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CLASSIFICATION OF COHEN-MACAULAY MODULES OF COVARIANTS FOR SYSTEMS OF BINARY FORMS

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ABSTRACT. For every module of covariants for a system of binary forms a formula is given, measuring to what extent Stanley's functional equation fails to be satisfied. As an application a new proof is given for the classification of the Cohen-Macaulay modules of covariants for systems of binary forms.

INTRODUCTION

Let V and M be two finite-dimensional SL_2-modules. The vector space of polynomial maps $V \to M$ commuting with the SL_2-actions is a finitely generated graded module $B(M)$ over the algebra B of invariant polynomial functions on V. It is called the module of covariants for V of type M. The ring of invariants is finitely generated and Hochster and Roberts [7] showed that it is Cohen-Macaulay, in fact, even Gorenstein.

If the highest weights of M are smaller than $s-2$, where s is an easily computed integer depending on V, Stanley [10] proved in 1979 that the generating function of $B(M)$ satisfies a functional equation of the kind

$$\mathscr{G}(B(M); t) = (-1)^d t^{-\dim V} \mathscr{G}(B(M), t^{-1})$$

and made some conjectures.

Early in 1989 Van den Bergh proved these conjectures for SL_2 (and later for general reductive groups, see [2]), by showing that these modules of covariants of small type are Cohen-Macaulay graded B-modules. In the spring of the same year we described [3] a new method of calculating generating functions and found some examples of modules of covariants *not* satisfying the functional equation together with a formula for the deviation from it. Furthermore, we classified the Cohen-Macaulay modules of covariants for certain classes of representations V. Subsequently, in the fall Van den Bergh [1] finished the classification for all V.

In this article we complete the picture by calculating the deviation from Stanley's functional equation for all modules of covariants and giving it an interpretation. Satisfying the functional equation turns out to be equivalent to being

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Cohen-Macaulay. We reprove the complete classification in a more elementary way.

The first sections are concerned with generating functions and complete the results in [3]. Next we show that there are at most two nonvanishing local cohomology groups. Finally we prove the classification of Cohen-Macaulay modules of covariants in full and give an interpretation of the deviation from Stanley's functional equation.

1. Notation

Let $\mathbf{k}$ be an algebraically closed field of characteristic zero. Let $G := \mathrm{SL}(2, \mathbf{k})$ and R_p be the irreducible representation on the binary forms of degree p. The subgroup of diagonal matrices is denoted by H. We fix a representation $V = R_{d_1} \oplus \cdots \oplus R_{d_m}$, where we suppose that d_i is even if and only if $i \leq e$, and $e = \dim V^H$.

Write $A := \mathbf{k}[V]$ for the algebra of functions on V, $B := A^G$ for the ring of invariants, and $B(M) := (A \otimes M)^G$ for the module of covariants of type M, where M is any G-module. They all have a natural $\mathbb{N}^m$-graded structure. For any $\mathbb{Z}^m$-graded vector space $W = \bigoplus_{\mathbf{i} \in \mathbb{Z}^m} W_\mathbf{i}$ we define the generating function

$$\mathscr{G}(W) = \mathscr{G}(W ; \mathbf{t}) := \sum_{\mathbf{i} \in \mathbb{Z}^m} \dim W_\mathbf{i} \mathbf{t}^\mathbf{i},$$

where $\mathbf{t}^\mathbf{i} := \prod_{n=1}^m t_n^{i_n}$. By $\mathbf{t}^{-1}$ we mean $(t_1^{-1}, \ldots, t_m^{-1})$.

Using the notation $\lceil z \rceil$ for the smallest integer greater than or equal to z, we define

$$f^{(n)}(t) := \prod_{i=0}^{\lceil n/2 \rceil - 1} (1 - x^{n-2i} t),$$

and $f(x, \mathbf{t}) := \prod_{k=1}^m f^{(d_k)}(x, t_k)$. Now write s for the degree of f with respect of x, and define β_i, ϕ_i, and α_i^p by

$$f(x, \mathbf{t}) = \sum_{i=0}^s \beta_i(\mathbf{t}) x^i, \quad \frac{1}{f(x, \mathbf{t})} = \sum_{i=0}^\infty \phi_i(\mathbf{t}) x^i, \quad \alpha_i^p = \beta_{p-i} - \beta_{i+p+2}.$$

Finally $E := \prod_{k=1}^e (1 - t_k)$, $T := \prod_{i=1}^m (-t_i)^{\lceil d_i/2 \rceil}$, and $\tau := \sum_{i-1}^m \lceil d_i/2 \rceil$. Then $\dim V = 2\tau + e$.

Lemma 1.1. *Let $p \in \mathbb{Z}$.*

(i) *We have*

$$\sum_i \alpha_i^p \mathscr{G}(B(R_i), \mathbf{t}) = \frac{\phi_{-p} - \phi_{-p-2}}{E}.$$

(ii) $\alpha_i^{s-2+p}(\mathbf{t}) = -T \alpha_i^{-p}(\mathbf{t}^{-1})$.

(iii) $\beta_0 = \phi_0 = 1$ *and* $\sum_{i=0}^k \phi_i \beta_{k-i} = 0$ *for* $k \geq 1$.

Proof. The first statement can be proved as [3, Proposition 3.1], the second statement is just [3, Lemma 4.2.3], and (iii) follows from $f \cdot \frac{1}{f} = 1$ and the Cauchy product of power series. □

2. Functional equations

We write

$$\widetilde{\mathscr{G}}_p = \widetilde{\mathscr{G}}_p(\mathbf{t}) := \mathscr{G}(B(R_p)\,;\,\mathbf{t}) + (-1)^{\dim V}(t_1^{d_1+1}\cdots t_m^{d_m+1})^{-1}\mathscr{G}(B(R_p)\,;\,\mathbf{t}^{-1})$$

for the deviation from Stanley's functional equation. We have the following basic result.

Proposition 2.1. (i) *For* $0 \leq p < s-2$ *we have* $\widetilde{\mathscr{G}}_p = 0$.

(ii) *For* $k \geq 0$ *we have* $\widetilde{\mathscr{G}}_{s-2+k} = -\psi_k/(ET)$, *where* $\psi_k(\mathbf{t})$ *is the coefficient of* x^k *in the Maclaurin expansion with respect to* x *of*

$$\frac{(1-x^2)}{f(x\,,\,\mathbf{t})f(x\,,\,\mathbf{t}^{-1})}.$$

Proof. The first statement is proved in [3, Proposition 6.4] and is well known. Using this fact and $\sum_{i\geq 0}\alpha_i^{s-2+k}(\mathbf{t})\mathscr{G}(B(R_i)\,;\,\mathbf{t}) = 0$ for $k \geq 0$, by Lemma 1.1(i), we have

$$\begin{aligned}
\sum_{i\geq s-2}\alpha_i^{s-2+k}(\mathbf{t})\widetilde{\mathscr{G}}_i &= \sum_{i\geq 0}\alpha_i^{s-2+k}(\mathbf{t})\widetilde{\mathscr{G}}_i \\
&= (-1)^{\dim V}(t_1^{d_1+1}\cdots t_m^{d_m+1})^{-1}\sum_{i\geq 0}\alpha_i^{s-2+k}(\mathbf{t})\mathscr{G}(B(R_i)\,;\,\mathbf{t}^{-1}) \\
&= (-1)^{\dim V}(t_1^{d_1+1}\cdots t_m^{d_m+1})^{-1}\sum_{i\geq 0}-T\alpha_i^{-k}(\mathbf{t}^{-1})\mathscr{G}(B(R_i)\,;\,\mathbf{t}^{-1}) \\
&= -T(-1)^{\dim V}(t_1^{d_1+1}\cdots t_m^{d_m+1})^{-1}\frac{(\phi_k(\mathbf{t}^{-1})-\phi_{k-2}(\mathbf{t}^{-1}))}{E(\mathbf{t}^{-1})} \\
&= \frac{-1}{ET}(\phi_k(\mathbf{t}^{-1})-\phi_{k-2}(\mathbf{t}^{-1})).
\end{aligned}$$

Using $\alpha_{s-2+j}^{s-2+k} = \beta_{k-j}$, for $j \geq 0$, we established for all $k \geq 0$

$$\sum_{j\geq 0}\beta_{k-j}(\mathbf{t})\widetilde{\mathscr{G}}_{s-2+j} = \frac{-1}{ET}(\phi_k(\mathbf{t}^{-1})-\phi_{k-2}(\mathbf{t}^{-1})).$$

So

$$\sum_{i=0}^{k}\sum_{j\geq 0}\phi_i\beta_{(k-i)-j}(\mathbf{t})\widetilde{\mathscr{G}}_{s-2+j} = \frac{-1}{ET}\sum_{i=0}^{k}\phi_i(\mathbf{t})(\phi_{k-i}(\mathbf{t}^{-1})-\phi_{k-i-2}(\mathbf{t}^{-1})),$$

and using Lemma 1.1(iii) we get

$$\widetilde{\mathscr{G}}_{s-2+k} = \frac{-1}{ET}\sum_{i=0}^{k}\phi_i(\mathbf{t})(\phi_{k-i}(\mathbf{t}^{-1})-\phi_{k-i-2}(\mathbf{t}^{-1})) = \frac{-\psi_k}{ET}.$$

This completes the proof. □

3. Degrees of generating functions

For a polynomial $p \in \mathbb{Z}[t_1, \ldots, t_m]$ we write $\deg_i p$ for the degree of p with respect to t_i and $\deg p$ for the total degree. For a rational function p/q

we define as usual $\deg_i(p/q) = \deg_i p - \deg_i q$ and $\deg(p/q) = \deg p - \deg q$. In this section we will calculate the degree of $\mathscr{G}(B(R_i), \mathbf{t})$ if $i \geq s-2$.

We start with a lemma.

Lemma 3.1. (i) *If* $e < m$ *then* $\deg \phi_k = k$; *if* $e = m$ *then* $\phi_{2k+1} = 0$ *and* $\deg \phi_{2k} = k$.

(ii) *If* $e < j \leq m$ *then* $\deg_j \phi_k = k$ *and if* $1 \leq j \leq e$ *then* $\deg_j \phi_{2k} = k$ *and* $\deg_j \phi_{2k+1} = k$ *provided that* $e \neq m$.

Proof. The Maclaurin expansion of $1/(1 - sx^n)$ is $1 + sx^n + s^2x^{2n} + s^3x^{3n} + \cdots$. Now $\frac{1}{f}$ is a product of such terms with various n and various t_i for s.

Suppose $e < j \leq m$. Then f contains a unique factor $1 - xt_j$. So ϕ_k contains t_j^k as the highest j-degree term. So $\deg_j \phi_k = k$.

If $1 \leq j \leq e$, then f contains the factor $1 - x^2 t_j$. So in ϕ_{2k} the term of highest j-degree is t_j^k and $\deg_j \phi_{2k} = k$. If $e < m$ then $t_j^k(t_{e+1} + \cdots + t_m)$ is part of the highest j-degree, and $\deg_j(\phi_{2k+1}) = k$. If $e = m$, then clearly $\phi_{2k+1} = 0$. This proves (ii); the first statement is proved analogously. □

The following proposition extends [3, Proposition 7.5.1], saying, for example, that the total degree of $\mathscr{G}(B(R_p), \mathbf{t})$ is less than or equal to $-\dim V$ if $p < s-2$. This follows easily from Lemma 2.1(i). Recall $\tau := \sum_{i-1}^{m} \lceil d_i/2 \rceil$.

Proposition 3.1. *Let* $k \geq 0$.

(i) *If* $e < m$, *then*

$$\deg \mathscr{G}(B(R_{s-2+k}), \mathbf{t}) = -\dim V + \tau + k.$$

(ii) *If* $e = m$, *then* $B(R_{2k+1}) = 0$ *and* $\deg \mathscr{G}(B(R_{s-2+2k}), \mathbf{t}) = -\dim V + \tau + k$.

(iii) *If* $e < i \leq m$, *then* $\deg_i \mathscr{G}(B(R_{s-2+k}), \mathbf{t}) = k - \lceil d_i/2 \rceil$.

(iv) *If* $1 \leq i \leq e$, *then* $\deg_i \mathscr{G}(B(R_{s-2+2k}), \mathbf{t}) = \deg_i \mathscr{G}(B(R_{s-2+2k+1}), \mathbf{t}) = k - d_1/2$.

Proof. The lowest degree of a term in the denominator minus the lowest degree of a term in the numerator of $\mathscr{G}(B(R_{s-2+k}); \mathbf{t}^{-1})$ is just the degree δ of $\mathscr{G}(B(R_{s-2+k}); \mathbf{t})$. Using Proposition 2.1(ii) one sees that $-\dim V - \delta$ equals the lowest degree of a term in ψ_k/T, which is just $-\tau - k$ if $e < m$, and $-\tau - k/2$ if $e = m$ and k is even.

So $\delta = -\dim V + \tau + k$ if $e < m$ and $\delta = -\dim V + \tau + k/2$ if $e = m$ and k is even. This proves (i) and (ii). The remaining statements are proved analogously. □

4. Depth estimates

From now on we only consider the total grading and write $\mathscr{G}(B(R_i)) = \mathscr{G}(B(R_i), t)$ for the corresponding single-variable Hilbert series. Let R be any finitely generated d-dimensional $\mathbb{N}$-graded $\mathbf{k}$-algebra, with $R_0 = \mathbf{k}$. Write R^+ for the unique maximal graded ideal and δ for the degree of $\mathscr{G}(R)$ as a rational function. Write $H^i_{R^+}(N)$ for the ith local cohomology group of the graded module N in the category of graded R-modules. See [6] for some of its properties; the most important being that $\dim N$ (resp. $\operatorname{depth}_{R^+}(N)$, i.e., the length of the longest regular sequence on N consisting of homogeneous elements of R^+) is the largest (resp. smallest) index i such that $H^i_{R^+}(N) \neq 0$. For any graded R-module N the *graded dual* is defined as

$$N^\vee := \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_{\mathbf{k}}(N_{-n}, \mathbf{k});$$

it is a graded R-module.

Lemma 4.1. *Suppose R is Gorenstein. For any finitely generated graded R-module N*

$$H^i_{R^+}(N)^\vee \simeq \mathrm{Ext}^{d-i}_R(N, R[\delta]),$$

where $R[\delta]_i := R_{\delta+i}$.

Proof. See [6, Proposition 2.1.6]. □

Using this graded duality, we prove that at most two local cohomology groups of modules of covariants do not vanish. We start with a lemma concerning the support. Write $\pi : V \to V//G$ for the quotient morphism.

Lemma 4.2. *The support of* $\mathrm{Ext}^i_B(A, B)$ *is contained in* $\pi(V^H)$.

Proof. Let $x \notin \pi(V^H)$ and $\tilde{x}$ be an element on the unique closed orbit in $\pi^{-1}(x)$. Then its stabilizer K is finite. Let N be the slice representation at $\tilde{x}$, i.e., a K-stable complement of $T_{\tilde{x}}G\tilde{x} \subset T_{\tilde{x}}V = V$. The quotient map $N \to N//K$ for K is equidimensional, so $\mathbf{k}[N]$ is a Cohen-Macaulay ($B' := \mathbf{k}[N]^K$)-module. B' is Gorenstein, since the finite group K acts with determinant equal to one on N. Using Lemma 4.1 it follows that $\mathrm{Ext}^i_{B'}(\mathbf{k}[N], B') = 0$ if $i \neq 0$.

According to Luna's étale slice theorem (see [9]), there is an $f \in B$ with $f(x) \neq 0$ and a Cartesian diagram

$$\begin{array}{ccc} \left(\mathbf{k}[G] \otimes_{\mathbf{k}} \mathbf{k}[N]_f\right)^K & \longleftarrow & A_f \\ \uparrow & & \uparrow \\ B'_{f'} & \longleftarrow & B_f \end{array}$$

where the horizontal maps are faithfully flat; K acts on $\mathbf{k}[G \times N] \simeq \mathbf{k}[G] \otimes_{\mathbf{t}} \mathbf{k}[N]$ by $(kx)(g, n) := x(gk, k^{-1}n)$, with $k \in K$, $g \in G$, $n \in N_{f'}$, and $x \in \mathbf{k}[G \times N]$. In particular,

$$\left(\mathbf{k}[G] \otimes \mathbf{k}[N]_f\right)^K \simeq A_f \otimes_{B_f} B'_f.$$

So for every G-module M, we have for $i \neq 0$

$$\begin{aligned} \mathrm{Ext}^i_B(B(M), B)_f \otimes_{B_f} B'_f &\simeq \mathrm{Ext}^i_{B'_f}((M \otimes_{\mathbf{k}} A_f \otimes_{B_f} B'_f)^G, B'_f) \\ &\simeq \mathrm{Ext}^i_{B'_f}((M \otimes_{\mathbf{k}} \mathbf{k}[G] \otimes_{\mathbf{k}} \otimes \mathbf{k}[N]_f)^{G\times H}, B'_f) \\ &\simeq \mathrm{Ext}^i_{B'_f}((M \otimes_{\mathbf{k}} k[N]_f)^H, B'_f) \simeq 0. \end{aligned}$$

So $\mathrm{Ext}^i_B(A, B)_f = 0$ and x is not in the support of $\mathrm{Ext}^i_B(A, B)$. □

The following lemma is very useful and is due to Brion [4].

Lemma 4.3. *Suppose* $\dim V//G - e \geq 2$. *Then* $B(M) \simeq B(M^*) \simeq B(M)^*$ *for all* G*-modules* M.

Proof. We sketch the proof. Let ϕ be the composition

$$B(M^*) \simeq \mathrm{Hom}_A(A \otimes M, A)^G \to \mathrm{Hom}_B(B(M), B) = B(M)^*,$$

where the second map is the restriction. As in the proof of Lemma 4.2 one checks that ϕ is an isomorphism outside $\pi(V^H)$. Since both $B(M^*)$ and $B(M)^*$ are reflexive and the codimension of $\pi(V^H)$ in $V//G$ is $\dim V//G - e \geq 2$, we conclude that ϕ is an isomorphism. □

We apply the foregoing lemmas to give a direct proof of a fact observed before by Van den Bergh [1].

Proposition 4.1. *Suppose* $\dim V//G - e \geq 2$ *and let* M *be any* G*-module. Then for* $i \neq 0$ *or* $\tau - 2$

$$\mathrm{Ext}^i_B(B(M), B) = H^{d-i}_{B^+}(B(M)) = 0.$$

Proof. Let P be the homogeneous prime ideal of B corresponding to $\pi(V^H)$. The variety $X := \pi^{-1}\pi(V^H)$ equals $G \cdot (\bigoplus_{j \geq 0} V_j)$, where V_j is the H-weight space of weight j. Since X is stabilized by the upper triangular matrices of G, the dimension of X is $\tau + e + 1$. Since A is Cohen-Macaulay, we have

$$\mathrm{depth}_P(A) = \mathrm{height}(P \cdot A) = \dim V - \dim X = \tau - 1.$$

Suppose there is a $j > 0$ such that $\mathrm{Ext}^j_B(B(M), B) \neq 0$, and take j minimal. By dualizing a free resolution of $B(M)$ we get a sequence

$$0 \to B(M)^* \to F_0^* \to \cdots \to F_j^* \to C \to 0,$$

where all F_i^* are free and $\mathrm{Ext}^j_B(B(M), B) \subset C$. Let $Q \supset P$ be an associated prime ideal of $\mathrm{Ext}^j_B(B(M), B)$; it is associated to C as well, $\mathrm{depth}_Q(C) = 0$, and by the depth lemma [5, Lemma 1.1] $\mathrm{depth}_Q(B(M)^*) = j + 1$.

Since $B(M) \simeq B(M)^*$ by Lemma 4.3, we have

$$\begin{aligned} j + 1 = \mathrm{depth}_Q(B(M)^*) &\geq \mathrm{depth}_P(B(M)^*) \\ &= \mathrm{depth}_P(B(M)) \geq \mathrm{depth}_P(A) = \tau - 1. \end{aligned}$$

So $j \geq \tau - 2$. By Lemma 4.1 we get $H^{d-j}_{B^+}(B(M)) = 0$ if $0 < j < \tau - 2$.

Since

$$\mathrm{depth}_{B^+} A = \mathrm{height}(B^+ A) = \mathrm{codim}_V \pi^{-1}\pi(0) = \dim V - \tau - 1,$$

we have $H^{d-j}_{B^+}(B(M)) = 0$ for $j > \tau - 2$. We conclude that $\mathrm{Ext}^j_B(B(M), B)$ vanishes for $j > \tau - 2$, by using Lemma 4.1 again. This proves the lemma. (This proof is inspired by the proof of [5, Theorem 3.8].) □

5. Classification of Cohen-Macaulay modules of covariants

Let C be the subalgebra of B generated by a homogeneous system of parameters of degrees $\sigma_1, \ldots, \sigma_d$, and write $\sigma := \dim V - \sum_{i=1}^d \sigma_i$. Since B is a graded Gorenstein domain, by the theorem of Hochster and Roberts, B is finitely generated and free as a C-module.

Proposition 5.1. *Suppose* $\dim V//G - e \geq 2$.

(i) *Let* $p \geq 0$, *and if* $e = m$ *let* p *be even. Then*

$$\mathscr{G}(\operatorname{Ext}_B^{\tau-2}(B(R_p), B); t) = (-1)^{\tau-2}\widetilde{\mathscr{G}}_p = \frac{\psi_{p-s+2}(t)t^{-\tau}}{(1-t)^e}.$$

(ii) *Now suppose that* $p \geq s-2$. *Then* $\operatorname{Ext}_B^{\tau-2}(B(R_p), B)$ *is a Cohen-Macaulay* B*-module of dimension* e.

(iii) *There is a minimal free graded* C*-resolution of* $B(R_p)$ *of the form*

$$0 \to F_{\tau-2} \to F_{\tau-3} \to \cdots \to F_1 \to F_0 \to B(R_p) \to 0.$$

This resolution and its dual, shifted over σ *degrees, patch together to a minimal free graded* C*-resolution of* $\operatorname{Ext}_B^{\tau-2}(B(R_p), B)$:

$$\begin{aligned}0 \to F_{\tau-2} \to \cdots \to F_0 \to F_0^*[\sigma] \to F_1^*[\sigma] \to \cdots \to F_{\tau-2}^*[\sigma]\\ \to \operatorname{Ext}_B^{\tau-2}(B(R_p), B) \to 0.\end{aligned}$$

Proof. Let for the moment $F_\bullet \to B(R_p)$ be any free graded C-resolution, which is always of finite length. The homology group $H_i(F_\bullet^*)$ of the dual complex is $\operatorname{Ext}_C^i(B(R_p), C)$. Since by standard arguments $H_{C^+}^{d-i}(B(R_p)) \simeq H_{B^+}^{d-i}(B(R_p))$, it follows by applying Lemma 4.1 twice that $\operatorname{Ext}_C^i(B(R_p), C)[\sigma] = \operatorname{Ext}_B^i(B(M), B)$ as B-modules. By Proposition 4.1 there are at most two non-vanishing Ext-groups of which $\operatorname{Ext}_B^0(B(R_p), B) \simeq R_p$, by Lemma 4.3.

From

$$\mathscr{G}(B(R_p), t) = \sum_i (-1)^i \mathscr{G}(F_i, t)$$

and $\mathscr{G}(F_i^*[\sigma]; t) = (-1)^d t^{-\dim V}\mathscr{G}(F_i; t^{-1})$ we get

$$(-1)^d t^{-\dim V}\mathscr{G}(B(R_p); t^{-1}) = \sum_i (-1)^i \mathscr{G}(F_i^*[-\sigma], t).$$

Since

$$\sum_i (-1)^i \mathscr{G}(F_i^*[-\sigma], t) = \mathscr{G}(B(R_p)^*, t) + (-1)^{\tau-2}\mathscr{G}(\operatorname{Ext}_B^{\tau-2}(B(R_p), B), t),$$

we get

$$\begin{aligned}&\mathscr{G}(\operatorname{Ext}_B^{\tau-2}(B(R_p), B); t)\\ &\quad = (-1)^{\tau-1}(\mathscr{G}(B(R_p); t) - (-1)^d t^{-\dim V}\mathscr{G}(B(R_p); t^{-1}))\\ &\quad = (-1)^{\tau-2}\widetilde{\mathscr{G}}_p = \frac{\psi_{p-s+2}(t)t^{-\tau}}{(1-t)^e},\end{aligned}$$

where we used Proposition 2.1. This proves (i).

By Lemma 3.1, $\operatorname{Ext}_B^{\tau-2}(B(R_p), B)$ does not vanish if and only if $p \geq s-2$ and, when $e = m$, p is even. If it vanishes and $B(R_p)$ is not zero, then by Lemma 4.1 the depth of $B(R_p)$ is d; hence, $B(R_p)$ is free as a C-module.

From now on suppose $p \geq s-2$ and, when $e = m$, p is even. Then the depth of $B(R_p)$ is $d - (\tau - 2)$, so a minimal free graded C-resolution exists of length $\tau - 2$. Taking its dual and shifting over σ degrees, we get that

$$0 \to B(R_p)^* \to F_0^*[\sigma] \to \cdots \to F_{\tau-2}^*[\sigma] \to \mathrm{Ext}_B^{\tau-2}(B(R_p), B) \to 0$$

is exact. Since $B(R_p)^* \simeq B(R_p)$, the two free complexes glue together to a minimal free graded C-resolution of $\mathrm{Ext}_B^{\tau-2}(B(R_p), B)$, and so the depth of it is $2\nu - d - 1 = e$. From the Hilbert series we see that its dimension is also e, so $\mathrm{Ext}_B^{\tau-2}(B(R_p), B)$ is a graded Cohen-Macaulay B-module. □

5.1. The representations V such that all its modules of covariants are Cohen-Macaulay are the same as those where the quotient map π is equidimensional, so those where

$$\dim V - \dim V//G = \dim \pi^{-1}\pi(0) = \sum_{i=1}^{m} \left\lceil \frac{d_i}{2} \right\rceil + 1 = \tau + 1.$$

So by some arithmetic, the result is that, up to trivial summands, V is one of the representations contained in

$$\{R_1, R_1 \oplus R_1, R_2, R_1 \oplus R_2, R_2 \oplus R_2, R_3, R_4\}.$$

In fact, in this case all modules of covariants are free (see [8]). They can also be characterized as the representations with $\dim V//G - e < 2$.

The full classification is given in the following theorem.

Theorem 5.1. (i) *If* $\dim V - d = \tau + 1$ *then all modules of covariants for* V *are free.*

(ii) *Suppose* $\dim V - d < \tau + 1$. *If* $e = m$ *then* $B(R_p) = 0$ *when* p *is odd. Let* $p \in \mathbb{N}$ *be even if* $e = m$ *and arbitrary if* $e \neq m$. *The following statements are equivalent*:

(1) $p < s - 2$;
(2) $B(R_p)$ *is a Cohen-Macaulay* B-*module*;
(3) *the deviation* $\widetilde{\mathscr{G}}_p$ *from Stanley's functional equation is zero*;
(4) *the degree of* $\mathscr{G}(B(R_p))$ *as a rational function is smallar than or equal to* $-\dim V$;
(5) $H_{B^+}^{\tau+e-1}(B(R_p)) = 0$; *and*
(6) $\mathrm{Ext}_B^{\tau-2}(B(R_p), B) = 0$.

Proof. We discussed (i) just before the statement of the theorem. The equivalences follow from Proposition 5.1 using Propositions 3.1 and 4.1 and Lemma 4.1. □

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