ON CONSTRUCTING BANACH SPACES
WITH NO UNCONDITIONAL BASIS

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ABSTRACT. There exists a weak Hilbert space with no unconditional basis.

The aim of this note is to show a rather general construction of Banach spaces with no unconditional basis. As a corollary, one obtains a weak Hilbert space with no unconditional basis, thus answering a question raised some years ago by several authors (cf., e.g., [CS, P]). Our approach is based on techniques first introduced by Johnson, Lindenstrauss, and Schechtman in [JLS] for the study of the Kalton-Peck space [KP]. These techniques were refined by Ketonen [K] and generalized further by Borzyszkowski [B]. The novelty of the present general approach consists of the use of a simple interpolation trick, which is nevertheless strong enough to allow for the construction of a weak Hilbert space which contains a subspace with no unconditional basis.

1. THE MAIN CONSTRUCTION

The standard notation from the Banach space theory used throughout this paper can be found, for example, in [P, TJ].

Let us recall that, if $(Z_k)_{k=1}^\infty$ is a family of finite-dimensional subspaces of a Banach space X, then the unconditional constant of $(Z_k)_{k=1}^\infty$, denoted by $\mathrm{unc}(Z_k)_{k=1}^\infty$, is the infimum of numbers $c > 0$ such that, for all finite sequences of vectors $(x_k)_k$, with $x_k \in Z_k$, and for all choices of signs $(\varepsilon_k)_k$, the following holds:

$$\left\| \sum_k \varepsilon_k x_k \right\| \leq c \left\| \sum_k x_k \right\|.$$

If $\mathrm{unc}(Z_k)_{k=1}^\infty$ is finite, we call $(Z_k)_{k=1}^\infty$ an unconditional decomposition.

A basis $(e_i)_{i=1}^\infty$ in a Banach space X is 1-conditional if $\mathrm{unc}(\mathrm{span}\{e_i\}_{i=1}^\infty) = 1$. If A is a set of positive integers, we denote $\mathrm{span}(\{e_i\}_{i \in A})$ in X by $X|_A$.

Throughout this paper we fix an interpolation functor which to any $0 < \theta < 1$ and any interpolation couple of Banach spaces (X_0, X_1), with norms $\|\cdot\|_0$ and $\|\cdot\|_1$ respectively, corresponds the space $(X_0, X_1)_\theta$, with the norm $\|\cdot\|_\theta$ such that:

(a) whenever (X_0, X_1) and (Y_0, Y_1) are two interpolation couples and T is a linear operator such that $\|T: X_j \to Y_j\| = M_j$, for $j = 0, 1$, then one has

$$(X_0, X_0)_\theta = X_0 \quad \text{and} \quad \|T: (X_0, X_1)_\theta \to (Y_0, Y_1)_\theta\| \le M_0^{1-\theta} M_1^\theta;$$

(b) whenever (X_0, X_1) is an interpolation couple and $(e_i)_{i=1}^\infty$ is a 1-unconditional basis in both X_0 and X_1 and A is a set of positive integers, then

$$(X_0|_A, X_1|_A)_\theta = (X_0, X_1)_\theta|_A.$$

It is well known (cf., e.g., [BL]) that the complex method satisfies conditions (a) and (b). The real K- and J-methods satisfy (b), but (a) holds only up to a factor depending on the parameters of the method. However, by multiplying the interpolation norms by suitable constants, one can actually satisfy condition (a) as well.

Condition (a) implies the following: for all $x \in X_0 \cap X_1$ one has

$$(1) \qquad\qquad\qquad \|x\|_\theta \le \|x\|_0^{1-\theta} \|x\|_1^\theta.$$

Let us observe the following fact crucial for further arguments.

Remark 1. Let $X_0 = X_1 = \mathbf{R}^n$ be two Banach spaces with norms $\|\cdot\|_0$ and $\|\cdot\|_1$, respectively. Let $\|\text{id}: X_0 \to X_1\| = D$, and let $z_0 \in \mathbf{R}^n$ be such that $\|z_0\|_0 = 1$ and $\|z_0\|_1 = D$. Then $\|z_0\|_\theta = D^\theta$.

Indeed, since $\|\text{id}: (X_0, X_1)_\theta \to X_1\| \le D^{1-\theta}$, we get, by (1),

$$D = \|z_0\|_1 \le D^{1-\theta} \|z_0\|_\theta \le D^{1-\theta} \|z_0\|_0^{1-\theta} \|z_0\|_1^\theta = D.$$

The main theorem of this section is

Theorem 1. *Let* $(X_0, \|\cdot\|_0)$ *and* $(X_1, \|\cdot\|_1)$ *be two Banach spaces of finite cotype, let* $X_1 \subset X_0$, *and let* $\|x\|_0 \le \|x\|_1$ *for* $x \in X_1$. *Let* $(e_i)_{i=1}^\infty$ *be a normalized 1-unconditional basis for* $(X_0, \|\cdot\|_0)$ *and* $(X_1, \|\cdot\|_1)$. *Assume that, for every infinite set* A *of positive integers, there exists a sequence* $(a_k)_{k \in A}$ *of scalars such that the series* $\sum_{k \in A} a_k e_k$ *is convergent in* X_0 *and divergent in* X_1. *Then for every increasing sequence* $(\theta_i)_{i=1}^\infty$ *in* $(0, 1)$, *and for every* $1 \le p \le \infty$, *the space* $(\sum \bigoplus (X_0, X_1)_{\theta_i})_p$ *has a subspace* Y *which has a basis but has no unconditional basis.*

Actually, the subspace Y even does not have the *local unconditional structure*.

The proof is based on a modification of the following criterion, due to Ketonen [K] and Borzyszkowski [B].

Proposition 1. *Let* Y *be a Banach space of finite cotype with a 2-dimensional unconditional decomposition* $Y = \overline{\text{span}(\{Z_k\}_{k=1}^\infty)}$. *Let* $Z_k = \text{span}\{x_k, y_k\}$ *for* $k = 1, 2, \dots$, *and assume there exists* $C \ge 1$ *such that, for every* $\alpha, \beta \in \mathbf{R}$ *and* $k = 1, 2, \dots$,

(i) $\max(|\alpha|, |\beta|) \le \|\alpha x_k + \beta y_k\| \le C(|\alpha| + |\beta|)$.

If $\text{unc}(Y) < \infty$, *then there exists an infinite set* A *of positive integers, a bounded linear operator* $T: \overline{\text{span}(\{Z_k\}_{k \in A})} \to \overline{\text{span}(\{Z_k\}_{k \in A})}$, *and* $a, b, c, d \in \mathbf{R}$ *such that:*

(ii) $T(\sum_{k \in A} \alpha_k x_k + \beta_k y_k) = \sum_{k \in A} (a\alpha_k + b\beta_k) x_k + (c\alpha_k + d\beta_k) y_k$;

(iii) *for every* $k \in A$ *and for every scalar* $\lambda \in \mathbf{R}$, *one has* $\|(T - \lambda \,\text{id})|_{Z_k}\| \ge C_1$, *where* $C_1 > 0$ *is an absolute constant.*

In fact, there is a function f such that if Y is of cotype q with the cotype q constant K_1, $K_2 = \text{unc}\{Z_k\}_{k=1}^{\infty}$, and C is a constant from (i), then $\|T\| \leq f(C, q, K_1, K_2, \text{unc}(Y))$.

We will use Proposition 1 by constructing, in the situation of Theorem 1, a subspace Y which does not admit a bounded operator T as above. This is done in the main technical proposition which states:

Proposition 2. *Let X_0, X_1, (θ_i), and p be as in Theorem 1. Then there exists a basic sequence $x_1, y_1, x_2, y_2, \ldots$ in $(\sum \bigoplus (X_0, X_1)_{\theta_i})_p$ and $C \geq 1$ such that, for every $\alpha, \beta \in \mathbf{R}$ and $k = 1, 2, \ldots$, one has*

   (i) *$\max(|\alpha|, |\beta|) \leq \|\alpha x_k + \beta y_k\| \leq C(|\alpha| + |\beta|)$;*

   (ii) *the decomposition $(Z_k)_{k=1}^{\infty}$, where $Z_k = \text{span}\{x_k, y_k\}$ for $k = 1, 2, \ldots$, is unconditional.*

Moreover, for every infinite set A of positive integers one has

   (iii) *the linear operator $\sum_{k \in A} \alpha_k x_k \rightarrow \sum_{k \in A} \alpha_k y_k$, acting from $\text{span}(\{x_k\}_{k \in A})$ to $\text{span}(\{y_k\}_{k \in A})$, is not bounded;*

   (iv) *the linear operator $\sum_{k \in A} \alpha_k y_k \rightarrow \sum_{k \in A} \alpha_k x_k$, acting from $\text{span}(\{y_k\}_{k \in A})$ to $\text{span}(\{x_k\}_{k \in A})$, is not bounded;*

   (v) *the projection $\sum_{k \in A} \alpha_k x_k + \beta_k y_k \rightarrow \sum_{k \in A} \alpha_k x_k$, acting from $\text{span}(\{Z_k\}_{k \in A})$ to $\text{span}(\{x_k\}_{k \in A})$, is not bounded.*

Assuming the truth of Propositions 1 and 2 we can prove Theorem 1.

Proof of Theorem 1. Let $(x_k)_{k=1}^{\infty}$ and $(y_k)_{k=1}^{\infty}$ be as in Proposition 2. Put $Z_k = \text{span}\{x_k, y_k\}$ and $Y = \overline{\text{span}(\{Z_k\}_{k=1}^{\infty})}$. Suppose that $\text{unc}(Y) < \infty$. By Proposition 1 there is an infinite set A of positive integers and a bounded operator $T: \overline{\text{span}(\{Z_k\}_{k=1}^{\infty})} \rightarrow \overline{\text{span}(\{Z_k\}_{k=1}^{\infty})}$ such that (i), (ii), and (iii) hold.

Assume $b \neq 0$. Suppose that $\sum_{k \in A} \beta_k y_k$ converges. Since T is continuous, $\sum_{k \in A} b\beta_k x_k + d\beta_k y_k$ converges too. It follows that $\sum_{k \in A} \beta_k x_k$ converges, which contradicts Proposition 2(iv). Similarly we get $c = 0$. Hence by Proposition 1(iii) $a \neq d$. It follows that the projection $\sum_{k \in A} \alpha_k x_k + \beta_k y_k \rightarrow \sum_{k \in A} \alpha_k x_k$ is continuous, but this contradicts Proposition 2(v).

Therefore, $\text{unc}(Y) = \infty$. □

We pass now to the proofs of Propositions 1 and 2.

Proof of Proposition 1. By [B, Proposition A], there is a constant $D < \infty$ and operators $T_n: Y \rightarrow Y$, for $n = 1, 2, \ldots$, such that:

(2) $$T_n(Z_k) \subset Z_k \quad \text{for each } k,$$

(3) $$\|T_n\| \leq D,$$

(4) $$\|(T_n - \lambda \, \text{id})|_{Z_k}\| \geq \tfrac{1}{8} \quad \text{for every } \lambda \in \mathbf{R}, \ k = 1, 2, \ldots, n.$$

Let $A_{n,k}$, for $n = 1, 2, \ldots$, $k = 1, 2, \ldots, n$, be the family of 2×2 matrices of operators $T_n|_{Z_k}$ with respect to the basis $\{x_k, y_k\}$. By (3) and condition (i), the coefficients of $A_{n,k}$ are bounded. Hence, by the Cantor diagonal process, we can extract an increasing sequence $(n_j)_{j=1}^{\infty}$ of integers such that for every k the limit $\lim_{j \to \infty} A_{n_j, k} = A_k$ exists. Let

$$A_k = \begin{bmatrix} a_k & b_k \\ c_k & d_k \end{bmatrix}.$$

Set

$$T_1\left(\sum_{k=1}^{\infty}\alpha_k x_k + \beta_k y_k\right) = \sum_{k=1}^{\infty}(a_k\alpha_k + b_k\beta_k)x_k + (c_k\alpha_k + d_k\beta_k)y_k.$$

By the Banach-Steinhaus theorem, $\|T_1\| \leq D$. Moreover,

(5) $\|(T_1 - \lambda\,\mathrm{id})|_{Z_k}\| \geq \frac{1}{8}$ for every $\lambda \in \mathbf{R}$, $k = 1, 2, \ldots$.

Fix $0 < C_0 < 1$ to be defined later, and choose a subsequence $(k_j)_{j=1}^{\infty}$ such that

$$\max(|a_{k_j} - a|, |b_{k_j} - b|, |c_{k_j} - c|, |d_{k_j} - d|) < C_0 2^{-j},$$

for $j = 1, 2, \ldots$. Of course, such a, b, c, d exist by compactness. Put $A = \{k_1, k_2, \ldots\}$, and define $T\colon \mathrm{span}(\{Z_k\}_{k\in A}) \to \mathrm{span}(\{Z_k\}_{k\in A})$ by

$$T\left(\sum_{k\in A}\alpha_k x_k + \beta_k y_k\right) = \sum_{k\in A}(a\alpha_k + b\beta_k)x_k + (c\alpha_k + d\beta_k)y_k.$$

Using condition (i) and the assumption that $(Z_k)_{k=1}^{\infty}$ is an unconditional decomposition of Y, it follows that $\|(T_1 - T)|_{\mathrm{span}(\{Z_k\}_{k\in A})}\|$ is bounded; so, T is bounded and hence it can be extended to the bounded operator on $\overline{\mathrm{span}(\{Z_k\}_{k\in A})}$. Finally, (5) and (i) imply that C_0 can be chosen such that (iii) holds. □

Before we start the proof of Proposition 2 we need the following lemma.

Lemma 1. *Let $(e_i)_{i=1}^{\infty}$ be a normalized 1-unconditional basis for two Banach spaces Y_0 and Y_1. Suppose that $\|\mathrm{id}\colon Y_0 \to Y_1\| = \infty$. Then for every $D \geq 1$ there exists a positive integer n_0 such that*

$$D \leq \|\mathrm{id}\colon Y_0|_{A_0} \to Y_1|_{A_0}\| \leq 2D,$$

where $A_0 = \{1, 2, \ldots, n_0\}$.

Proof. For $n = 1, 2, \ldots$, put $A_n = \{1, 2, \ldots, n\}$ and $d_n = \|\mathrm{id}\colon Y_0|_{A_n} \to Y_1|_{A_n}\|$. Observe that d_n is an increasing sequence and $\lim_{n\to\infty} d_n = \infty$. Now, the lemma follows from the inequality $d_{n+1} \leq 1 + d_n$. □

Proof of Proposition 2. First, observe that there exist two sequences $(D_n)_{n=3}^{\infty}$ and $(\varepsilon_n)_{n=3}^{\infty}$ such that $D_n \geq 1$, $0 < \varepsilon_n < 1$, and that, for every $n \geq 3$,

(6) $\varepsilon_n(8D_n^{\theta_{n-1}-\theta_n} + 4\varepsilon_{n+1}D_n)^{-1} > n$ and $\varepsilon_{n+1} < \varepsilon_n/2$.

This can be shown by an easy induction, since $\theta_{n-1} - \theta_n < 0$.

Set $\varepsilon_1 = \varepsilon_2 = 1$. By $(e_{ik})_{k=1}^{\infty}$ denote the normalized 1-unconditional basis in $(X_0, X_1)_{\theta_i}$ for $i = 1, 2, \ldots$. Fix any $a_i, b_i \in \{0, 1\}$, for $i = 1, 2, \ldots$, such that $a_1 = b_1 = 1$, $b_1 = a_2 = 0$, and

(7) $a_i = b_i = 1$ infinitely many times,

(8) $a_i = 0$ and $b_i = 1$ infinitely many times,

(9) $a_i = 1$ and $b_i = 0$ infinitely many times.

Then set

$$x_k = \sum_{i=1}^{\infty}\varepsilon_i a_i e_{ik}, \qquad y_k = \sum_{i=1}^{\infty}\varepsilon_i b_i e_{ik}.$$

To see (ii) it is enough to observe that, for $k \neq k'$, the spaces Z_k and $Z_{k'}$ have disjoint supports. Since $\|x_k\| < 2$, $\|y_k\| < 2$, and $\|\alpha x_k + \beta y_k\| \geq \|\alpha e_{1k} + \beta e_{2k}\| \geq \max(|\alpha|, |\beta|)$, we obtain (i). Now, it is easy to see that $x_1, y_1, x_2, y_2, \ldots$ is a basic sequence.

Next, let us prove part (iii). Let A be an infinite set of positive integers. Fix an arbitrary $m \geq 3$ such that $b_m = 1$ and $a_m = 0$. By the assumption on A and by Lemma 1, with $D = D_m$, we conclude that there exists a positive integer n_0 such that for $d_0 = \|\operatorname{id}: X_0|_{A_0} \to X_1|_{A_0}\|$ where $A_0 = A \cap \{1, 2, \ldots, n_0\}$, one has

$$(10) \qquad D_m \leq d_0 \leq 2D_m.$$

Choose $(\beta_k)_{k \in A_0}$ such that $\|\sum_{k \in A_0} \beta_k e_k\|_0 = 1$ and $\|\sum_{k \in A_0} \beta_k e_k\|_1 = d_0$. Denote $z_0^i = \sum_{k \in A_0} \beta_k e_{ik} \in (X_0, X_1)_{\theta_i}$. By property (c) and Remark 1, we obtain $\|z_0^i\|_{\theta_i} = d_0^{\theta_i}$, for $i = 1, 2, \ldots$. Thus, by (10), $D_m^{\theta_i} \leq \|z_0^i\|_{\theta_i} \leq 2D_m^{\theta_i}$.

We have

$$\frac{\|\sum_{k \in A_0} \beta_k y_k\|}{\|\sum_{k \in A_0} \beta_k x_k\|} = \frac{\|\sum_{i=1}^\infty \varepsilon_i b_i \sum_{k \in A_0} \beta_k e_{ik}\|}{\|\sum_{i \neq m} \varepsilon_i a_i \sum_{k \in A_0} \beta_k e_{ik}\|}$$

$$\geq \frac{\varepsilon_m \|z_0^m\|_{\theta_m}}{\sum_{i \neq m} \varepsilon_i \|z_0^i\|_{\theta_i}} \geq \frac{\varepsilon_m D_m^{\theta_m}}{2(\sum_{i \neq m} \varepsilon_i D_m^{\theta_i})}$$

$$\geq \frac{\varepsilon_m D_m^{\theta_m}}{8 D_m^{\theta_{m-1}} + 4\varepsilon_{m+1} D_m} \geq \varepsilon_m (8 D_m^{\theta_{m-1} - \theta_m} + 4\varepsilon_{m+1} D_m)^{-1} > m.$$

Since, by (8), m can be taken arbitrarily large, the obtained estimate implies (iii).

Similarly, using (9), we obtain (iv). To see (v), fix an arbitrary $m \geq 3$ such that $b_m = a_m = 1$. As in the proof of (iii), there exists a finite subset A_0 of A and a sequence $(\beta_k)_{k \in A_0}$ such that for $z_0^i = \sum_{k \in A_0} \beta_k e_{ik} \in (X_0, X_1)_{\theta_i}$ one has

$$\frac{\|\sum_{k \in A_0} \beta_k x_k\|}{\|\sum_{k \in A_0} \beta_k (x_k - y_k)\|} \geq \frac{\varepsilon_m \|z_0^m\|_{\theta_m}}{\sum_{i \neq m} \varepsilon_i \|z_0^i\|_{\theta_i}} > m.$$

Therefore, (v) is concluded by (7). □

Letting $X_j = l_{q_j}$ for $j = 0, 1$ and $1 \leq q_1 < q_0 \leq \infty$ and using a suitable interpolation functor, we get $(X_0, X_1)_\theta = l_q$ where $1/q = (1 - \theta)/q_0 + \theta/q_1$. As an immediate consequence of Theorem 1 we obtain

Corollary 1. *For any decreasing sequence $(q_i)_{i=1}^\infty$ in $(1, \infty)$ and any $1 \leq p \leq \infty$, the space $(\sum \oplus l_{q_i})_p$ has a subspace with no unconditional basis.*

Taking arbitrary $1 \leq p < q_i < 2$ and observing that the space $(\sum \oplus l_{q_i})_p$ is then isomorphic to a subspace of $L_p([0, 1])$, we reconstruct Ketonen's result.

Corollary 2. *For any $1 \leq p < 2$, the space $L_p([0, 1])$ has a subspace with no unconditional basis.*

2. WEAK HILBERT SPACES WITHOUT AN UNCONDITIONAL BASIS

In this section we shall use some notation and facts contained in [P]. The main result of this section is

Theorem 2. *There exists a weak Hilbert space with no unconditional basis.*

Let T be any version of the 2-convexified Tsirelson space, presented in [CS; P, Chapter 13]. To fix attention, we set, in the notation from [P], $T = X_\delta$, with $\delta = 2^{-1/2}$. Then T is a weak Hilbert space and the standard unit vector basis is normalized and 1-unconditional in T.

An interpolation functor required in this section will be assumed to satisfy, besides conditions (a) and (b) of §1, also the following condition:

(c) whenever (X_0, X_1) is an interpolation couple, $(l_2(X_0), l_2(X_1))_\theta = l_2((X_0, X_1)_\theta)$.

It is well known (cf., e.g., [BL]) that the complex method satisfies (c). An appropriate modification of K- and J-functionals shows that real interpolation methods, with suitably chosen parameters, also satisfy (c).

We require the following proposition, based on some rather general interpolation properties of T. To make the notation more compact, we set, for $0 < \theta < 1$, $T_\theta = (T, \ell_2)_\theta$.

Proposition 3. *There exists an increasing sequence $(\theta_k)_{k=1}^\infty$ in $(0, 1)$ such that the space $(\sum_{k=1}^\infty \oplus T_{\theta_k})_2$ is a weak Hilbert space.*

Now, the proof of Theorem 2 follows immediately from Theorem 1, Proposition 3, and the well-known fact that no subsequence of the basis of T is equivalent to the standard unit vector basis in ℓ_2 (see, e.g., [CS] or [P]).

Proof of Proposition 3. Fix a positive integer n, and let $J_n = \{n+1, n+2, \dots\}$.

First, we shall show that there exists an increasing sequence θ_k in $(0, 1)$ such that, if F is an $(n2^k)$-dimensional subspace of $T_{\theta_k}|_{J_n}$, then $d(F, \ell_2^{n2^k}) \leq 2$.

Indeed, it is easy to see from the definition of the norm on T (cf., e.g., [P, (3.9) and (3.10)]) that arbitrary vectors $y_1, \dots, y_{n2^k}$ in $T|_{J_n}$ satisfy

$$2^{-(k+1)/2} \left(\sum_{s=1}^{n2^k} \|y_s\|^2 \right)^{1/2} \leq \left\| \left(\sum_{s=1}^{n2^k} |y_s|^2 \right)^{1/2} \right\| \leq \left(\sum_{s=1}^{n2^k} \|y_s\|^2 \right)^{1/2}.$$

This yields, as in [P, Lemma 13.3], that for all orthogonal $n2^k \times n2^k$ matrices (a_{is}) and for all $n2^k$-tuples $(y_s)_{s=1}^{n2^k}$ in $T|_{J_n}$ we have

$$(11) \qquad \left(\sum_{i=1}^{n2^k} \left\| \sum_{s=1}^{n2^k} a_{is} y_s \right\|^2 \right)^{1/2} \leq c \cdot 2^{(k+1)/2} \left(\sum_{s=1}^{n2^k} \|y_s\|^2 \right)^{1/2},$$

where c is an absolute constant.

Now, we use an interpolation procedure that is well known in the present context (cf., e.g., [TJ, §28]). Consider the operator $R: (y_s)_{s=1}^{n2^k} \to (\sum_{s=1}^{n2^k} a_{is} y_s)_{i=1}^{n2^k}$. By (11), $\|R: \ell_2^{n2^k}(T) \to \ell_2^{n2^k}(T)\| \leq c \cdot 2^{(k+1)/2}$, and clearly $\|R: \ell_2^{n2^k}(\ell_2) \to \ell_2^{n2^k}(\ell_2)\| = 1$. Interpolating these estimates and using the fact that, by (d), $\ell_2^{n2^k}(T_\theta) = (\ell_2^{n2^k}(T), \ell_2^{n2^k}(T))_\theta$, we can choose θ_k such that

$$\|R: \ell_2^{n2^k}(T_{\theta_k}) \to \ell_2^{n2^k}(T_{\theta_k})\| \leq (c \cdot 2^{(k+1)/2})^{1-\theta_k} \leq 2.$$

This is equivalent to an inequality analogous to (11) and is known to imply the required distance estimate (cf., e.g., [P, Lemma 13.3] or [TJ, §27]).

For every $m = 2^\ell$, $\ell = 1, 2, \ldots,$ set

$$Y_m = T_{\theta_1}|_{J_{m/2}} \oplus_2 T_{\theta_2}|_{J_{m/2^2}} \oplus_2 T_{\theta_3}|_{J_{m/2^3}} \oplus_2 \cdots \oplus_2 T_{\theta_\ell}|_{J_{m/2^\ell}} \oplus_2 \left(\sum_{k > \ell} \bigoplus T_{\theta_k} \right)_2.$$

Observe that the codimension of Y_m is $m - 1$ and that every m-dimensional subspace F of Y_m satisfies $d(F, \ell_2^m) \leq 2$. A similar argument as in the proof of Lemma 13.5 in [P] shows that $(\sum_{k=1}^\infty \bigoplus T_{\theta_k})_2$ is a weak Hilbert space. □

Remark 2. Recall that the class of weak Hilbert spaces can be characterized by a linear behaviour of various functions associated with finite-dimensional structure of Banach spaces. Such functions are, for instance, the codimension of a nicely complemented Euclidean subspace in any n-dimensional subspace of X (cf., e.g., [P]), or, as shown in [JP], the uniformity function of the uniform approximation property of X. It is well known [P] that, for examples related to the Tsirelson space, these functions have much slower growth than linear, and the same has recently been shown in [NT] for an arbitrary weak Hilbert space with an unconditional basis. Obviously, for the weak Hilbert space constructed in Theorem 2, the mentioned functions, as well as many other functions discussed in [P], which do not necessarily characterize weak Hilbert spaces, have growth as slow as for the Tsirelson space.

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