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HERMITIAN $*$-EINSTEIN SURFACES

GEO GRANTCHAROV AND OLEG MUŠKAROV

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ABSTRACT. We study the problem when a compact Hermitian $*$-Einstein surface M is Kählerian and show that it is true if M is additionally assumed to be either Einstein or anti-self-dual. We also prove that if the $*$-scalar curvature of M is positive then M is a conformally Kähler surface with positive first Chern class.

1. INTRODUCTION

A well-known conjecture of Goldberg [Go] says that any compact almost Kähler Einstein manifold is Kählerian. This conjecture is still open, but some progress has been made under additional curvature conditions [Go, O1, S1, SV]. In particular, Sekigawa and Vanhecke [SV] have proved that the Goldberg conjecture is true for four-dimensional $*$-Einstein manifolds with constant $*$-scalar curvature.

The purpose of this note is to obtain Hermitian analogs of the above-mentioned result. More specifically, we study the problem when a compact Hermitian $*$-Einstein surface is Kählerian. Our interest in this problem was also motivated by the following observations. First, any compact Hermitian $*$-Einstein surface with positive $*$-scalar curvature is globally conformally Kähler (cf. Theorem 3.1). Second, the Page metric on the complex surface F_1 [B, 11.82] is Einstein $*$-Einstein and with nonconstant positive $*$-scalar curvature. Since the Page metric is non-Kähler, the following conjecture seems to be true: *Any compact Hermitian $*$-Einstein surface M with constant positive $*$-scalar curvature is Kählerian.*

In this note we prove the above conjecture under the additional assumption that M is either Einstein or anti-self-dual (Corollaries 4.2 and 4.3). These results are consequences of the following

Theorem 1.1. *Let M be a compact Hermitian Einstein and $*$-Einstein surface. Then either*

(i) *M is Kählerian, or*

(ii) *the metric of M is conformal to an extremal Kähler metric with nonconstant positive scalar curvature. Moreover, the scalar curvature of M is a positive constant, and the $*$-scalar curvature of M is positive and nonconstant.*

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The converse statement is also true.

We should note that the proof of Theorem 1.1 relies heavily on some results of Derdzinski [D], Vaisman [V], Sekigawa [S2], and Kodâ [K].

2. Preliminaries

Let $M = (M, J, g)$ be a Hermitian surface (i.e., a Hermitian manifold of real dimension four) with complex structure J and compatible Riemannian metric g. Denote by Ω the Kähler form of M given by $\Omega(X, Y) = g(X, JY)$ for $X, Y \in \chi(M)$, where $\chi(M)$ is the Lie algebra of all smooth vector fields on M. We assume that M is oriented by the volume form $dM = \frac{1}{2}\Omega^2$. It is well known [V] that $d\Omega = \omega \wedge \Omega$, where $\omega = \delta\Omega \circ J$ is the Lee form of M. Recall [V] that M is called locally conformally Kähler provided $d\omega = 0$.

Let ∇, R, ρ, and τ be the Riemannian connection, the Riemannian curvature tensor, the Ricci tensor, and the scalar curvature of M, respectively. Here

$$R(X, Y, Z, W) = g([\nabla_X, \nabla_Y]Z - \nabla_{[X, Y]}Z, W),$$
$$\rho(X, Y) = \sum_s R(E_s, X, Y, E_s), \qquad \tau = \sum_s \rho(E_s, E_s)$$

for $X, Y, Z, W \in \chi(M)$, where $\{E_s\}$, is an orthonormal frame. Furthermore we denote by ρ^* and τ^* the $*$-Ricci tensor and the $*$-scalar curvature of M defined respectively by

$$\rho^*(X, Y) = \sum_s R(E_s, X, JY, JE_s), \qquad \tau^* = \sum_s \rho^*(E_s, E_s).$$

By using the first Bianchi identity we get

$$\rho^*(X, Y) = -\frac{1}{2}\sum_s R(X, JY, E_s, JE_s). \tag{2.1}$$

Note that on a Kähler manifold the $*$-Ricci tensor and the Ricci tensor coincide. We shall say that M is a $*$-Einstein surface if the $*$-Ricci tensor is a functional multiple of the metric g, i.e., if $\rho^* = \tau^*/4 \cdot g$. Note that in contrast to Einstein manifolds the $*$-scalar curvature of a $*$-Einstein surface need not be a constant. On any Hermitian surface the Ricci tensor and the $*$-Ricci tensor are related by

$$\rho(X, Y) + \rho(JX, JY) - \rho^*(X, Y) - \rho^*(JX, JY) = \frac{(\tau - \tau^*)}{2} \cdot g(X, Y) \tag{2.2}$$

for $X, Y \in \chi(M)$ (cf., e.g., [TV]). We also have [V]

$$\tau - \tau^* = 2\delta\omega + \|\omega\|^2. \tag{2.3}$$

Denote by γ_1 the generalized first Chern form of M [S2]. Then

$$8\pi\gamma_1(X, Y) = -\sum_s (2R(X, Y, E_s, JE_s) + g((\nabla_X J)(E_s), (\nabla_{JY} J)(E_s))),$$

and by (2.1) we get

$$8\pi\gamma_1(X, Y) = -4\rho^*(X, JY) - \sum_s g((\nabla_X J)(E_s), (\nabla_{JY} J)(E_s)). \tag{2.4}$$

Note that when M is compact the 2-form γ_1 represents the first Chern class of M in the de Rham cohomology group.

The Riemannian metric g induces a metric on the bundle $\bigwedge^2 TM$ of 2-vectors by $\langle X_1 \wedge X_2, X_3 \wedge X_4 \rangle = \det(g(X_i, X_j))$. The curvature operator $\mathscr{R}$ is the selfadjoint endomorphism of $\bigwedge^2 TM$ defined by $\langle \mathscr{R}(X \wedge Y), Z \wedge W \rangle = R(X, Y, Z, W)$. The Hodge star operator defines an endomorphism $*$ of $\bigwedge^2 TM$ with $*^2 = \mathrm{Id}$. Hence $\bigwedge^2 TM = \bigwedge^2_+ TM \oplus \bigwedge^2_- TM$, where $\bigwedge^2_\pm TM$ are the subbundles of $\bigwedge^2 TM$ corresponding to the (± 1)-eigenvalues of $*$. The block-decomposition of $\mathscr{R}$ with respect to the above splitting of $\bigwedge^2 TM$ is

$$\mathscr{R} = \begin{bmatrix} \tau/12 \cdot \mathrm{Id} + \mathscr{W}_+ & \mathscr{B} \\ {}^t\mathscr{B} & \tau/12 \cdot \mathrm{Id} + \mathscr{W}_- \end{bmatrix} \tag{2.5}$$

where τ is the scalar curvature; $\mathscr{B}$ and $\mathscr{W} = \mathscr{W}_+ + \mathscr{W}_-$ represent the traceless Ricci tensor and the Weyl conformal tensor, respectively. Recall that M is called (anti-) self-dual if $(\mathscr{W}_+)\mathscr{W}_- = 0$.

3. Conformally Kähler $*$-Einstein surfaces

Recall [V] that a Hermitian surface (M, J, g) is called (globally) conformally Kähler if the metric g is conformal to a Kähler metric with respect to J.

Theorem 3.1. *Any compact Hermitian $*$-Einstein surface with positive $*$-scalar curvature is conformally Kähler.*

Proof. We first prove the following lemma.

Lemma 3.2. *Any compact Hermitian $*$-Einstein surface is locally conformally Kähler.*

Proof of the lemma. Let $M = (M, J, g)$ be a compact Hermitian $*$-Einstein surface with Lee form ω and Riemannian curvature tensor R. For any orthonormal J-frame $\{E_s\}$ (i.e., $JE_1 = E_2$, $JE_2 = -E_1$, $JE_3 = E_4$, $JE_4 = -E_3$) we set

$$F_1 = (E_1 - iE_2)/\sqrt{2}, \qquad F_{\bar{1}} = (E_1 + iE_2)/\sqrt{2},$$
$$F_2 = (E_3 - iE_4)/\sqrt{2}, \qquad F_{\bar{2}} = (E_3 + iE_4)/\sqrt{2},$$

and $K_{abcd} = R(F_a, F_b, F_c, F_d)$, where the curvature tensor R is continued by complex linearity. Then a direct computation involving (2.1) shows that M is $*$-Einstein iff

$$K_{121\bar{1}} + K_{122\bar{2}} = 0, \qquad K_{1\bar{2}1\bar{1}} + K_{1\bar{2}2\bar{2}} = 0, \qquad K_{1\bar{1}1\bar{1}} - K_{2\bar{2}2\bar{2}} = 0. \tag{3.1}$$

The first identity of (3.1) together with a result of Kodâ [K, Lemma 6.3] implies that $d\omega$ is an anti-self-dual 2-form. Hence $d\omega = 0$ since M is compact.

Now we are ready to prove the theorem. Let M be a Hermitian $*$-Einstein surface with $\tau^* > 0$ everywhere on M. Consider the Riemannian metric g^* on M defined by

$$g^*(X, Y) = \tau^* g(X, Y) + \sum_s g((\nabla_X J)(E_s), (\nabla_Y J)(E_s)), \tag{3.2}$$

where $\{E_s\}$ is an orthonormal frame. It follows easily that the metric g^* is compatible with the complex structure J since for any Hermitian manifold we have $(\nabla_{JX}J)(Y) = J(\nabla_X J)(Y)$ (cf., e.g., [Gr]). From (3.2), (2.4), and the fact that M is a $*$-Einstein surface we deduce that the Kähler form Ω^* of (M, J, g^*) is given by $\Omega^* = -8\pi\gamma_1$, where γ_1 is the generalized first Chern form of M. Hence $d\Omega^* = 0$, which shows that g^* is a Kähler metric on M with respect to J. In particular, the first Betti number of M is even, and the theorem follows by a result of Vaisman [V].

Remarks. (1) From (2.4) it follows that the first Chern class of M is positive, i.e., M is an algebraic surface. This is another way to prove the existence of a Kähler metric on M.

(2) By the classification of compact complex surfaces with positive first Chern class (cf., e.g., [B, 11.13]) it follows that the only surfaces on which the existence of $*$-Einstein metrics with positive $*$-scalar curvature can be expected are $CP^1 \times CP^1$ and Σ_r, $0 \leq r \leq 8$.

Theorem 3.1 reduces the problem of describing the compact Hermitian $*$-Einstein surfaces with positive $*$-scalar curvature to the problem when the conformal class of a Kähler metric contains a $*$-Einstein metric. The next theorem gives a partial answer to this question.

Theorem 3.3. *Let M be a compact Kähler surface with Kähler form Ω, Ricci form γ, and scalar curvature τ. Let $F > 0$ be a smooth function on M. Then*

(i) *the metric $\tilde{g} = F^{-2} \cdot g$ is $*$-Einstein iff*

$$(3.3) \qquad (\tau/4 - \Delta F/F) \cdot \Omega = \gamma - 2i\partial\overline{\partial}F/F,$$

where Δ is the Laplace operator. Moreover, the $*$-scalar curvature of $\tilde{g}$ is given by

$$(3.4) \qquad \tilde{\tau}^* = F^2 \cdot \tau - 2F\Delta F - 4\|dF\|^2.$$

(ii) *The conformal class of g contains at most one (up to a constant) $*$-Einstein metric.*

Proof. To prove (i) we continue all tensor fields on M by complex linearity. For a tensor field Z of type $(0, 2)$ we denote by $Z^{1,1}$ its $(1, 1)$-part given by $Z^{1,1}(X, Y) = \frac{1}{2}(Z(X, Y) + Z(JX, JY))$ for all complex vector fields X, Y on M. Let $\rho_0(\tilde{\rho}_0)$ and $\rho_0^*(\tilde{\rho}_0^*)$ be the traceless Ricci tensor and the traceless $*$-Ricci tensor of the metric $g(\tilde{g})$, respectively.

Lemma 3.4. *The metric $\tilde{g}$ is $*$-Einstein iff $\tilde{\rho}_0^{1,1} = 0$.*

Proof of the lemma. Denote by ω the 1-form on M defined by $\omega = d(\log F^2)$. It is well known (cf., e.g., [O2]) that the $*$-Ricci tensors of the metrics g and $\tilde{g}$ are related by

$$(3.5) \qquad \tilde{\rho}^*(X, Y) = \rho^*(X, Y) + 1/2(P(X, Y) + P(JX, JY))$$

for $X, Y \in \chi(M)$, where

$$(3.6) \qquad P(X, Y) = (\nabla_X\omega)(Y) + 1/2 \cdot \omega(X)\omega(Y) - 1/4 \cdot \|\omega\|^2 \cdot g(X, Y).$$

Moreover,

$$(3.7) \qquad F^2\tilde{\tau}^* = \tau^* - \delta\omega - 1/2 \cdot \|\omega\|^2.$$

Assume that $\tilde{g}$ is a $*$-Einstein metric. Then $(\tilde{\rho}_0^*)^{1,1} = 0$, and (2.2) shows that $(\tilde{\rho}_0)^{1,1} = 0$. Conversely, let $(\tilde{\rho}_0)^{1,1} = 0$. Then from (2.2) it follows that $(\tilde{\rho}_0^*)^{1,1} = 0$. On the other hand from (3.5) and the fact that g is a Kähler metric (i.e., $\rho = \rho^*$) we get $\tilde{\rho}_0^*(X, Y) = \tilde{\rho}_0^*(JX, JY)$. Hence $\tilde{\rho}_0^* = 0$, which simply means that $\tilde{g}$ is a $*$-Einstein metric.

To prove statement (i) we first recall that the traceless Ricci tensors of the metrics g and $\tilde{g} = F^{-2}g$ are related by (cf. [B, (1.161b)])

$$\tilde{\rho}_0 = 2/F \cdot (D\,dF + \Delta F/F \cdot g) + \rho_0. \tag{3.8}$$

Hence the first part of (i) follows from (3.8) and Lemma 3.4 using the fact that for Kähler manifolds the following identities hold:

$$(D\,dF)^{1,1}(X, JY) = -i(\partial\overline{\partial}F)(X, Y), \qquad (\rho_0)^{1,1} = \rho_0.$$

Formula (3.4) is an easy consequence of (3.7).

To prove (ii) let F and H be positive smooth functions on M such that the metrics $F^{-2}g$ and $H^{-2}g$ are both $*$-Einstein. Then (3.3) implies $(H\Delta F - F\Delta H)\Omega = 2i(H\partial\overline{\partial}F - F\partial\overline{\partial}H)$, and taking into account the fact that the 2-forms Ω and $H\partial\overline{\partial}F - F\partial\overline{\partial}H$ are $\partial\overline{\partial}$-closed, we get $\partial\overline{\partial}(H\Delta F - F\Delta H)\wedge\Omega = 0$. On the other hand,

$$\begin{aligned}\Delta(H\Delta F - F\Delta H)\cdot dM &= 2i\langle\partial\overline{\partial}(H\Delta F - F\Delta H), \Omega\rangle \cdot dM \\ &= 2i\partial\overline{\partial}(H\Delta F - F\Delta H)\wedge\Omega\end{aligned}$$

since $*\Omega = \Omega$. Hence $\Delta(H\Delta F - F\Delta H) = 0$, which implies $(H\Delta F - F\Delta H) \equiv$ const, since M is compact. Now integrating over M gives

$$(H\Delta F - F\Delta H) = 0. \tag{3.9}$$

An easy computation involving the definition of the Laplace operator shows that $\Delta(F/H) = 2\cdot\langle d(F/H), d(\log H)\rangle + 1/h^2\cdot(H\Delta F - F\Delta H)$. This, together with (3.9), implies that $\Delta U - 2\cdot\langle dU, d(\log H)\rangle = 0$, where $U = F/H$. Now by the maximum principle for second order strongly elliptic differential operators we conclude that $U \equiv$ const, and statement (ii) is proved.

An immediate consequence of Theorem 3.3(ii) is the following.

Corollary 3.4. *Let (M, J, g) be a Kähler-Einstein surface. Then the only $*$-Einstein metrics in the conformal class of g are the constant multiples of g.*

4. Proof of Theorem 1.1 and applications

We begin the proof of Theorem 1.1 with the following

Lemma 4.1. *A compact Hermitian Einstein surface M is $*$-Einstein iff it is locally conformally Kähler.*

Proof of the lemma. Using the same notation as in the proof of Lemma 3.2 we check easily that M is Einstein iff

$$\begin{aligned} &K_{121\bar{2}} = K_{12\bar{1}2} = 0, \qquad K_{1\bar{1}1\bar{1}} = K_{2\bar{2}2\bar{2}}, \\ &K_{1\bar{2}1\bar{1}} + K_{1\bar{2}2\bar{2}} = 0, \qquad K_{121\bar{1}} = K_{122\bar{2}}. \end{aligned} \tag{4.1}$$

On the other hand by [K, Lemma 6.3] and the fact that M is compact it follows that M is locally conformally Kähler iff

$$K_{121\bar{1}} + K_{122\bar{2}} = 0. \tag{4.2}$$

Hence the lemma follows from (3.1), (4.1), and (4.2).

Now let M be a compact Hermitian Einstein and $*$-Einstein surface. Then by a result of Derdzinski [D, Proposition 5] it follows that either $\mathscr{W}_+ = 0$ identically or $\mathscr{W}_+ \neq 0$ everywhere on M. Consider the case $\mathscr{W}_+ \equiv 0$. Then by [K] we get $\tau = 3\tau^*$, i.e., τ^* is a constant. Assume that $\tau^* < 0$. Then a result of Sekigawa [S2] says that M is a Kähler surface. Hence $\tau = \tau^* = 0$, which is a contradiction. If $\tau^* = 0$ then $\tau = 0$ and (2.3) implies $2\delta\omega + \|\omega|^2 = 0$. Integrating over M gives $\int_M \|\omega\|^2 \, dM = 0$, which shows that M is a Kähler surface. It remains to consider the case $\tau^* > 0$. Then by Theorem 3.1 it follows that M is conformally Kähler, i.e., $\omega = dF$, where F is a positive smooth function on M. Using (2.3) and $\tau = 3\tau^*$ we get

$$2\tau^* = 2\Delta F + \|dF\|^2. \tag{4.3}$$

Let $F(p_0) = \min_{p\in M} F(p)$. Then $(dF)_{p_0} = 0$, $(\Delta F)_{p_0} \leq 0$, and 4.3 implies $\tau^* \leq 0$, which is a contradiction. The arguments above show that if $\mathscr{W}_+ = 0$ then M is a Ricci-flat Kähler surface.

Now consider the case when $\mathscr{W}_+ \neq 0$ everywhere on M. We shall show that $\|\mathscr{W}_+\|^{2/3} \cdot g$ is a Kähler metric on M. By Lemma 4.2 we know that M is locally conformally Kähler. This means that for any point $p \in M$ there exist a connected neighbourhood U of p and a positive smooth function F on U such that $F \cdot g$ is a Kähler metric with respect to J on U. Since M is Einstein by a result of Derdzinski [D, Proposition 5 (ii)], it follows that there is a constant C_U such that $F = C_U \|\mathscr{W}_+\|^{2/3}$. Hence $\|\mathscr{W}_+\|^{2/3} \cdot g$ is a Kähler metric on M. Suppose that the tensor field $\mathscr{W}_+$ is parallel. Then from [D, Proposition 5 (iii)] it follows that $\|\mathscr{W}_+\|^{2/3}$ is a constant, i.e., g is a Kähler metric. It remains to consider the case when $\mathscr{W}_+$ is not parallel. Now from [D, Theorem 2(ii)] we deduce that $\tilde{g} = \|\mathscr{W}_+\|^{2/3} \cdot g$ is an extremal Kähler metric whose scalar curvature $\tilde{\tau}$ is nonconstant and positive everywhere on M. Moreover

$$\tau = \tilde{\tau}^3 - 6\tilde{\tau}\tilde{\Delta}\tilde{\tau} - 12\tilde{g}(d\tilde{\tau}, d\tilde{\tau}) > 0.$$

On the other hand, from (3.4) it follows that

$$\tau^* = \tilde{\tau}^3 - 2\tilde{\tau}\tilde{\Delta}\tilde{\tau} - 4\tilde{g}(d\tilde{\tau}, d\tilde{\tau}).$$

Hence $3\tau^* = 2\tilde{\tau}^3 + \tau > 0$, and statement (ii) is proved. The converse statement follows by Lemma 4.2 and the results of Derdzinski [D].

From Theorem 4.1 we obtain the following Hermitian analogs of the result of Sekigawa and Vanhecke [SV] mentioned in the introduction.

Corollary 4.2. *Any compact Hermitian Einstein and $*$-Einstein surface with constant $*$-scalar curvature is Kählerian.*

Corollary 4.3. *Any compact Hermitian anti-self-dual and $*$-Einstein surface with constant $*$-scalar curvature is Kählerian and Ricci-flat.*

Proof. From (2.2) it follows that the Ricci tensor ρ satisfies the following identity: $\rho(X, Y) + \rho(JX, JY) = \tau/2 \cdot g(X, Y)$ for all $X, Y \in \chi(M)$. This, together with (2.5) and an integral formula of Sekigawa [S2, (3.29)], gives

$$\int_M (\|\mathscr{W}_+\|^2 + \tau^2/48 - (\tau^*)^2/16 - \|\mathscr{B}\|^2) \, dM = 1/16 \cdot \int_M \tau^* \|\omega\|^2 \, dM. \tag{4.4}$$

On the other hand, $\tau = 3\tau^*$ since $\mathscr{W}_+ = 0$ (cf. [K]). Hence from (2.3), (4.4), and the fact that τ^* is a constant we get $\int_M \|\mathscr{B}\|^2 \, dM = 0$. Therefore, g is an Einstein metric, and the corollary follows by [K] and Corollary 4.2.

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INSTITUTE OF MATHEMATICS, BULGARIAN ACADEMY OF SCIENCES, UL. "ACAD. G. BONTCHEV", BL.8, 1090 SOFIA, BULGARIA