r^{\tau+1} e^{(\tau-1)\theta i} B^\tau \lambda + B \quad \text{if } a_1^2 < 4a_2, \end{aligned}$$

we see by Basic Facts (ii) and (iii) that

$$[a_1 A^{k_1}, a_2 A^{2k_1} + r^{-1} e^{-i\theta} A^{1-k_1}, A] \in \mathcal{A}_3.$$

Since  $1 - k_1 < \frac{2}{3}$ , we claim using Theorem 2.5 again that

$$(3.4) \quad [a_1 A^{k_1}, a_2 A^{2k_1}, A] \in \mathcal{A}_3, \quad \frac{1}{3} < k_1 < \frac{1}{2}.$$

In conclusion, Corollary 2.2, combined with (3.1), (3.3), and (3.4), shows the “if part”. For the “only if part”, apply Theorems 3.1 and 2.5 and see that

$$[a_1 A^{k_1}, a_2 A^{k_2}, A] \notin \mathcal{A}_3 \quad \text{if } k_1 < \frac{1}{3} \text{ or } k_2 > \frac{2}{3}.$$

Furthermore, Corollary 2.4, together with Theorem 3.1, gives that

$$[a_1 A^{k_1}, a_2 A^{k_2}, A] \notin \mathcal{A}_3 \quad \text{if } k_2 < \frac{1}{2}(1 + k_1) \text{ or } k_2 > 2k_1.$$

Then referring to (3.2) ends the proof. Q.E.D.

*Remark.* If  $a_1, a_2 > 0$ ,  $k_1 \geq k_2$ , then  $[a_1 A^{k_1}, a_2 A^{k_2}, A] \notin \mathcal{A}_3$ . Indeed by virtue of Theorem 2.6  $[a_1 A^{k_1}, a_2 A^{k_2}, A] \in \mathcal{A}_3$  implies  $[a_1 A^{k_1}, 0, A] \in \mathcal{A}_3$ , which contradicts Theorem 3.1. Again by Theorem 2.6, if  $k_2 \geq 1$ , then  $[a_1 A^{k_1}, a_2 A^{k_2}, A] \in \mathcal{A}_3$  iff  $[a_1 A^{k_1}, a_2 A^{k_2}] \in \mathcal{A}_2$ .

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