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HYPERSURFACES SATISFYING THE EQUATION $\Delta x = Rx + b$

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ABSTRACT. We prove that a hypersurface in a space form or in Lorentzian space whose immersion x satisfies $\Delta x = Rx + b$ is minimal or isoparametric. In particular, we locally classify such hypersurfaces which are not minimal.

1. INTRODUCTION

In [T] Takahashi proved that, if an isometric immersion $x: M^n \to \mathbb{R}^{n+k}$ satisfies $\Delta x = -\lambda x$, then M is either minimal in $\mathbb{R}^{n+k}$ (for $\lambda = 0$) or minimal in $\mathbb{S}^{n+k-1}(r)$ with $r^2 = n/\lambda$, where $x = (x_1, \dots, x_{n+k})$, $\Delta x = (\Delta x_1, \dots, \Delta x_{n+k})$, and Δ is the Laplacian on M given by the induced metric. Garay generalized this theorem for the hypersurfaces $x: M^n \to \mathbb{R}^{n+1}$ satisfying $\Delta x = Dx$, where D is a constant $(n+1) \times (n+1)$ diagonal matrix. He proved that such a hypersurface is either minimal or an open subset of a sphere or of a cylinder [G].

Let $\mathbb{R}^{n+1,1}$ be Lorentzian space with the metric

$$(x, y) = x_1y_1 + \cdots + x_{n+1}y_{n+1} - x_{n+2}y_{n+2}.$$

Then $\{x \in \mathbb{R}^{n+1,1} | (x, x) = -1\} \subset \mathbb{R}^{n+1,1}$ gives a natural isometric embedding of the hyperbolic space $\mathbb{H}^{n+1}$. So there exist $n+2$ coordinate functions on an immersed hypersurface M in $\mathbb{H}^{n+1}$ or in $\mathbb{S}^{n+1}$, i.e., $x: M^n \to \mathbb{H}^{n+1} \subset \mathbb{R}^{n+1,1}$ or $x: M^n \to \mathbb{S}^{n+1} \subset \mathbb{R}^{n+2}$. In particular, we let $\Delta x = (\Delta x_1, \dots, \Delta x_{n+2})$.

The purpose of this paper is to classify the isometric immersions of hypersurfaces in a simply connected space form $N^{n+1}(c)$ or in Lorentzian space $\mathbb{R}^{n,1}$ satisfying $\Delta x = Rx + b$, where R is a constant square matrix and b is a constant vector. This has been done in [AFL] for immersed surfaces in $\mathbb{R}^{2,1}$ or $\mathbb{R}^3$. Our main results are:

Theorem 1.1. *Let $x: M^n \to \mathbb{R}^{n+1}$ (or $\mathbb{R}^{n,1}$) be an isometric immersion with nondegenerate induced metric satisfying $\Delta x = Rx + b$, where R is a constant $(n+1) \times (n+1)$ matrix and b is a constant vector in $\mathbb{R}^{n+1}$ (or $\mathbb{R}^{n,1}$). Then M is minimal or isoparametric.*

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Theorem 1.2. *Let $x: M^n \to \mathbb{S}^{n+1}$ (or $\mathbb{H}^{n+1}$) be an isometric immersion satisfying $\Delta x = Rx$, where R is a constant $(n+2) \times (n+2)$ matrix. Then M is minimal or isoparametric.*

Theorem 1.3. *In the above theorems, if M is isoparametric and not minimal, then M has at most two distinct principal curvatures. Moreover, it is an open piece of one of the following, up to rigid motions*:

(a) $\mathbb{S}^n(r)$, $\mathbb{S}^k(r) \times \mathbb{R}^{n-k}$ *in* $\mathbb{R}^{n+1}$;
(b) $\mathbb{S}^n(r)$, $\mathbb{S}^k(r_1) \times \mathbb{S}^{n-k}(r_2)$ *in* $\mathbb{S}^{n+1}$;
(c) $\mathbb{R}^n$, $\mathbb{H}^n(r)$, $\mathbb{S}^n(r)$, $\mathbb{S}^k(r_1) \times \mathbb{H}^{n-k}(r_2)$ *in* $\mathbb{H}^{n+1}$;
(d) $\mathbb{S}^k(r) \times \mathbb{R}^{n-k-1,1}$, $\mathbb{R}^{n-k} \times \mathbb{H}^k(r)$, $\mathbb{R}^{n-k} \times \mathbb{S}_1^k(r)$ *in* $\mathbb{R}^{n,1}$.

To prove these, we will show first that M has a constant mean curvature H, and then M will turn out to be isoparametric when $H \neq 0$. Isoparametric submanifold theory in space forms has been much developed recently [M, Ma, PT, W]. Münzner showed that the number of principal curvatures of isoparametric hypersurfaces in the sphere has to be $g = 1, 2, 3, 4$, or 6, and it is known that, when $g = 1$ or 2, the hypersurface must be a hypersphere or a product of two spheres.

This paper is organized as follows. In §2 we set up notation and review basic facts about submanifold geometry in space forms and derive a differential identity relating the Laplacian of the mean curvature vector to the shape operator. This is needed for the proof of our main results. In §3 examples of hypersurfaces satisfying $\Delta x = Rx + b$ will be listed, providing the converse of the above theorems. In the last section, we prove the main theorems in each ambient space.

2. Preliminaries

Suppose $x: M^n \to \mathbb{R}^{n+k}$ is an isometric immersion. A local orthonormal frame field $e_1, \dots, e_{n+k}$ in $\mathbb{R}^{n+k}$ is said to be *adapted* to M, if when restricted to M, $e_1, \dots, e_n$ are tangent to M. From now on, we shall use the following index convention:

$$1 \le A, B, C \le n+k, \qquad 1 \le i, j, k \le n, \qquad n+1 \le \alpha, \beta \le n+k.$$

Let ω_A be the dual coframe on $\mathbb{R}^{n+k}$, and let ω_{AB} be the connection form corresponding to the differential d on $\mathbb{R}^{n+k}$. This induces the Levi-Civita connection ∇ on M by

$$\nabla e_i = \sum_j \omega_{ij} \otimes e_j$$

and defines the shape operator A_{e_α} in the direction e_α by

$$A_{e_\alpha} = \sum_j \omega_{j\alpha} \otimes e_j = \sum_{i,j} h_{i\alpha j} \omega_i \otimes e_j.$$

The shape operator $A = \sum_{j,\alpha} \omega_{j\alpha} \otimes \omega_\alpha \otimes e_j$ is identified with the second fundamental form II under the metric isomorphism $T^*M \simeq TM$:

$$\mathrm{II} = \sum_{j,\alpha} \omega_{j\alpha} \otimes \omega_j \otimes e_\alpha = \sum_{i,j,\alpha} h_{i\alpha j} \omega_i \otimes \omega_j \otimes e_\alpha.$$

The mean curvature vector $\vec{H}$ is then defined as

$$\vec{H} = \sum_{\alpha} H_\alpha e_\alpha = \operatorname{tr} \mathrm{II} = \sum_{i,\alpha} h_{i\alpha i} e_\alpha.$$

Now, let f be a C^∞ function on M. The *Laplacian* of f is defined by

$$\Delta f = \operatorname{tr} \nabla\nabla f = \operatorname{tr} \nabla df.$$

Locally, it is defined as follows: Let

$$df = \sum_i f_i \omega_i, \qquad \nabla df = \sum_{i,j} f_{ij} \omega_j \otimes \omega_i.$$

Then $\sum_j f_{ij}\omega_j = df_i + \sum_m f_m \omega_{mi}$. Hence,

$$\Delta f = \sum_i f_{ii} = \sum_i df_i(e_i) + \sum_{i,m} f_m \omega_{mi}(e_i).$$

This operator can be extended to

$$\Delta\colon C^\infty(M \times \mathbb{R}^{n+k}) \to C^\infty(M \times \mathbb{R}^{n+k})$$

by taking the Laplacian coordinatewise. For example, $\Delta x = \operatorname{tr} \nabla dx = \vec{H}$ (cf. [PT]).

We now review some geometry of a pseudo-Riemannian hypersurface M^n in $\mathbb{R}^{n,1}$. Choose an adapted orthonormal frame e_A on M and its dual ω^A, i.e., $\omega^A(e_B) = \delta_{AB}$. The differential $\overline{\nabla} = d$ on $\mathbb{R}^{n,1}$ induces the Levi-Civita connection ∇ on M,

$$\overline{\nabla} e_A = \sum_B \omega_B^A \otimes e_B, \qquad \overline{\nabla}\omega^A = -\sum_B \omega_B^A \otimes \omega^B,$$
$$\nabla e_i = \sum_j \omega_i^j \otimes e_j, \qquad \nabla \omega^i = -\sum_j \omega_j^i \otimes \omega^j,$$

and the structure equations are

$$\varepsilon_A \omega_A^B + \varepsilon_B \omega_B^A = 0, \quad \text{where } \varepsilon_A = \langle e_A, e_A\rangle,$$
$$d\omega^i = -\sum_j \omega_j^i \wedge \omega^j.$$

Local invariants on M are defined as follows:

$$\mathrm{II} = \sum_j \omega_j^{n+1} \otimes \omega^j \otimes e_{n+1} = \sum_{i,j} h_{ij}\omega^i \otimes \omega^j \otimes e_{n+1},$$
$$A_{e_{n+1}} = -\sum_j \omega_{n+1}^j \otimes e_j = \varepsilon \sum_{i,j} \varepsilon_j h_{ij} \omega^i \otimes e_j,$$
$$\vec{H} = He_{n+1} = \sum_i \varepsilon_i h_{ii} \cdot e_{n+1},$$

where $\varepsilon = \varepsilon_{n+1}$. The covariant derivative of II satisfies

$$\sum_k h_{ijk}\omega^k = dh_{ij} - \sum_m h_{mj}\omega_i^m - \sum_m h_{im}\omega_j^m.$$

As in the Euclidean case, it is easy to check that h_{ijk} is symmetric in i, j, k and $\Delta x = \vec{H} = He_{n+1}$. Let $f \in C^\infty(M)$ and $df = \sum_i f_i \omega^i$. Then ∇f, the gradient of f, and ∇df are given by

$$\nabla f = \sum_i \varepsilon_i \langle \nabla f, e_i \rangle e_i = \sum_i \varepsilon_i f_i e_i, \qquad \nabla df = \sum_{i,j} f_{ij} \omega^i \otimes \omega^j,$$

where $\sum_j f_{ij}\omega^j = df_i - \sum_j f_j \omega_i^j$. The Laplacian of f is

$$\Delta f = \sum_i \varepsilon_i f_{ii} = \sum_i \varepsilon_i \left\{ e_i(f_i) - \sum_m f_m \omega_i^m(e_i) \right\}.$$

Theorem 2.1. *Let M be a hypersurface in N, and H its mean curvature. Then*

$$\Delta(He_{n+1}) = \begin{cases} -(2A_{e_{n+1}} + H \cdot I)(\nabla H) + (\Delta H - H\|A_{e_{n+1}}\|^2)e_{n+1} + cH^2 x \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for } N = N^{n+1}(c), \\ -(2A_{e_{n+1}} + \varepsilon H \cdot I)(\nabla H) + (\Delta H - \varepsilon H\|A_{e_{n+1}}\|^2)e_{n+1} \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for } N = \mathbb{R}^{n,1}. \end{cases}$$

Proof. We will give the proof of the theorem for the case $c = 0$, and the other cases can be proved in a similar manner. We first calculate Δe_{n+1}. From

$$de_{n+1} = \sum_i (e_{n+1})_i \omega_i = \sum_j \omega_{n+1,j} \otimes e_j = -\sum_{i,j} h_{ij} \omega_i \otimes e_j,$$

we have $(e_{n+1})_i = -\sum_j h_{ij} e_j$. Then

$$\begin{aligned} &\sum_j (e_{n+1})_{ij} \omega_j \\ &\quad = d(e_{n+1})_i + \sum_m (e_{n+1})_m \omega_{mi} \\ &\quad = -\sum_j dh_{ij} \otimes e_j - \sum_{j,k} h_{ij}(\omega_{jk} \otimes e_k + \omega_{j,n+1} \otimes e_{n+1}) - \sum_{m,k} h_{mk}\omega_{mi} \otimes e_k \\ &\quad = -\sum_j \left(dh_{ij} + \sum_m h_{mj}\omega_{mi} + \sum_m h_{im}\omega_{mj} \right) \otimes e_j - \sum_{j,k} h_{ij}h_{jk}\omega_k \otimes e_{n+1} \\ &\quad = -\sum_{j,k} (h_{ijk}\omega_k \otimes e_j + h_{ij}h_{jk}\omega_k \otimes e_{n+1}). \end{aligned}$$

Hence, $\Delta e_{n+1} = -\sum_{i,j} h_{iji} e_j + h_{ij}^2 e_{n+1}$. Now the theorem follows:

$$\begin{aligned} \sum_i (He_{n+1})_{ii} &= \sum_i H_{ii} e_{n+1} + 2H_i(e_{n+1})_i + H(e_{n+1})_{ii} \\ &= \Delta H \cdot e_{n+1} - 2\sum_{i,j} h_{ij} H_i e_j - H\left(\sum_j H_j e_j + \sum_{i,j} h_{ij}^2 \cdot e_{n+1} \right) \end{aligned}$$

since $H_j = \sum_i h_{iij} = \sum_i h_{iji}$. $\square$

3. Examples

If $x: M^n \to N^{n+1}(c)$ is minimal, then it is well known (cf. [PT]) that $\Delta x = -cnx$, and if $x: M^n \to \mathbb{R}^{n,1}$ is minimal, then $\Delta x = 0$.

In the following, we denote by e_{n+1} a unit normal vector on M, and we will give examples of hypersurfaces that satisfy the differential equation $\Delta x = Rx+b$ for some R and b.

3.1. **Euclidean case.** (a) $M = \{x \in \mathbb{R}^{n+1} : \|x - c\|^2 = r^2\}$, a standard sphere. Then $e_{n+1} = (x-c)/r$, and

$$\Delta x = -\frac{n}{r^2} I \cdot x + \frac{n}{r^2} c.$$

(b) Let $M = \{(y, z) \in \mathbb{R}^{k+1} \times \mathbb{R}^{n-k+1} : \|y - c\|^2 = r^2\}$, a cylinder. Then $e_{n+1} = ((y-c)/r, 0)$, and

$$\Delta x = \begin{pmatrix} -kI/r^2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix} + \begin{pmatrix} kc/r^2 \\ 0 \end{pmatrix}.$$

3.2. **Spherical case.** (a) $M = \mathbb{S}^n(\sin\theta) \subset \mathbb{S}^{n+1}$. Let $x = y + v$ where $y = (x_1, \dots, x_{n+1}, 0)$ and $v = (0, \dots, 0, \cos\theta)$. Then $e_{n+1} = \cot\theta \cdot y - \tan\theta \cdot v$ and

$$\Delta x = \begin{pmatrix} -n\csc^2\theta I & 0 \\ 0 & 0 \end{pmatrix} x.$$

(b) $M = \mathbb{S}^k(\cos\theta) \times \mathbb{S}^{n-k}(\sin\theta) \subset \mathbb{S}^{n+1}$. Let $x = (x_1, x_2) \in \mathbb{R}^{k+1} \times \mathbb{R}^{n-k}$; then $e_{n+1} = (-\tan\theta \cdot x_1, \cot\theta \cdot x_2)$ and

$$\Delta x = \begin{pmatrix} -k\sec^2\theta I & 0 \\ 0 & -(n-k)\csc^2\theta I \end{pmatrix} x.$$

3.3. **Hyperbolic case.** (a) $M = \mathbb{S}^n(\sinh\theta) \subset \mathbb{H}^{n+1}$. Let $x = (y, \cosh\theta)$, where $y = (y_1, \dots, y_{n+1})$; then $e_{n+1} = (\coth\theta \cdot y, \sinh\theta)$,

$$\Delta x = \begin{pmatrix} -n\,\mathrm{csch}^2\theta I & 0 \\ 0 & 0 \end{pmatrix} x.$$

(b) $M = \mathbb{H}^n(\cosh\theta) \subset \mathbb{H}^{n+1}$. Let $x = (\sinh\theta, y)$, where $y = (y_1, \dots, y_{n+1})$; then $e_{n+1} = (\cosh\theta, \tanh\theta \cdot y)$,

$$\Delta x = \begin{pmatrix} 0 & 0 \\ 0 & n\,\mathrm{sech}^2\theta I \end{pmatrix} x.$$

(c) $M = \mathbb{S}^k(\sinh\theta) \times \mathbb{H}^{n-k}(\cosh\theta) \subset \mathbb{H}^{n+1}$. Let $x = (x_1, x_2)$; then $e_{n+1} = (\coth\theta \cdot x_1, \tanh\theta \cdot x_2)$,

$$\Delta x = \begin{pmatrix} -k\,\mathrm{csch}^2\theta I & 0 \\ 0 & (n-k)\,\mathrm{sech}^2\theta I \end{pmatrix} x.$$

(d) $M = \{x = (y, \frac{1}{2}|y|^2, \frac{1}{2}|y|^2 + 1) : y \in \mathbb{R}^n\} \subset \mathbb{H}^{n+1}$. ($M$ is isometric to $\mathbb{R}^n$.)

In this case, $e_{n+1} = (y, \frac{1}{2}|y|^2 - 1, \frac{1}{2}|y|^2)$,

$$\Delta x = \begin{pmatrix} \mathbf{O} & 0 & 0 \\ 0 & -n & n \\ 0 & -n & n \end{pmatrix} x.$$

3.4. **Lorentzian case.** (a) $M = \mathbb{S}^k(r) \times \mathbb{R}^{n-k-1,1}$, $1 \le k < n$. Let $x = (y, z)$; then $e_{n+1} = (y/r, 0)$,

$$\Delta x = \begin{pmatrix} -kI/r^2 & 0 \\ 0 & \mathbf{O} \end{pmatrix} x.$$

(b) $M=\mathbb{R}^{n-k}\times\mathbb{H}^k(r)$. Let $x=(y, z)$; then $e_{n+1}=(0, z/r)$,

$$\Delta x=\begin{pmatrix}\mathbf{O} & 0\\ 0 & kI/r^2\end{pmatrix}x.$$

(c) $M=\mathbb{R}^{n-k}\times\mathbb{S}^k_1(r)$. Let $x=(y, z)$; then $e_{n+1}=(0, z/r)$,

$$\Delta x=\begin{pmatrix}\mathbf{O} & 0\\ 0 & -kI/r^2\end{pmatrix}x.$$

(d) A B-scroll with $H\neq 0$. Take a null curve $\gamma(t)$ in $\mathbb{R}^{r+s+2,1}$ with a pseudo-orthonormal frame along $\gamma(t)$

$$\{X(t), Y(t), C(t), W_1(t), \dots, W_s(t), Z_1(t), \dots, Z_r(t)\}$$

such that

$$\begin{cases}\dot{\gamma}(t)=X(t);\\ \dot{C}(t)=-B(t)Y(t),\ B(t)\neq 0;\\ \dot{Z}_a(t)\in\operatorname{Span}\{Y(t), Z_1(t), \dots, Z_r(t)\}.\end{cases}$$

Then

$$\begin{aligned}x(t, y, w_1, \dots, w_s, z_1, \dots, z_r)&=\gamma(t)+yY(t)+\sum w_iW_i(t)+\sum z_aZ_a(t)\\ &\quad+\frac{1}{\lambda}C(t)-\sqrt{\frac{1}{\lambda^2}-\sum z_a^2}C(t)\end{aligned}$$

defines a pseudo-Riemannian isoparametric hypersurface with $m(u)=u^2(u-\lambda)$ as the minimal polynomial of $A_{e_{n+1}}$ (cf. [Ma]).

In this case, $e_{n+1}=\lambda(\sqrt{1/\lambda^2-\sum z_a^2}C(t)-\sum z_aZ_a(t))$. Suppose $\Delta x=Rx+b$ for some $R\in\mathrm{gl}(r+s+3)$ and $b\in\mathbb{R}^{r+s+2,1}$; then

$$\begin{cases}RY=RW_i=0,\\ R\gamma+b=r\lambda C,\\ RZ_a=-r\lambda^2Z_a,\\ RC=-r\lambda^2C.\end{cases}$$

The differential of the last equation gives $0=BRY=-r\lambda^2BC\neq 0$, which is a contradiction. Hence this example does not satisfy the desired p.d.e., and, in fact, it is easy to check that an isoparametric hypersurface whose shape operator is not diagonalizable cannot be of the desired type.

4. Proofs of theorems

Throughout this section, the immersion x of M into $N^{n+1}(c)$ or $\mathbb{R}^{n,1}$ will be assumed to satisfy $\Delta x=Rx+b$ ($b=0$ when $c\neq 0$), and $A_{e_{n+1}}$ is denoted by A.

For $N=N^{n+1}(c)$, let e_A ($e_{n+2}=x$ when $c=1$ or -1) be a local frame and ω_A its coframe on M. Define $r_{AB}=\langle Re_A, e_B\rangle$. Taking d of $Rx+b=\Delta x=He_{n+1}-cnx$,

$$R\,dx=\sum_i R\omega_i\otimes e_i=\sum_i\omega_i\otimes r_{ij}e_j+\omega_i\otimes r_{i,n+1}e_{n+1}+|c|\omega_i\otimes r_{i,n+2}x$$

and

$$dH\otimes e_{n+1}+H\,de_{n+1}-cn\,dx=\sum_i H_i\omega_i\otimes e_{n+1}-H\omega_{i,n+1}\otimes e_i-cn\omega_i\otimes e_i$$

are equal; hence, we get

$$\text{(4.1)} \qquad \begin{cases} r_{ij} = -Hh_{ij} - cn\delta_{ij}\,, \\ r_{i\,,\,n+1} = H_i \\ (r_{i\,,\,n+2} = 0 \quad \text{for } c \neq 0). \end{cases}$$

To calculate $r_{n+1\,,\,A}$, from $(\Delta + cn)x = (R + cn)x + b = He_{n+1}$, we have

$$R(He_{n+1}) = \Delta(He_{n+1})\,,$$

and, from Theorem 2.1, we get

$$\text{(4.2)} \qquad \begin{cases} Hr_{n+1\,,\,i} = -2\sum_j h_{ij}H_j - HH_i\,, \\ Hr_{n+1\,,\,n+1} = \Delta H - H\|A\|^2 \\ (Hr_{n+1\,,\,n+2} = cH^2 \quad \text{for } c = \pm 1). \end{cases}$$

On the other hand, from $Rx = -cnx + He_{n+1}$ when $c = \pm 1$,

$$\text{(4.3)} \qquad \begin{cases} r_{n+2\,,\,i} = 0\,, \\ r_{n+2\,,\,n+1} = H\,, \\ r_{n+2\,,\,n+2} = -cn. \end{cases}$$

For the Lorentzian case, define r_{AB} by $R = \sum_{AB} r_{AB}\omega^A \otimes e_B$. Then by a similar argument we obtain

$$\text{(4.4)} \qquad \begin{cases} r_{ij} = -\varepsilon\varepsilon_j Hh_{ij}\,, \\ r_{i\,,\,n+1} = H_i\,, \end{cases}$$

$$\text{(4.5)} \qquad \begin{cases} Hr_{n+1\,,\,i} = -2\varepsilon\varepsilon_i \sum_j \varepsilon_j h_{ij}H_j - \varepsilon\varepsilon_i HH_i\,, \\ Hr_{n+1\,,\,n+1} = \Delta H - \varepsilon H\|A\|^2. \end{cases}$$

From (4.1) and (4.4), $(r_{ij}) = -H \cdot A$ in $N^{n+1}(c)$, or $(\varepsilon_j r_{ij}) = -H \cdot A$ in $\mathbb{R}^{n,1}$, is a symmetric operator on TM ; hence, by taking d of

$$\text{(4.6)} \qquad \langle Re_i\,, e_j\rangle = \langle e_i\,, Re_j\rangle\,,$$

we have

Proposition 4.1.

$$r_{n+1\,,\,j}h_{ik} - r_{n+1\,,\,i}h_{jk} = r_{j\,,\,n+1}h_{ik} - r_{i\,,\,n+1}h_{jk} \quad \textit{for } N = N^{n+1}(c)\,,$$
$$\varepsilon_j r_{n+1\,,\,j}h_{ik} - \varepsilon_i r_{n+1\,,\,i}h_{jk} = \varepsilon r_{j\,,\,n+1}h_{ik} - \varepsilon r_{i\,,\,n+1}h_{jk} \quad \textit{for } N = \mathbb{R}^{n,1}.$$

Proof. We will prove these only for $N = \mathbb{R}^{n+1}$. Apply d to both sides of (4.6):

$$\sum_A \omega_{iA}r_{Aj} + r_{iA}\omega_{jA} = \sum_A \omega_{jA}r_{Ai} + r_{jA}\omega_{iA}\,,$$

and then use (4.1) and (4.2). □

Now define a field of selfadjoint operators $T \in C^\infty(T^*M \otimes TM)$ by $T = A + H \cdot I$. $(T = A + \varepsilon H \cdot I$, when $N = \mathbb{R}^{n,1}$.)

Proposition 4.2. $\langle T(\nabla H), v\rangle A(w) = \langle T(\nabla H), w\rangle A(v)\quad \forall v, w \in C^\infty(TM)$.

Proof. Using Proposition 4.1, (4.1), (4.2), (4.4), and (4.5), we see that for $N = N^{n+1}(c)$,

$$\left(\sum_l h_{jl}H_l + HH_j\right) h_{ik} = \left(\sum_l h_{il}H_l + HH_i\right) h_{jk}, \forall i, j, k;$$

for $N = \mathbb{R}^{n,1}$

$$\left(\sum_l \varepsilon_l h_{jl}H_l + HH_j\right) h_{ik} = \left(\sum_l \varepsilon_l h_{il}H_l + HH_i\right) h_{jk}, \forall i, j, k. \quad \square$$

Proposition 4.3. *M has constant mean curvature.*

Proof. To prove this, we will show that $\nabla H^2(p) = 0\ \ \forall p \in M$. Suppose not. Then $H(p)\nabla H(p) \neq 0$ for some $p \in M$.

Case 1. $T(\nabla H)(p) \neq 0$. Then, by the nondegeneracy of the metric, there exists a local tangent vector field v such that $\langle T(\nabla H), v\rangle \neq 0$. This implies that A has rank 1 on a neighborhood U of p by Proposition 4.2. We will get a contradiction in each ambient space case.

(i) $N = \mathbb{R}^{n+1}$. Choose e_i so that

$$\begin{cases} Ae_i = \lambda_i e_i, \\ \lambda_1 = H, \\ \lambda_i = 0, \qquad i > 1, \end{cases}$$

on U; then

$$\omega_{i,n+1} = \lambda_i \omega_i. \tag{4.7}$$

We will show that $\omega_{1j} = 0$. Put $d\lambda_i = \sum_k \lambda_{ik}\omega_k$, $\omega_{ij} = \sum_k \gamma_{ijk}\omega_k$.

From Proposition 4.2 when $i = k = 1$, $j > 1$, we have $H_j = 0$. Hence, $\lambda_{1j} = H_j = 0$, i.e.,

$$\lambda_{ij} = 0 \quad \text{when } i > 1, \text{ or } i = 1 \text{ and } j > 1. \tag{4.8}$$

Take d of (4.7), and using the Codazzi and the structure equations,

$$d\lambda_i \wedge \omega_i = \sum_j (\lambda_j - \lambda_i)\omega_{ij} \wedge \omega_j.$$

This gives

$$\begin{aligned} (\lambda_j - \lambda_i)\gamma_{ijk} &= (\lambda_k - \lambda_i)\gamma_{ikj} \quad \text{for distinct } i, j, k, \\ \lambda_{ik} &= (\lambda_i - \lambda_k)\gamma_{iki} \quad \text{for } i \neq k. \end{aligned}$$

Using $\lambda_i = \lambda_{ij} = \lambda_{1k} = 0$ when $i, j, k > 1$, we have

$$\begin{cases} \gamma_{i1j} = 0 & \text{when } i \neq j > 1, \\ \gamma_{1k1} = \gamma_{i1i} = 0 & \text{when } i, k > 1. \end{cases}$$

From $\gamma_{ijk} = -\gamma_{jik}$, we can easily show that

$$\gamma_{1ij} = 0 \quad \forall i, j. \tag{4.9}$$

Now,

$$\Delta H = \sum_i \nabla_{e_i}\nabla_{e_i} H + \sum_{i,k} H_i \omega_{ik}(e_k) = e_1(H_1) + \sum_k H_1\gamma_{1kk} = e_1(H_1) \quad (\text{by (4.9)}).$$

Hence the matrix of R with respect to e_A is of the form

$$\begin{pmatrix} -H^2 & 0 & \cdots & 0 & H_1 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ -3H_1 & 0 & \cdots & 0 & e_1(H_1)/H - H^2 \end{pmatrix}$$

by (4.1), (4.2), and $\|A\|^2 = H^2$. Hence, we have

$$e_1(H_1)/H - 2H^2 = c_1, \tag{4.10}$$

$$-He_1(H_1) + H^4 + 3H_1^2 = c_2, \tag{4.11}$$

where the constants c_1 and c_2 come from

$$\det(tI - R) = \sum_i (-1)^i c_i t^{n-i}.$$

Eliminate $e_1(H_1)$ in (4.10) and (4.11) to obtain

$$3H_1^2 - H^4 = c_1 H^2 + c_2.$$

Differentiate this; then, since $H_1 \neq 0$,

$$3e_1(H_1) - 2H^3 = c_1 H. \tag{4.12}$$

By (4.10) and (4.12), we conclude that H is a constant, which is a contradiction.

(ii) $N = N^{n+1}(c)$, $c = \pm 1$. Choose e_i so that

$$\begin{cases} Ae_i = \lambda_i e_i, & \\ \lambda_n = H, & \qquad \text{on } U. \\ \lambda_i = 0, \quad i < n, & \end{cases}$$

Then by Proposition 4.2, we have $H_i = 0$ when $i < n$; hence, the matrix of $R + cnI$ with respect to e_A is of the form

$$\begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \\ 0 & & \begin{pmatrix} -H^2 & H_n & 0 \\ -3H_n & * & cH \\ 0 & H & 0 \end{pmatrix} \end{pmatrix};$$

hence, $cH^4 = -c_3$, where c_3 is the coefficient of t^3 in $\det((R + cnI) - tI)$. We conclude that H is a constant, which is a contradiction.

(iii) $N = \mathbb{R}^{n,1}$. From the canonical forms of selfadjoint operators (see [O]), A is conjugate to the diagonal matrix

$$\begin{pmatrix} 0 & & & \\ & 0 & & \\ & & \ddots & \\ & & & \varepsilon H \end{pmatrix}$$

with respect to some orthonormal frame e_i. Hence, we have $-\omega_{n+1}^i = \lambda_i \omega^i$, where $\lambda_i = 0$ for $i < n$, and $\lambda_n = \varepsilon H$. In a way similar to the Euclidean case, we can prove that $\omega_j^n = 0$ and $\Delta H = \varepsilon_n e_n(H_n)$. Hence, the matrix of R with respect to e_A is of the form

$$\begin{pmatrix} 0 & & & & \\ & \ddots & & & \\ & & 0 & & \\ & & & -\varepsilon H^2 & H_n \\ & & & -3\varepsilon\varepsilon_n H_n & \varepsilon_n\varepsilon_n(H_n)/H - \varepsilon H^2 \end{pmatrix}.$$

Again as in the Euclidean case, by solving a differential equation, we conclude that H is a constant, which is a contradiction.

Case 2. $T(\nabla H)(p) = 0$. By Case 1, this holds on a neighborhood U of p. We will discuss the spherical and the Lorentzian cases only.

(i) For $N = \mathbb{S}^{n+1}$. It implies that $A(\nabla H) = -H\nabla H$, i.e., $-H$ is a principal curvature. Therefore, we can choose e_i so that

$$\begin{cases} A(e_i) = \lambda_i e_i, \\ \lambda_n = -H, & \text{on } U. \\ \nabla H = H_n e_n, \end{cases}$$

Now $R + nI$ is represented as the matrix

$$\begin{pmatrix} -H\lambda_1 & \cdots & \cdots & 0 & 0 & 0 \\ \vdots & -H\lambda_2 & & \vdots & \vdots & \vdots \\ \vdots & & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & \cdots & -H\lambda_n & H_n & 0 \\ 0 & \cdots & \cdots & H_n & * & H \\ 0 & \cdots & \cdots & 0 & H & 0 \end{pmatrix}$$

with

$$H = \sum_{i=1}^{n} \lambda_i, \qquad \lambda_n = -H.$$

This implies that $-H\lambda_1, \dots, -H\lambda_{n-1}$ are eigenvalues of the constant matrix $R + nI$ and hence all of them are constants, so $-\sum_{i=1}^{n-1} H\lambda_i = -H(H - \lambda_n) = -2H^2$ is a constant, which is a contradiction.

(ii) For $\mathbb{R}^{n,1}$. If $|\nabla H| \neq 0$, then

$$A = \begin{pmatrix} A' & 0 \\ 0 & -\varepsilon H \end{pmatrix} \quad \text{on } (\nabla H)^\perp \oplus \mathbb{R}(\nabla H)$$

where A' is selfadjoint. By choosing $e_n = \nabla H/|\nabla H|$, R can be written as

$$\begin{pmatrix} -H \cdot A' & \\ & \begin{pmatrix} -\varepsilon H^2 & H_n \\ \varepsilon\varepsilon_n H_n & * \end{pmatrix} \end{pmatrix}.$$

Hence the eigenvalues of $-H \cdot A'$ are constants and so is $H \cdot \operatorname{tr} A' = H \cdot \operatorname{tr} A - (-\varepsilon H^2) = 2\varepsilon H^2$, which is a contradiction.

If $|\nabla H| = 0$ locally, then choose a pseudo-orthonormal frame $e_1, \dots, e_n$ such that $|e_{n-1}| = |e_n| = 0$, $\langle e_{n-1}, e_n \rangle = 1$, and e_n is parallel to ∇H. Then

$$A = \begin{pmatrix} D_{n-k} & \\ & A'_k \end{pmatrix}$$

where D_{n-k} is an $(n-k)$-diagonal matrix and

$$A_2' = \begin{pmatrix} -\varepsilon H & \pm 1 \\ 0 & -\varepsilon H \end{pmatrix}, \qquad A_3' = \begin{pmatrix} -\varepsilon H & 0 & 1 \\ 1 & -\varepsilon H & 0 \\ 0 & 0 & -\varepsilon H \end{pmatrix}.$$

This gives the matrix for R

$$\begin{pmatrix} -H \cdot D_{n-k} & 0 & 0 \\ 0 & -H \cdot A_k' & * \\ 0 & * & * \end{pmatrix}.$$

This implies that $H \cdot \operatorname{tr} D_{n-k} = H \cdot (\operatorname{tr} A - \operatorname{tr} A_k') = (k+1)\varepsilon H^2$ is a constant, which is a contradiction.

Hence, we have

$$H(p)\nabla H(p) = \tfrac{1}{2}\nabla H(p)^2 = 0 \quad \forall p \in M,$$

i.e., H is a constant. □

Proposition 4.4. *If $H \neq 0$, then M is isoparametric.*

Proof. If $H \neq 0$, then $B = \frac{1}{H}(R + cnI)$ is conjugate to $\left(\begin{smallmatrix} -A & 0 \\ 0 & * \end{smallmatrix}\right)$. Since B is a constant matrix, A is conjugate to a constant matrix, so M is isoparametric. □

Now we prove the main theorems. All the isoparametric hypersurfaces in $\mathbb{R}^{n+1}$ are known to be hyperplanes, standard spheres, and circular cylinders [S], where the first one is minimal. For the hyperbolic case, Cartan [C] showed that the examples in §2, (2.3) are the only possible cases. The isoparametric hypersurfaces in $\mathbb{R}^{n,1}$ are classified [Ma] and the examples in §2 are all of them, but in our situation, (a)–(c) are the only possible cases when $H \neq 0$. For the spherical case, we need

Theorem 4.5. *If $x\colon M \to \mathbb{S}^{n+1}$ satisfies $\Delta x = Rx$ and $H \neq 0$, then the number of principal curvatures is either 1 or 2.*

Proof. Let $TM = E_1 \oplus \cdots \oplus E_g$ be the eigendecomposition corresponding to the principal curvatures λ_i, $i = 1, 2, \ldots, g$, and let E_i' be the eigenspace of B corresponding to λ_i. Then $E_i(p) \subset E_i'$, $\forall p \in M$. But we know the leaf of E_i is a sphere, which should belong to E_i', so $\dim E_i < \dim E_i'$. From

$$n = \sum_{i=1}^{g} \dim E_i < \sum_{i=1}^{g} \dim E_i' \leq n + 2 = \dim \mathbb{R}^{n+2},$$

we obtain $g = 1$ or 2. □

As mentioned before, the isoparametric hypersurfaces in $\mathbb{S}^{n+1}$ with $g = 1$ or 2 are hyperspheres or products of two spheres. This completes the proofs of the theorems in §1.

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