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PARABOLICITY OF A CLASS OF HIGHER-ORDER ABSTRACT DIFFERENTIAL EQUATIONS

XIO TIJUN AND LIANG JIN

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ABSTRACT. Let E be a complex Banach space, $c_i \in \mathbb{C}$ $(1 \le i \le n-1)$, and A be a nonnegative operator in E. We discuss the parabolicity of the higher-order abstract differential equations

$$(*) \qquad u^{(n)}(t) + \sum_{i=1}^{n-1} c_i A^{k_i} u^{(n-i)}(t) + Au(t) = 0$$

and some perturbation cases of $(*)$. A sufficient and necessary condition for $(*)$ to be parabolic is obtained, provided $k_1 > k_2 - k_1 > \cdots > 1 - k_{n-1} > 0$, $c_i \neq 0$ $(1 \le i \le n-1)$. For A strictly nonnegative (Definition 1.3), $n = 3$, $c_1, c_2 \ge 0$, a sharp criterion is given.

1. Introduction and preliminaries

Throughout this paper, E will be a complex Banach space and $n \in N$ (the set of natural numbers). For $\theta \in (0, \pi/2]$ and $\omega \in R$ (the set of real numbers), write

$$\sum(\theta, \omega) = \{z \in \mathbb{C}: z \neq \omega, |\arg(z-\omega)| < \tfrac{\pi}{2} + \theta\},$$
$$\sum\nolimits_\theta = \{z \in \mathbb{C}: z \neq 0, |\arg z| < \theta\}.$$

Definition 1.1. Suppose $A_1, \dots, A_n$ are closed linear operators in E and $\theta \in (0, \pi/2]$. We say $[A_1, \dots, A_n] \in \mathscr{A}_n(\theta)$, if for each $\theta' \in (0, \theta)$ there exist $C_{\theta'}, \omega_{\theta'} > 0$ such that

$$(1.1) \qquad \left\| \lambda^{n-i} A_i \left(\lambda^n + \sum_{i=1}^{n} \lambda^{n-i} A_i \right)^{-1} \right\| \le C_{\theta'},$$

whenever $\lambda \in \sum(\theta', \omega_{\theta'})$, $1 \le i \le n$.

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Write $\mathscr{A}_n = \bigcup_{\theta\in(0,\pi/2]}\mathscr{A}_n(\theta)$. When $[A_1, \dots, A_n] \in \mathscr{A}_n$, we also say the abstract differential equation

$$u^{(n)}(t) + \sum_{i=1}^{n} A_i u^{(n-i)}(t) = 0 \tag{1.2}$$

is parabolic.

Clearly, when $[A_1, \dots, A_n] \in \mathscr{A}_n(\theta)$ $(\theta \in (0, \pi/2])$, (1.1) also holds for $i = 0$ $(A_0 = I$, the identity operator); $[A_1] \in \mathscr{A}_1(\theta)$ for some $\theta \in (0, \pi/2]$ iff $-A_1$ is the generator of an exponentially bounded holomorphic semigroup.

We note that parabolicity of equation (1.2) is 'comparable' with existence of an analytic exponentially bounded semigroup for the corresponding first-order system

$$v'(t) + G_n v(t) = 0, \tag{1.3}$$

in a proper B-space, where $v = (u_0, u_1, \dots, u_{n-1}) = (u, u', \dots, u^{(n-1)})$ and

$$G_n = \begin{pmatrix} 0 & -I & 0 & \cdots & 0 \\ 0 & 0 & -I & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & -I \\ A_n & A_{n-1} & A_{n-2} & \cdots & A_1 \end{pmatrix}. \tag{1.4}$$

It is known that the parabolicity of (1.2) ensures the existence and uniqueness of the solution of the Cauchy problem for (1.2) (see, e.g., [10, 11]). But under what condition is equation (1.2) parabolic? For second-order equations, this problem has been studied by many authors (see, e.g., [1–3, 5–9] and references therein); here (1.2) amounts to

$$u''(t) + A_1 u'(t) + A_2 u(t) = 0, \tag{1.5}$$

and (1.3) to

$$G_2 = \begin{pmatrix} 0 & -I \\ A_2 & A_1 \end{pmatrix}. \tag{1.6}$$

In [1] Chen and Russell posed two conjectures in their study of linear elastic systems with structure damping, which state, qualitatively, that $-G_2$ (in a suitable product space) generates an exponentially bounded analytic semigroup in the case where A_1, A_2 are positive and selfadjoint operators in a Hilbert space and the dissipation operator A_1 is 'comparable' with the $\frac{1}{2}$th power of the elastic operator A_2. Huang [6, 7] and, independently, Chen and Triggiani [2] proved these two conjectures; furthermore, they [3, 8, 9] discussed the general case where A_1 is 'comparable' with the αth power of A_2 over the entire range $0 \le \alpha \le 1$ of the parameter α. Recently, in the framework of Banach spaces, Favini and Obrecht [10] studied sufficient and necessary conditions ensuring equation (1.5) with $A_1 = \rho A_2^{\alpha}$ $(\rho \in \mathbb{C}, 0 < \alpha < 1)$ parabolic.

This paper aims at investigating the parabolicity of (1.2) for any n, but under the special condition $A_n = A > 0$, $A_i = c_i A^{k_i}$, $c_i \in \mathbb{C}$ $(1 \le i \le n-1)$, that is,

$$u^{(n)}(t) + \sum_{i=1}^{n-1} c_i A^{k_i} u^{(n-i)}(t) + Au(t) = 0. \tag{1.7}$$

First (in §2), assuming $c_i \neq 0$ $(1 \leq i \leq n-1)$, $k_1 > k_2 - k_1 > \cdots > 1 - k_{n-1} > 0$, we obtain a sufficient and necessary condition for (1.7) to be parabolic. Furthermore, some perturbation theorems are presented. Following this (in §3), we specialize to the case where A is strictly nonnegative, $n = 3$, $c_1, c_2 \geq 0$, and give a complete and clear answer for the problem of whether (1.7) is parabolic.

Definition 1.2. Suppose S is an arbitrary linear operator in E. S is nonnegative, if for each $\lambda > 0$, $\lambda \in \rho(-S)$ and

$$\sup\{\|\lambda(\lambda + S)^{-1}\| \colon \lambda > 0\} < +\infty.$$

It can be shown (cf. [4, Lemma 6.4.1]) that, if S is a nonnegative operator in E, then there exists $\theta \in (0, \pi/2]$ such that $\lambda \in \rho(-S)$ for each $\lambda \in \sum_\theta$ with $\{\|\lambda(\lambda + S)^{-1}\| \colon \lambda \in \sum_\theta\}$ bounded.

Let S be a nonnegative operator in E. Set as in [5]

$$\begin{aligned}
\theta_\infty^+(S) = \inf\{\theta \in (-\pi, \pi) \colon & \text{ there exist } C, \omega > 0 \text{ such that,} \\
& \text{for each } \lambda \text{ with } |\lambda| \geq \omega \text{ and } \theta \leq \arg\lambda \leq \pi, \\
& \qquad \lambda \in \rho(S) \text{ and } \|\lambda(\lambda - S)^{-1}\| \leq C\}, \\
\theta_\infty^-(S) = \sup\{\theta \in (-\pi, \pi) \colon & \text{ there exist } C, \omega > 0 \text{ such that,} \\
& \text{for each } \lambda \text{ with } |\lambda| \geq \omega \text{ and } -\pi \leq \arg\lambda \leq \theta, \\
& \qquad \lambda \in \rho(S) \text{ and } \|\lambda(\lambda - S)^{-1}\| \leq C\}.
\end{aligned}$$

Obviously, $\theta_\infty^+(S) \geq \theta_\infty^-(S)$; $[S] \in \mathscr{A}_1(\theta)$ $(\theta \in (0, \pi/2])$ iff $\theta_\infty^+(S) \leq \pi/2 - \theta$ and $\theta_\infty^-(S) \geq -\pi/2 + \theta$. It is not difficult to verify that, for $c \in \mathbb{C}$, cS is nonnegative iff either

(i) $\arg c < -\pi - \theta_\infty^+(S)$, or
(ii) $-\pi - \theta_\infty^-(S) < \arg c < \pi - \theta_\infty^+(S)$, or
(iii) $\arg c > \pi - \theta_\infty^-(S)$,

and if cS is nonnegative, we have

(1.8)
$$\theta_\infty^\pm(cS) = \begin{cases} \arg c + \theta_\infty^\pm(S) + 2\pi & \text{if } \arg c < -\pi - \theta_\infty^+(S), \\ \arg c + \theta_\infty^\pm(S) & \text{if } -\pi - \theta_\infty^-(S) < \arg c < \pi - \theta_\infty^+(S), \\ \arg c + \theta_\infty^\pm(S) - 2\pi & \text{if } \arg c > \pi - \theta_\infty^-(S). \end{cases}$$

Finally, for each $0 < \alpha < 1$, as pointed out in the proof of [5, Lemma 3.3],

$$\theta_\infty^\pm(S^\alpha) = \alpha\theta_\infty^\pm(S). \tag{1.9}$$

Definition 1.3. We say that S is strictly nonnegative if $\theta_\infty^\pm(S) = 0$.

2. Results for arbitrary order

Throughout this section, A will be a densely defined and nonnegative operator in E, $c_i \in \mathbb{C}$ $(1 \leq i \leq n-1)$, and

$$P_0(\lambda) = \lambda^n + \sum_{i=1}^{n-1} c_i A^{k_i} \lambda^{n-i} + A.$$

First, we state the well-known Moment inequality:

Let $0 \leq \alpha < \beta < \varepsilon \leq 1$. Then there exists a constant $C = C(\alpha, \beta, \varepsilon)$ such that

$$\|A^\beta u\| \leq C\|A^\varepsilon u\|^{(\beta-\alpha)/(\varepsilon-\alpha)}\|A^\alpha u\|^{(\varepsilon-\beta)/(\varepsilon-\alpha)} \qquad (u \in D(A^\varepsilon)). \tag{2.1}$$

Theorem 2.1. *Let* $k_1 > k_2 - k_1 > \cdots > k_{n-1} - k_{n-2} > 1 - k_{n-1} > 0$, $c_i \neq 0$ *for each* $1 \leq i \leq n-1$. *Then* $[c_1 A^{k_1}, \ldots, c_{n-1}A^{k_{n-1}}, A] \in \mathscr{A}_n(\theta)$ $(\theta \in (0, \pi/2])$ *iff, for each* $1 \leq i \leq n$, $[c_{i-1}^{-1}c_i A^{k_i - k_{i-1}}] \in \mathscr{A}_1(\theta)$, *where* $c_0 = c_n = 1$.

Proof. Sufficiency. Set $c_{i-1}^{-1}c_i = \tilde{c}_i$, $k_i - k_{i-1} = t_i$ $(1 \leq i \leq n)$,

$$P_1(\lambda) = \prod_{i=1}^{n}(\lambda + \tilde{c}_i A^{t_i}),$$

$$Q(\lambda) = \sum_{m=1}^{n-1} \sum_{(i_1, \ldots, i_m)\in I_m} \tilde{c}_{i_1} \cdots \tilde{c}_{i_m} A^{t_{i_1}+\cdots+t_{i_m}} \lambda^{n-m},$$

where, for each $1 \leq m \leq n-1$,

$$I_m = \{(i_1, \ldots, i_m) \colon 1 \leq i_1 < \cdots < i_m \leq n, (i_1, \ldots, i_m) = (1, \ldots, m)\}.$$

Then

$$t_1 > t_2 > \cdots > t_n, \quad k_m = \sum_{i=1}^{m} t_i \qquad (1 \leq m \leq n-1).$$

By hypothesis, for each $\theta' \in (0, \theta)$, there exist $C_{\theta'}, \omega_{\theta'} > 0$ such that

$$\|\lambda^{n-m}A^{t_{i_1}+\cdots+t_{i_m}}P_1^{-1}(\lambda)\| \leq C_{\theta'}, \qquad \|\lambda^n P_1^{-1}(\lambda)\| \leq C_{\theta'} \tag{2.2}$$

whenever $\lambda \in \sum(\theta', \omega_{\theta'})$, $1 \leq i_1 < \cdots < i_m \leq n$, $1 \leq m \leq n$. This together with (2.1) yields that, for each $\theta' \in (0, \theta)$, there exist $C, C_{\theta'}, \omega_{\theta'} > 0$ such that, for $\lambda \in \sum(\theta', \omega_{\theta'})$, $(i_1, \ldots, i_m) \in I_m$, $1 \leq m \leq n-1$,

$$\begin{aligned}
&\|\lambda^{n-m}A^{t_{i_1}+\cdots+t_{i_m}}P_1^{-1}(\lambda)\| \\
&\quad \leq C|\lambda|^{n-m}\|A^{k_m}P_1^{-1}(\lambda)\|^{(t_{i_1}+\cdots+t_{i_m})k_m^{-1}}\|P_1^{-1}(\lambda)\|^{1-(t_{i_1}+\cdots+t_{i_m})k_m^{-1}} \\
&\quad \leq C(C_{\theta'}|\lambda|)^{[(t_{i_1}+\cdots+t_{i_m})k_m^{-1}-1]m},
\end{aligned}$$

which approaches 0 as $|\lambda| \to \infty$.

Therefore, for each $\theta' \in (0, \theta)$, there is $\tilde{\omega}_{\theta'} > \omega_{\theta'}$ such that for $\lambda \in \sum(\theta', \tilde{\omega}_{\theta'})$

$$\|Q(\lambda)P_1^{-1}(\lambda)\| < \tfrac{1}{2}. \tag{2.3}$$

Thus using (2.2) again we obtain that, for each $\theta' \in (0, \theta)$, $\lambda \in \sum(\theta', \tilde{\omega}_{\theta'})$, $1 \leq m \leq n$,

$$\|c_m\lambda^{n-m}A^{k_m}P_0^{-1}(\lambda)\| = \|c_m\lambda^{n-m}A^{k_m}P_1^{-1}(\lambda)[I - Q(\lambda)P_1^{-1}(\lambda)]^{-1}\| \leq 2c_m C_{\theta'}.$$

In conclusion, $[c_1A^{k_1}, \ldots, c_{n-1}A^{k_{n-1}}, A] \in \mathscr{A}_n(\theta)$.

Necessity. Making use of (2.1) as in the proof of sufficiency, we obtain that, for each $\theta' \in (0, \theta)$, there exists $\omega_{\theta'} > 0$ such that, for $\lambda \in \sum(\theta', \omega_{\theta'})$, $(i_1, \ldots, i_m) \in I_m$, $1 \leq m \leq n$,

$$\|\lambda^{n-m}A^{t_{i_1}+\cdots+t_{i_m}}P_0^{-1}(\lambda)\| \leq C_{\theta'}|\lambda|^{[(t_{i_1}+\cdots+t_{i_m})k_m^{-1}-1]m}, \tag{2.4}$$

and therefore there exist $\tilde{\omega}_{\theta'} > \omega_{\theta'}$, $M_{\theta'} > 0$ such that, for $\lambda \in \sum(\theta', \tilde{\omega}_{\theta'})$, $1 \le m \le n$,

$$\begin{cases} \|Q(\lambda)P_0^{-1}(\lambda)\| < \frac{1}{2}, \\ \|\lambda(\lambda + \tilde{c}_m A^{t_m})^{-1} P_1(\lambda) P_0^{-1}(\lambda)\| \le M_{\theta'}. \end{cases}$$

Accordingly, for each $\theta' \in (0, \theta)$, $\lambda \in \sum(\theta', \omega_{\theta'})$, $1 \le m \le n$,

$$\|\lambda(\lambda + \tilde{c}_m A^{t_m})^{-1}\| = \|\lambda(\lambda + \tilde{c}_m A^{t_m})^{-1} P_1(\lambda) P_0^{-1}(\lambda)[I + Q(\lambda)P_0^{-1}(\lambda)]^{-1}\| \le 2M_{\theta'}.$$

This ends the proof. Q.E.D.

Corollary 2.2. *Let* $\theta_\infty^{\pm}(A) = 0$, $c_i > 0$ $(1 \le i \le n-1)$, *and* $k_1 > k_2 - k_1 > \cdots > k_{n-1} - k_{n-2} > 1 - k_{n-1} > 0$. *Then* $[c_1 A^{k_1}, \ldots, c_{n-1} A^{k_{n-1}}, A] \in \mathscr{A}_n(\pi/2)$.

The following are some perturbation cases.

Theorem 2.3. *Let* $B_1, \ldots, B_{n-1}$ *be closed linear operators in* E *satisfying that, for each* $1 \le m \le n-1$, *there is* l_m *with* $k_{m-1} < l_m < \frac{1}{2}(k_{m-1} + k_{m+1})$ *such that* $D(B_m) \supset D(A^{l_m})$. *Then if* $[c_1 A^{k_l}, \ldots, c_{n-1} A^{k_{n-1}}, A] \in \mathscr{A}_n(\theta)$ $(\theta \in (0, \pi/2])$, *so does* $[c_1 A^{k_1} + B_1, \ldots, c_{n-1} A^{k_{n-1}} + B_{n-1}, A]$.

Proof. By hypothesis, there is $C > 0$ such that, for each $1 \le m \le n-1$, $u \in D(A^{l_m})$,

$$\|B_m u\| \le C\|x\| + C\|A^{l_m} u\|.$$

So using (2.1) yields that, for each $\theta' \in (0, \theta)$, there exist $C_{\theta'}, \omega_{\theta'} > 0$ such that, for each $1 \le m \le n-1$, $\lambda \in \sum(\theta', \omega_{\theta'})$,

$$\begin{aligned} \|\lambda^{n-m} B_m P_0^{-1}(\lambda)\| &\le C|\lambda|^{n-m}\|P_0^{-1}(\lambda)\| + C|\lambda|^{n-m}\|A^{l_m} P_0^{-1}(\lambda)\| \\ &\le CC_{\theta'}|\lambda|^{-m} + C|\lambda|^{n-m}\|A^{k_{m-1}} P_0^{-1}(\lambda)\|^{\tau}\|A^{k_{m+1}} P_0^{-1}(\lambda)\|^{1-\tau} \\ &\le CC_{\theta'}(|\lambda|^{-m} + |\lambda|^{n-m}|\lambda|^{(m-n-1)\tau}|\lambda|^{(m-n+1)(1-\tau)}) \\ &= CC_{\theta'}(|\lambda|^{-m} + |\lambda|^{1-2\tau}) \end{aligned}$$

which approaches 0 as $|\lambda| \to \infty$, where $\tau = (l_m - k_{m-1})(k_{m+1} - k_{m-1})^{-1} < 1$. Consequently, for each $\theta' \in (0, \theta)$ there is $\tilde{\omega}_{\theta'} > \omega_{\theta'}$ such that, for $\lambda \in \sum(\theta', \tilde{\omega}_{\theta'})$,

$$\left\| \sum_{m=1}^{n-1} \lambda^{n-m} B_m P_0^{-1}(\lambda) \right\| < \frac{1}{2}.$$

This leads to the result as claimed. Q.E.D.

Corollary 2.4. *Let* $0 < k_1 < \cdots < k_{n-1} < 1$ *and* $k_j < \frac{1}{2}(k_{j-1} + k_{j+1})$ *for some* $1 \le j \le n-1$. *Then* $[c_1 A^{k_1}, \ldots, c_{n-1} A^{k_{n-1}}, A] \in \mathscr{A}_n(\theta)$ $(\theta \in (0, \pi/2])$ *implies* $[c_1 A^{k_1}, \ldots, c_{j-1} A^{k_{j-1}}, 0, c_{j+1} A^{k_{j+1}}, \ldots, c_{n-1} A^{k_{n-1}}, A] \in \mathscr{A}_n(\theta)$.

Theorem 2.5. *Let* $\tilde{c}_i \in \mathbb{C}$, $0 < l_i < i/n$ *for each* $1 \le i \le n-1$, $\theta \in (0, \pi/2]$. *Then* $[c_1 A^{k_1}, \ldots, c_{n-1} A^{k_{n-1}}, A] \in \mathscr{A}_n(\theta)$ *iff* $[c_1 A^{k_1} + \tilde{c}_1 A^{l_1}, \ldots, c_{n-1} A^{k_{n-1}} + \tilde{c}_{n-1} A^{l_{n-1}}, A] \in \mathscr{A}_n(\theta)$.

This theorem is an immediate consequence of the following (general) result by taking $i_m = n$, $\varepsilon_m = l_m$ $(1 \le m \le n-1)$, and using (2.1).

Theorem 2.6. *Assume* $[A_1, \dots, A_n] \in \mathscr{A}_n(\theta)$ *for some* $\theta \in (0, \pi/2]$, *and* B_m, $A_m + B_m$ *are closed and densely defined linear operators in* E $(1 \le m \le n)$. *If, for each* $1 \le m \le n$, *there exist* i_m, ε_m *with* $1 \le i_m \le n$, $0 < \varepsilon_m \le 1$ *such that* $D(B_m) \supset D(A_{i_m})$ *and, for each* $u \in D(A_{i_m})$,

$$\|B_m u\| \le C\|u\| + C\|A_{i_m} u\|^{\varepsilon_m} \|u\|^{1-\varepsilon_m} \quad \textit{for some } C > 0,$$

then, for each $\theta' \in (0, \theta)$, *there is* $\omega_{\theta'} > 0$ *such that, for* $1 \le m \le n$,

$$\sup_{\lambda \in \Sigma(\theta', \omega_{\theta'})} \|\lambda^{n-m} B_m P^{-1}(\lambda)\| < \begin{cases} \frac{1}{2} & \textit{if } i_m \varepsilon_m < m, \\ +\infty & \textit{if } i_m \varepsilon_m = m, \end{cases} \tag{2.5}$$

where $P(\lambda) = \lambda^n + \sum_{i=1}^n \lambda^{n-i} A_i$. *Furthermore, when* $i_m \varepsilon_m < m$, $[A_1 + B_1, \dots, A_n + B_n] \in \mathscr{A}_n(\theta)$.

Proof. Observing that, for each $\theta' \in (0, \theta)$, there exist $C_{\theta'}$, $\omega_{\theta'} > 0$ such that, for $\lambda \in \sum(\theta', \omega_{\theta'})$, $1 \le m \le n$,

$$\begin{aligned} &\|\lambda^{n-m} B_m P^{-1}(\lambda)\| \\ &\quad \le C|\lambda|^{n-m} \|P^{-1}(\lambda)\| + C|\lambda|^{n-m} \|A_{i_m} P^{-1}(\lambda)\|^{\varepsilon_m} \|P^{-1}(\lambda)\|^{1-\varepsilon_m} \\ &\quad \le C C_{\theta'} (|\lambda|^{-m} + |\lambda|^{n-m} |\lambda|^{(i_m - n)\varepsilon_m} |\lambda|^{-n(1-\varepsilon_m)}) \\ &\quad = C C_{\theta'} (|\lambda|^{-m} + |\lambda|^{i_m \varepsilon_m - m}), \end{aligned}$$

we obtain (2.5). The remaining part follows from the plain equality

$$\left(P(\lambda) + \sum_{m=1}^n \lambda^{n-m} B_m \right)^{-1} = P^{-1}(\lambda) \left[I + \sum_{m=1}^n \lambda^{n-m} B_m P^{-1}(\lambda) \right]^{-1}. \quad \text{Q.E.D.}$$

3. The case of $n = 3$

Throughout this section, A will be densely defined, unbounded, and strictly nonnegative. First we state several basic facts.

Basic Facts. For $0 < \beta \le 1$, $a > 0$, $\operatorname{Re} c > 0$, we have:

(i) $\theta_\infty^\pm(aA^\beta) = 0$;
(ii) $[cA^\beta] \in \mathscr{A}_1$, $[-cA^\beta] \notin \mathscr{A}_1$;
(iii) for $b \in R$, $[bA^{\beta/2}, aA^\beta] \in \mathscr{A}_2$ iff $b > 0$;
(iv) (see [5, Theorem 3.7]) let $\frac{1}{2} < \beta < 1$; then $[cA^\beta, A] \in \mathscr{A}_2$ iff

$$\begin{cases} \arg c > -\pi/2 + \max\{(1-\beta)\theta_\infty^+(A), -\beta\theta_\infty^-(A)\}, \\ \arg c < \pi/2 - \max\{\beta\theta_\infty^+(A), -(1-\beta)\theta_\infty^-(A)\}. \end{cases}$$

Theorem 3.1. *Let* $a_1, a_2 > 0$ *and* $0 < k_1, k_2 < 1$. *Then*

$$[a_1 A^{k_1}, 0, A], [0, a_2 A^{k_2}, A], [0, 0, A] \notin \mathscr{A}_3.$$

Proof. Observe that, for each $y_1 \ge 0$, the function

$$y(x) = x^{-1} + x(y_1 - x)$$

is continuous in $(0, +\infty)$, and $y \to +\infty$ as $x \to 0^+$, $y \to -\infty$ as $x \to +\infty$. Hence, for each $y_1, y_2 \ge 0$, there exists $x_1 > 0$ such that

$$y_2 = x_1^{-1} + x_1(y_1 - x_1).$$

Set $x_2 = y_1 - x_1$. If $x_2 > 0$, i.e., $y_1 > x_1$, then $y_2 > x^{-1}$; therefore, $y_1 y_2 > y_1 x_1^{-1} > 1$. If $x_2 \le 0$, i.e., $y_1 \le x_1$, then $y_2 \le x_1^{-1}$; therefore, $y_1 y_2 > y_1 x_1^{-1} > 1$. In other words,

$$x_2 > 0 \quad \text{if } y_1 y_2 > 1; \qquad x_2 \le 0 \quad \text{if } y_1 y_2 \le 1.$$

So from the equality

$$\lambda^3 + y_1 A^{1/3}\lambda^2 + y_2 A^{2/3}\lambda + A = (\lambda + x_1 A^{1/3})(\lambda^2 + x_2 A^{1/3}\lambda + x_1^{-1} A^{2/3}),$$

we see by Basic Facts (ii) and (iii)

$$[y_1 A^{1/3}, y_2 A^{2/3}, A] \in \mathscr{A}_3 \quad \text{if } y_1 y_2 > 1. \tag{3.1}$$

But

$$[y_1 A^{1/3}, y_2 A^{2/3}, A] \notin \mathscr{A}_3 \quad \text{if } y_1 y_2 \le 1. \tag{3.2}$$

In fact, if $[y_1 A^{1/3}, y_2 A^{2/3}, A] \in \mathscr{A}_3$ $(y_1 y_2 \le 1)$, then by virtue of (2.5) we have that there are C, $\omega > 0$, $\theta \in (0, \pi/2]$ such that, for $\lambda \in \sum(\theta, \omega)$, $i = 1, 2, 3$,

$$\|\lambda^{3-i} A^{i/3}(\lambda^3 + y_1 A^{1/3}\lambda^2 + y_2 A^{2/3}\lambda + A)^{-1}\| \le C.$$

According to this, the equality

$$(\lambda^2 + x_2 A^{1/3}\lambda + x_1^{-1} A^{2/3})^{-1} = (\lambda + x_1 A^{1/3})(\lambda^3 + y_1 A^{1/3}\lambda^2 + y_2 A^{2/3}\lambda + A)^{-1}$$

shows $[x_2 A^{1/3}, x_1^{-1} A^{2/3}] \in \mathscr{A}_2$, which contradicts Basic Fact (iii). So (3.2) holds. (3.2) indicates $[a_1 A^{1/3}, 0, A]$, $[0, a_2 A^{2/3}, A]$, $[0, 0, A] \notin \mathscr{A}_3$. Since $[0, 0, A] \notin \mathscr{A}_3$, using Theorem 2.5 yields that $[a_1 A^{k_1}, 0, A]$, $[0, a_2 A^{k_2}, A] \notin \mathscr{A}_3$ if $k_1 < \frac{1}{3}$, $k_2 < \frac{2}{3}$. Finally, we have that, for each $a > 0$, $0 < \beta < 1$,

$$[aA^\beta, a^{-1} A^{1-\beta}, A] \notin \mathscr{A}_3.$$

Indeed, if not, then the equality

$$(\lambda^2 + a^{-1} A^{1-\beta})^{-1} = (\lambda + aA^\beta)(\lambda^3 + aA^\beta \lambda^2 + a^{-1} A^{1-\beta}\lambda + A)^{-1}$$

yields $[0, a^{-1} A^{1-\beta}] \in \mathscr{A}_2$, which contradicts Basic Facts (iii). Thus, we conclude by Theorem 2.5 again that

$$[aA^\beta, 0, A] \notin \mathscr{A}_3 \quad \text{if } \beta > \tfrac{1}{3},$$
$$[0, a^{-1} A^{1-\beta}, A] \notin \mathscr{A}_3 \quad \text{if } \beta < \tfrac{1}{3}.$$

The proof is then complete. Q.E.D.

Theorem 3.2. *Let* $a_1, a_2 > 0$ *and* $0 < k_1 < k_2 < 1$. *Then* $[a_1 A^{k_1}, a_2 A^{k_2}, A] \in \mathscr{A}_3$ *iff either*

(i) $k_1 > \frac{1}{3}$, $\frac{1}{2}(1 + k_1) \le k_2 \le 2k_1$, *or*
(ii) $k_1 = \frac{1}{3}$, $k_2 = \frac{2}{3}$, $a_1 a_2 > 1$.

Proof. Observing

$$\begin{aligned} &\lambda^3 + (a_2 a_1^{-1} A^{(1-k_1)/2} + a_1 A^{k_1})\lambda^2 + (a_1^{-1} A^{1-k_1} + a_2 A^{(1+k_1)/2})\lambda + A \\ &\quad = (\lambda + a_1 A^{k_1})(\lambda^2 + a_2 a_1^{-1} A^{(1-k_1)/2}\lambda + a_1^{-1} A^{1-k_1}), \end{aligned}$$

we obtain

$$[a_2 a_1^{-1} A^{(1-k_1)/2} + a_1 A^{k_1}, a_1^{-1} A^{1-k_1} + a_2 A^{(1+k_1)/2}, A] \in \mathscr{A}_3.$$

Thus appealing to Theorem 2.5 gives

(3.3) $$[a_1A^{k_1}, a_2A^{(1+k_1)/2}, A] \in \mathscr{A}_3 \qquad (k_1 > \tfrac{1}{3}).$$

Next, let $\frac{1}{3} < k_1 < \frac{1}{2}$. Set $\tau = k_1(1-k_1)^{-1}$;

$$b_1 = \begin{cases} \frac{1}{2}[a_1 + (a_1^2 - 4a_2)^{1/2}] & \text{if } a_1^2 \geq 4a_2, \\ re^{i\theta} & \text{if } a_1^2 < 4a_2; \end{cases}$$

$$b_2 = \begin{cases} \frac{1}{2}[a_1 - (a_1^2 - 4a_2)^{1/2}] & \text{if } a_1^2 \geq 4a_2, \\ re^{-i\theta} & \text{if } a_1^2 < 4a_2, \end{cases}$$

where $\theta = \arccos(\frac{1}{2}a_1a_2^{-1/2})$, $r = a_2^{1/2}$,

$$B = r^{-1}e^{-i\theta}A^{1-k_2}.$$

Then $\theta_\infty^\pm(B) = -\theta$, $\frac{1}{2} < \tau < 1$, $b_1 + b_2 = a_1$, and $b_1b_2 = a_2$. Therefore, if $a_1^2 < 4a_2$,

$$\max\{(1-\tau)\theta_\infty^+(B), -\tau\theta_\infty^-(B)\} = \theta\tau,$$
$$\max\{\tau\theta_\infty^+(B), -(1-\tau)\theta_\infty^-(B)\} = \theta(1-\tau),$$

which implies by Basic Facts (iv) that

$$[r^{\tau+1}e^{(\tau-1)\theta i}B^\tau, B] \in \mathscr{A}_3.$$

Consequently, using

$$\begin{aligned} &\lambda^3 + a_1A^{k_1}\lambda^2 + (a_2A^{2k_1} + b_1^{-1}A^{1-k_1})\lambda + A \\ &\quad = (\lambda + b_1A^{k_1})(\lambda^2 + b_2A^{k_1}\lambda + b_1^{-1}A^{1-k_1}), \end{aligned}$$

$$\lambda^2 + b_2A^{k_1}\lambda + b_1^{-1}A^{1-k_1} = \lambda^2 + r^{\tau+1}e^{(\tau-1)\theta i}B^\tau\lambda + B \quad \text{if } a_1^2 < 4a_2,$$

we see by Basic Facts (ii) and (iii) that

$$[a_1A^{k_1}, a_2A^{2k_1} + r^{-1}e^{-i\theta}A^{1-k_1}, A] \in \mathscr{A}_3.$$

Since $1 - k_1 < \frac{2}{3}$, we claim using Theorem 2.5 again that

(3.4) $$[a_1A^{k_1}, a_2A^{2k_1}, A] \in \mathscr{A}_3, \qquad \tfrac{1}{3} < k_1 < \tfrac{1}{2}.$$

In conclusion, Corollary 2.2, combined with (3.1), (3.3), and (3.4), shows the "if part". For the "only if part", apply Theorems 3.1 and 2.5 and see that

$$[a_1A^{k_1}, a_2A^{k_2}, A] \notin \mathscr{A}_3 \quad \text{if } k_1 < \tfrac{1}{3} \text{ or } k_2 > \tfrac{2}{3}.$$

Furthermore, Corollary 2.4, together with Theorem 3.1, gives that

$$[a_1A^{k_1}, a_2A^{k_2}, A] \notin \mathscr{A}_3 \quad \text{if } k_2 < \tfrac{1}{2}(1+k_1) \text{ or } k_2 > 2k_1.$$

Then referring to (3.2) ends the proof. Q.E.D.

Remark. If $a_1, a_2 > 0$, $k_1 \geq k_2$, then $[a_1A^{k_1}, a_2A^{k_2}, A] \notin \mathscr{A}_3$. Indeed by virtue of Theorem 2.6 $[a_1A^{k_1}, a_2A^{k_2}, A] \in \mathscr{A}_3$ implies $[a_1A^{k_1}, 0, A] \in \mathscr{A}_3$, which contradicts Theorem 3.1. Again by Theorem 2.6, if $k_2 \geq 1$, then $[a_1A^{k_1}, a_2A^{k_2}, A] \in \mathscr{A}_3$ iff $[a_1A^{k_1}, a_2A^{k_2}] \in \mathscr{A}_2$.

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DEPARTMENT OF MATHEMATICS, YUNNAN TEACHERS' UNIVERSITY, AND YUNNAN INSTITUTE OF APPLIED MATHEMATICS, KUNMING, PEOPLE'S REPUBLIC OF CHINA

TEACHING AND RESEARCH SECTION OF MATHEMATICS, KUNMING INSTITUTE OF TECHNOLOGY, AND YUNNAN INSTITUTE OF APPLIED MATHEMATICS, KUNMING, PEOPLE'S REPUBLIC OF CHINA