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TORSION UNITS IN INTEGRAL GROUP RINGS

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ABSTRACT. Let $G = \langle a \rangle \rtimes X$ where $\langle a \rangle$ is a cyclic group of order n, X is an abelian group of order m, and $(n, m) = 1$. We prove that if $\mathbb{Z}G$ is the integral group ring of G and H is a finite group of units of augmentation one of $\mathbb{Z}G$, then there exists a rational unit γ such that $H^\gamma \subseteq G$.

Let G be a finite group, $\mathbb{Z}G$ the integral group ring of G, and $U_1\mathbb{Z}G$ the group of units of augmentation one in $\mathbb{Z}G$. It has been conjectured by Zassenhaus that if H is a finite subgroup of $U_1\mathbb{Z}G$, then H is conjugate to a subgroup of G by a rational unit, i.e., there exists $\gamma \in U\mathbb{Q}G$ such that $H^\gamma \subseteq G$.

This conjecture has been confirmed by Weiss in [8] for p-groups. In this note we shall prove this conjecture for a certain class of metabelian groups. More precisely, we will establish the following result:

Theorem. *Let G be a split extension $\langle a \rangle \rtimes X$, where $\langle a \rangle$ is a cyclic group of order n, X is an abelian group of order m, and $(n, m) = 1$. If $H \subseteq U_1\mathbb{Z}G$ is a finite group, then there exists $\gamma \in U\mathbb{Q}G$ such that $H^\gamma \subseteq G$.*

Before proceeding to the proof of the theorem we record some useful facts that will be needed.

If N is a normal subgroup of G, let us denote by $\Delta(G, N)$ the kernel of the natural map $\mathbb{Z}G \to \mathbb{Z}(G/N)$. Also we briefly write $u \sim g$ to indicate that u is conjugate in $\mathbb{Q}G$ to g.

Lemma 1. *Let $G = A \rtimes X$, where A is an abelian normal p-group and X is any group with $(|A|, |X|) = 1$. Let $u \in U_1\mathbb{Z}G$ be a unit of the form $u = vw$, where $v \in U(1 + \Delta(G, A))$, $w \in U_1\mathbb{Z}X$. If u has finite order not divisible by p, then $u \sim w$.*

Proof. See [3, Lemma 2].

Lemma 2. *Let $G = \langle a \rangle \rtimes X$, where $o(a) = n$, $|X| = m$, and $(n, m) = 1$. If an element $a^i x$ of G, where $x \in X$, is of order divisible by all primes dividing n, then x is central in G.*

Proof. It is a consequence of [4, Lemma 2.3].

Lemma 3. *Let G be a split extension $\langle a \rangle \rtimes X$, where $\langle a \rangle$ is a cyclic group of order n, X is an abelian group of order m, and $(n, m) = 1$. If G_0 is a*

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subgroup of G, *then* $G_0 = \langle b \rangle \rtimes X_0$, *where* $b \in \langle a \rangle$ *and* X_0 *is isomorphic to a subgroup of* X.

Proof. If $\varphi\colon G_0 \to X$ is such that $a^i x \to x$, then φ is a homomorphism and $\operatorname{Ker}\varphi = \langle b \rangle$ for some $b \in \langle a \rangle$. Hence $G_0/\operatorname{Ker}\varphi \cong \varphi(G_0)$ and $G_0 = \langle b \rangle \rtimes X_0$, with X_0 isomorphic to a subgroup of X.

The proof of the theorem will be based on the following reduction:

Lemma 4. *Let* G *be a finite group,* G_0 *a subgroup of* G, *and* H *a finite subgroup of* $U\mathbb{Z}G$. *Suppose that there exists an isomorphism* $\varphi\colon H \to G_0$ *such that, for all* $h \in H$ *and for all complex irreducible characters* χ *of* G, $\chi(h) = \chi(\varphi(h))$. *Then* $G_0 \sim H$.

Proof. This has been proved when H is a cyclic subgroup in [4, 5]. The same argument gives a proof in general. See also [7, Lemma 4.6].

We can now prove the main result of this note.

Proof of the theorem. Let $G = \langle a \rangle \rtimes X$, where $o(a) = n$, $|X| = m$, and $(n, m) = 1$; also let H be a finite subgroup of $U_1\mathbb{Z}G$. In order to prove the theorem we will construct an isomorphism φ of H onto a subgroup of G satisfying the criterion of Lemma 4; that is, φ will be such that, for all irreducible characters χ of G, $\chi(h) = \chi(\varphi(h))$ for all $h \in H$.

By Whitcomb's argument given on [6, p. 103] it follows that, for all $h \in H$, there exists an element g_h of G such that $h \equiv g_h \pmod{\Delta G \Delta\langle a \rangle}$. Since in the metabelian group case being considered it has been proved by Cliff, Sehgal, and Weiss in [1] that $U(1 + \Delta(G, \langle a \rangle))$ is torsion-free, it follows that the torsion subgroup H of $U_1\mathbb{Z}G$ is isomorphic to a subgroup G_0 of G. Let $\alpha\colon H \to G_0$ be the above isomorphism defined by $\alpha(h) = g_h$. By Lemma 3, $G_0 = \langle b \rangle \rtimes X_0$ where $b \in \langle a \rangle$ and X_0 is a group isomorphic to a subgroup of X. Taking preimages, we can write $H = \langle u \rangle \rtimes K$, where $\alpha(u) = b$ and $\alpha(K) = X_0$.

Since $o(u)$ divides $o(a)$, by [2, Theorem 1.1], we have that $u \sim g$ for some $g \in \langle a \rangle$. Also, since α is an isomorphism and $u \sim g$, we have $o(g) = o(u) = o(b)$. Hence $g = b^i$ for some i and $u \sim b^i$.

Now let $k \in K$. We have $\alpha(k) = a^j x$ for some j and some $x \in X$. Note that if $k_1 \neq k_2$ and $\alpha(k_1) = a^{j_1} x$, $\alpha(k_2) = a^{j_2} x$, then $x_1 \neq x_2$ (since otherwise $\alpha(k_1 k_2^{-1}) \in \langle a \rangle$, contradicting the relative primeness of $|K|$ and $o(a)$).

We shall prove by induction on the number of different primes dividing $o(a)$ that $k \sim x$. Since $k \equiv a^j x \pmod{\Delta G \Delta\langle a \rangle}$, it follows that $k = (1 + \delta)x$ for some $\delta \in \Delta G \Delta\langle a \rangle$. Hence, if $\langle a \rangle$ is a p-group, since $(o(k), o(a)) = 1$, we obtain, by Lemma 1, that $k \sim x$, and we are done in this case.

Now let in general $o(a) = p_1^{n_1} \cdots p_t^{n_t}$, and write $\langle a \rangle = \langle a_1 \rangle \times \langle a_2 \rangle$, where $o(a_1) = p_1^{n_1}$ and $o(a_2) = p_2^{n_2} \cdots p_t^{n_t}$. Then $G = \langle a_1 \rangle \rtimes G_2$, where $G_2 = \langle \langle a_2 \rangle, X \rangle$, and we have that $U_1\mathbb{Z}G = U(1 + \Delta(G, \langle a_1 \rangle)) \rtimes U_1\mathbb{Z}G_2$. If we write $k = (1 + \delta_1)\gamma$, where $(1 + \delta_1) \in U(1 + \Delta(G, \langle a_1 \rangle))$ and $\gamma \in U_1\mathbb{Z}G_2$, then, by Lemma 1, it follows that $k \sim \gamma$. On the other hand, we can write $\gamma = (1+\delta_2)y$, where $(1 + \delta_2) \in U(1 + \Delta(G_2, \langle a_2 \rangle))$ and $y \in U_1\mathbb{Z}X = X$. But then by the inductive hypothesis $\gamma \sim y$, and this forces $k \sim y$. We now claim $x = y$. In fact, since $(1 + \delta_1) \equiv a_1^t \pmod{\Delta G \Delta\langle a_1 \rangle}$ for some t and $\gamma = (1 + \delta_2)y \equiv a_2^s y \pmod{\Delta G_2 \Delta\langle a_2 \rangle}$ for some s, we have $k = (1 + \delta_1)(1 + \delta_2)y \equiv a_1^t a_2^s y \pmod{\Delta G \Delta\langle a \rangle}$. In particular, $a_1^t a_2^s y \equiv a^j x \pmod{\Delta G \Delta\langle a \rangle}$. By [1] it follows that $a_1^t a_2^s y = a^j x$, and so $x = y$, as claimed.

We now define a map $\varphi\colon H \to G$ by setting $\varphi(u) = b^i$ and $\varphi(k) = x$, where $k \in K$ and $\alpha(k) = a^j x$ for some j. It is easy to see that φ is a homomorphism, and our earlier argument tells us that φ is one-to-one. The conjugacy results obtained above allow us to conclude that $\chi(u) = \chi(\varphi(u))$ and $\chi(k) = \chi(\varphi(k))$ for all $k \in K$ and all irreducible characters χ of G.

We want to prove that, for every $h \in H$ and for every irreducible character χ of G, $\chi(h) = \chi(\varphi(h))$. With this, the proof of the theorem will be completed according to Lemma 4. To this effect we use induction on the number of different primes dividing $o(a)$ but not dividing $o(h)$.

Let $h = u^t k \in H$ for some $k \in K$. If $o(h)$ is divisible by all primes dividing $o(a)$, then, since $o(b^{it}x) = o(\varphi(h)) = o(h)$, by Lemma 2, x is central in G. Hence, since $k \sim x$, $k = x$. Thus $h = u^t k$ and $\varphi(h) = b^{it}x$ are conjugate, and this says that $\chi(h) = \chi(\varphi(h))$ for all irreducible characters χ of G. Therefore, we may assume that at least one of the primes dividing $o(a)$, say p, does not divide $o(h)$.

As above write $\langle a\rangle = \langle a_1\rangle \times \langle a_2\rangle$, where $\langle a_1\rangle$ is a p-group and $\langle a_2\rangle$ is a p'-group. Then $G = \langle a_1\rangle \rtimes G_2$, where $G_2 = \langle\langle a_2\rangle, X\rangle$, and we have $U_1\mathbb{Z}G = U(1+\Delta(G, \langle a_1\rangle)) \rtimes U_1\mathbb{Z}G_2$. Moreover, $h = (1+\delta)\gamma$, where $(1+\delta) \in U(1+\Delta(G, \langle a_1\rangle))$ and $\gamma \in U_1\mathbb{Z}G_2$. By Lemma 1, it follows that $h \sim \gamma$, so $\chi(h) = \chi(\gamma)$ for all irreducible characters χ of G.

Let us denote by $\overline{\varphi}$ the homomorphism induced by φ when we factorize by $\langle a_1\rangle$, i.e.,

$$\overline{\varphi}\colon \overline{H} = \langle\overline{u}\rangle \rtimes \overline{K} \to \overline{G} = G_2.$$

Our map behaves well with respect to factoring by $\langle a_1\rangle$; namely $\overline{\varphi}(\overline{u}) = \overline{b}^i$ and $\overline{\varphi}(\overline{k}) = \overline{x}$.

Since $h = (1+\delta_1)\gamma$, where $(1+\delta_1) \in U(1+\Delta(G, \langle a_1\rangle))$ and $\gamma \in U_1\mathbb{Z}G_2$, by factoring by $\langle a_1\rangle$ we get $\overline{h} = \gamma$, so $\overline{\varphi}(\overline{h}) = \overline{\varphi}(\overline{\gamma}) = \overline{b}^{it}\overline{x}$.

By the inductive hypothesis, $\psi(\overline{\varphi}(\overline{h})) = \psi(\overline{h}) = \psi(\gamma) = \psi(\overline{b}^{it}\overline{x})$ for every irreducible character ψ of $G_2 = A_2 \rtimes X$. But if χ is an irreducible character of G and χ_{G_2} is the induced character on G_2, then χ_{G_2} is a linear combination of irreducible G_2-characters, and by the previous argument it follows that $\chi(\gamma) = \chi_{G_2}(\gamma) = \chi_{G_2}(\overline{b}^{it}\overline{x}) = \chi(\overline{b}^{it}\overline{x})$. Finally, by writing $b^{it}x = a_1^l \overline{b}^{it}\overline{x}$, for some l, since p does not divide $o(h) = o(b^{it}x)$, it follows, by Lemma 1, that $b^{it}x \sim \overline{b}^{it}\overline{x}$. Thus $\chi(h) = \chi(\gamma) = \chi(\overline{b}^{it}\overline{x}) = \chi(b^{it}x)$ for every irreducible character χ of G. This completes the proof of the theorem.

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