

**HIGH-DIMENSIONAL KNOTS WITH $\pi_1 \cong \mathbb{Z}$
 ARE DETERMINED BY THEIR COMPLEMENTS
 IN ONE MORE DIMENSION THAN FARBER'S RANGE**

WILLIAM RICHTER

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ABSTRACT. The surgery theory of Browder, Lashof, and Shaneson reduces the study of high-dimensional smooth knots $\Sigma^n \hookrightarrow S^{n+2}$ with $\pi_1 \cong \mathbb{Z}$ to homotopy theory. We apply Williams's Poincaré embedding theorem to a highly connected Seifert surface. Then such knots are determined by their complements if the \mathbb{Z} -cover of the complement is $[(n+2)/3]$ -connected; we improve Farber's work by one dimension.

0. INTRODUCTION

A high-dimensional knot will mean a smooth, oriented, codimension two embedding $\Sigma^n \hookrightarrow S^{n+2}$ of an exotic sphere, with $n \geq 5$. See the survey of Kervaire and Weber [KW] for more details. For our purposes, two knots $\Sigma_i^n \hookrightarrow S^{n+2}$ are said to be *equivalent* if there is diffeomorphism $\phi: S^{n+2} \xrightarrow{\cong} S^{n+2}$ such that $\phi(\Sigma_1^n) = \Sigma_2^n$. A knot $\Sigma^n \hookrightarrow S^{n+2}$ has a *complement* $X = S^{n+2} - \Sigma^n \times D^2$, and is *determined by its complement* if it is equivalent to any other knot with diffeomorphic complement. The orientation of Σ^n and the trivialization of the normal bundle neighborhood give a preferred diffeomorphism $\beta: \Sigma^n \times S^1 \xrightarrow{\cong} \partial X$. We will call the composite

$$\alpha: S^n \times S^1 \xrightarrow{\varpi \times \text{id}} \Sigma^n \times S^1 \xrightarrow{\beta} \partial X \xrightarrow{\iota} X \quad (\varpi: S^n \xrightarrow{\cong} \Sigma^n \text{ the degree one map})$$

the *attaching map* of the knot. Let $\tau: S^n \times S^1 \xrightarrow{\cong} S^n \times S^1$ be the homotopy equivalence (actually a diffeomorphism) given by the generator of $\pi_1(\text{SO}(n+1)) \cong \mathbb{Z}/2$.

A knot $\Sigma^n \hookrightarrow S^{n+2}$ is called *r-simple* [Ke2, Fa1] if the \mathbb{Z} -cover \tilde{X} of the complement is r -connected. Levine [Le1], Kearton [Ke1], and Kojima [Ko] showed that $[(n-1)/2]$ -simple knots were determined by their complements, using ambient surgery on the Seifert surface. Using Wall's thickening theory [Wa3], Farber [Fa2] proved this for $([n/3]+1)$ -simple knots. We extend Farber's result by one dimension.

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Theorem A. For $n+3 \leq 3q$ and $n \geq 5$, $(q-1)$ -simple smooth knots $\Sigma^n \hookrightarrow S^{n+2}$ are determined by their complements.

Theorem B. Let $\Sigma^n \hookrightarrow S^{n+2}$ be a knot with complement X and attaching map $\alpha: S^n \times S^1 \rightarrow X$. There exists a homotopy equivalence $\zeta: X \xrightarrow{\cong} X$ so that the diagram

$$(1) \quad \begin{array}{ccc} S^n \times S^1 & \xrightarrow{\alpha} & X \\ \tau \downarrow & & \downarrow \zeta \\ S^n \times S^1 & \xrightarrow{\alpha} & X \end{array}$$

commutes up to homotopy, if the knot is $(q-1)$ -simple, $n+3 \leq 3q$, and $n \geq 5$.

We prove Theorem B via Williams’s Theorem 1.7; in our range the Poincaré embedding $M \times I \hookrightarrow S^{n+2}$ of a Seifert surface is determined by its “unstable normal invariant” $\rho \in \pi_{n+2}(\Sigma(M/\partial M))$. We construct another Poincaré embedding $M \times I \hookrightarrow S^{n+2}$ suggested by $\alpha \cdot \tau$ (Lemma 1.4), with the same unstable normal invariant ρ (Theorem 1.6). Theorem 1.7 implies the two Poincaré embeddings are “concordant”. We use this concordance to construct the homotopy automorphism ζ .

That Theorem B implies Theorem A is well known for PL knots, by the work of Browder [Br1], and Lashof and Shaneson [LS], which we extend to smooth knots in §2.

Our proof requires Levine’s result [Le3], that there exists highly connected Seifert surfaces. Together with Barratt [BR], circa ‘82, we have a purely homotopy proof which uses \mathbb{Z} -equivariant Hopf invariants and Ranicki’s [Ra] equivariant S -duality.

We conjecture that high-dimensional knots with $\pi_1 \cong \mathbb{Z}$ are determined by their complements. If $\pi_1 \not\cong \mathbb{Z}$, there exist counterexamples due to Cappell and Shaneson, Gordon, and Suciu [CS, Go, Su]. Theorem 1.6, which is true outside our range $n+3 \leq 3q$, and the appendix provide some evidence for the conjecture.

We have a homotopy theoretic proof [Ri] of Theorem 1.7, completing Williams’s program [Wi2], proving the result using Browder-Quinn Poincaré surgery [Br4, Qu].

Given a subspace $A \subset X$ and a map $f: A \rightarrow Y$, we will write the identification space $X \cup_f Y$ as $\varinjlim (X \xleftarrow{\iota} A \xrightarrow{f} Y)$, the *colimit* or *pushout* of the diagram.

1. POINCARÉ EMBEDDINGS AND THE PROOF OF THEOREM B

In Theorem B we can eliminate the condition that ζ be a homotopy equivalence.

Lemma 1.1. Any self-map ζ of the knot complement X making diagram (1) commute up to homotopy is a homotopy equivalence.

Proof. Diagram (1) implies that ζ is a self-map of the Poincaré pair $(X, \partial X)$, and $\zeta_*[X] = [X] \in H_{n+2}(X, \partial X) \xrightarrow{\cong} H_{n+1}(\partial X)$. Furthermore, ζ induces the identity on $\pi_1(X) \cong \mathbb{Z}$. By naturality of \mathbb{Z} -equivariant Poincaré duality [Le2,

Wa2], the composite

$$H_{\Lambda}^{n+2-*}(\tilde{X}, \partial\tilde{X}; \Lambda) \xrightarrow{\zeta_*} H_{\Lambda}^{n+2-*}(\tilde{X}, \partial\tilde{X}; \Lambda) \xrightarrow{[X] \cap \cdot} H_*(\tilde{X}) \xrightarrow{\zeta_*} H_*(\tilde{X})$$

is the cap product isomorphism $[X] \cap \cdot$, where $\Lambda = \mathbb{Z}[\mathbb{Z}]$. Hence ζ_* is surjective. Since the group ring Λ is Noetherian and X is a finite complex, $\zeta_*: H_*(\tilde{X}) \rightarrow H_*(\tilde{X})$ must be an isomorphism; hence, ζ is a homotopy equivalence by the Whitehead theorem. \square

For a smooth knot $\Sigma^n \hookrightarrow S^{n+2}$ with complement $(X, \partial X)^{n+2}$, Alexander duality and relative transversality give a map $h: X \rightarrow S^1$ which is transverse to the point $1 \in S^1$, with inverse image $h^{-1}(1) = (M, \partial M)$, and $\partial M = \Sigma^n$. M^{n+1} is called a *Seifert surface* for the knot. By the relative tubular neighborhood theorem, there is a codimension zero embedding $M \times I \subset X$ extending the embedding $\Sigma^n \times I \subset \partial X = \Sigma^n \times S^1 = \Sigma^n \times I \cup \Sigma^n \times I$, where I is the interval $[-1, 1]$. Let $A = \partial(M \times I)$.

Let $W = \overline{X - M \times I}$ be the *Seifert surface complement*. The knot complement X is then obtained by gluing together $M \times [-1, 1]$ and W along their common boundary $M \times \{-1, 1\}$. By writing X as the union of the Seifert surface and its complement, we obtain the decomposition $S^{n+2} = M \times I \cup W \cup \Sigma^n \times D^2$. Let \widehat{W} be the manifold with corners $\widehat{W} = W \cup \Sigma^n \times D^2$, so that $A = \partial\widehat{W}$. There is a deformation retraction $\varepsilon: \widehat{W} \rightarrow W$ which maps $\Sigma^n \times D^2$ onto $\Sigma^n \times I$. Let $f: A \rightarrow W$ be the composite of the inclusion $A = \partial(M \times I) = \partial\widehat{W} \subset \widehat{W}$ and the retraction $\varepsilon: \widehat{W} \rightarrow W$. Note that f is a cofibration, since ε restricts to a homeomorphism $\varepsilon: \partial\widehat{W} \rightarrow \partial W$.

Williams [Wil] studies codimension zero *Poincaré embeddings* [Br1, Br2, Br3, Wa1] of an m -dimensional oriented finite Poincaré pair $(Y, \partial Y)$ in the sphere S^m , which consist of a *complement* Z along with an *attaching map* $f: \partial Y \rightarrow Z$, such that the pushout $Y \cup_f Z$ is homotopy equivalent to S^m . Williams [Wil] defines two Poincaré embeddings with attaching maps $f_1: \partial Y \rightarrow Z_1$ and $f_2: \partial Y \rightarrow Z_2$ to be *concordant* if there exists a homotopy equivalence $\xi: Z_1 \rightarrow Z_2$ so that $f_2 \simeq \xi \cdot f_1: \partial Y \rightarrow Z_2$.

For the above Seifert surface embedding, $(M, \partial M)$ is an $(n+1)$ -dimensional oriented Poincaré pair, with an oriented homeomorphism $\varpi: S^n \rightarrow \partial M$, given by the orientation of the knot. $(M \times I, A)$ is an $(n+2)$ -dimensional Poincaré pair, with $A = M \times \{\pm 1\} \cup \Sigma^n \times I$, and the attaching map $f: A \rightarrow W$ gives a Poincaré embedding $(M \times I, A) \hookrightarrow S^{n+2}$, by the homotopy equivalence $S^{n+2} = M \times I \cup \widehat{W} \xrightarrow{1 \cup \varepsilon} M \times I \cup_f W$. Let $\omega: S^n \rightarrow M$ be the composite of ϖ and the inclusion $\iota: \partial M \rightarrow M$. Let $v_{\pm}: M \rightarrow W$ be the restriction of f to $M \times \pm 1 \subset A$. Let $\iota_{\pm}: M \rightarrow A$ be the inclusions $M \times \{\pm 1\} \subset A$. For any map $g: A \rightarrow W$ we denote by g_{\pm} the restrictions $g_{\pm}: M \times \{\pm 1\} \subset A \xrightarrow{f} W$ and $g_{\pm}: S^n \times I \subset A \xrightarrow{f} W$.

Definition 1.2. Define $\delta: S^n \times I \xrightarrow{\cong} S^n \times I$ by $\delta(x, t) = (e^{i\pi(t+1)} \cdot x, t)$, using the standard action of $S^1 = \text{SO}(2) \subset \text{SO}(n+1)$ on S^n . We define the diffeomorphism τ of $S^n \times S^1 = \varinjlim (S^n \times I \xleftarrow{\iota} S^n \times \{\pm 1\} \xrightarrow{\iota} S^n \times I)$ to be the identity on the left $S^n \times I$ and δ on the right $S^n \times I$. We define the self-map

γ of $A = \varinjlim (M \times \{\pm 1\} \xrightarrow{\omega \times \text{id}} S^n \times \{\pm 1\} \xrightarrow{l} S^n \times I)$ to be the identity on $M \times \{\pm 1\}$ and δ on $S^n \times I$.

Lemma 1.3. (1) *The self-map τ of $S^n \times S^1$ is homotopic to the composite*

$$\tau': S^n \times S^1 \xrightarrow{\text{coaction}} S^n \times S^1 \vee S^{n+1} \xrightarrow{\text{id} \vee \eta} S^n \times S^1 \vee S^n \xrightarrow{\text{id} \vee \iota_1} S^n \times S^1.$$

(2) *The self-map γ of A is homotopic to the composite*

$$A \xrightarrow{\text{coaction}} A \vee S^{n+1} \xrightarrow{\text{id} \vee \eta} A \vee S^n \xrightarrow{\text{id} \vee \omega} A \vee M \xrightarrow{\text{id} \vee \iota_*} A.$$

Proof. By the Barratt-Puppe sequence of the CW-complex $S^n \times S^1 = S^n \vee S^1 \cup e^{n+1}$, the self-map τ' is characterized up to homotopy by the property that: τ' induces the identity in homology, and the Hopf construction of the composite $S^n \times S^1 \xrightarrow{\tau'} S^n \times S^1 \xrightarrow{\pi_1} S^n$ is the generator $\eta \in \pi_{n+2}(S^{n+1}) \cong \mathbb{Z}/2$. But τ clearly satisfies both of these properties; hence (1). Now consider the rel boundary coaction map

$$S^n \times I \xrightarrow{\text{coaction}} S^n \times I \vee S^{n+1}$$

of $S^n \times I$. We can define the coaction map of $A = M \times I \cup S^n \times I$ onto its top cell by glueing the identity map on the left half $M \times I$ to the above rel boundary coaction map. Furthermore, the self-map δ of $S^n \times I$ is the identity on $S^n \times \{\pm 1\} \cup N \times I$ for some point N . Therefore, δ is homotopic, rel boundary, to the composite

$$S^n \times I \xrightarrow{\text{coaction}} S^n \times I \vee S^{n+1} \xrightarrow{\text{id} \vee g} S^n \times I \vee S^n \xrightarrow{\text{id} \vee \iota_1} S^n \times I$$

for some map $g \in \pi_{n+1}(S^n)$. By part (1), we see that $g = \eta \in \pi_{n+1}(S^n)$. By glueing in this rel boundary homotopy, the second assertion follows. \square

Lemma 1.4. *Let $f: A \rightarrow W$ be the attaching map of a Poincaré embedding $(M \times I, A) \hookrightarrow S^{n+2}$. Then the composite $A \xrightarrow{\gamma} A \xrightarrow{f} W$ is also the attaching map of a Poincaré embedding $(M \times I, A) \hookrightarrow S^{n+2}$. Furthermore, $(f \cdot \gamma)_+ = f_+ : M \times \{\pm 1\} \rightarrow W$ and $(f \cdot \gamma)_- = f_- \cdot \delta : S^n \times I \rightarrow W$.*

Consider a Poincaré embedding $(M, A) \hookrightarrow S^m$ of an oriented, finite m -dimensional Poincaré pair (M, A) , with complement W and attaching map $f: A \rightarrow W$. Let $\nu: S^m \xrightarrow{\cong} M \cup_f W$ be the homotopy equivalence for which the composite $\rho: S^m \xrightarrow{\nu} M \cup_f W \rightarrow M/A$ is orientation preserving. Williams [Wi1] calls $\rho \in \pi_m(M/A)$ the *unstable normal invariant*. Williams [Wi2] shows that Browder's cofibration [Br3] $S^m \xrightarrow{\rho} M/A \xrightarrow{\Sigma f \cdot \partial} \Sigma W$ is split by the degree one map $M/A \xrightarrow{\text{pinch}} S^m$, that the composite

$$(2) \quad M/A \xrightarrow{\partial} \Sigma A \xrightarrow{\Sigma(f) \cdot \partial \vee \text{pinch}} \Sigma W \vee S^m$$

is a homotopy equivalence. From this we deduce

Lemma 1.5. *Let $\rho, \rho' \in \pi_m(M/A)$ be the unstable normal invariants of the Poincaré embeddings $(M, A) \hookrightarrow S^m$ with attaching maps $f: A \rightarrow W$ and $f': A \rightarrow W'$. Then:*

- (1) $\rho = \rho'$ if and only if the composite $S^m \xrightarrow{\rho'} M/A \xrightarrow{\Sigma f' \cdot \partial} \Sigma W'$ is nullhomotopic.

- (2) If $W' = W$ and the suspensions of the attaching maps $f, f': A \rightarrow W$ are homotopic, then the unstable normal invariants are equal, $\rho = \rho' \in \pi_m(M/A)$.

Now consider our Seifert surface Poincaré embedding $(M \times I, A) \hookrightarrow S^{n+2}$, with attaching map $f: A \rightarrow W$, and unstable normal invariant $\rho \in \pi_{n+2}((M \times I)/A)$.

Theorem 1.6. *The Poincaré embeddings $(M \times I, A) \hookrightarrow S^{n+2}$ with attaching maps $f \cdot \gamma, f: A \rightarrow W$ have equal unstable normal invariant.*

Proof. This follows from the codimension one framed embedding of the Seifert surface, which implies the vanishing of $\Sigma\omega \in \pi_{n+1}(\Sigma M)$. That is, the cofibration sequence

$$S^{n+1} = \Sigma S^n \xrightarrow{\Sigma\omega} \Sigma M \xrightarrow{\Sigma\iota} \Sigma(M/S^n) \xrightarrow{\Sigma\partial} \Sigma^2 S^n = S^{n+2}$$

splits; we have a homotopy equivalence $\Sigma M \vee S^{n+2} \xrightarrow{\Sigma\iota \vee \rho} \Sigma(M/S^n)$. Hence $\Sigma\iota: \Sigma M \rightarrow \Sigma(M/S^n)$ has a left homotopy inverse, which implies that $S^{n+1} \xrightarrow{\Sigma\omega} \Sigma M$ is nullhomotopic. By Lemma 1.3(2), $\Sigma\gamma \simeq \text{id}: \Sigma A \rightarrow \Sigma A$. Hence, the maps $\Sigma f, \Sigma(f \cdot \gamma): \Sigma A \rightarrow \Sigma W$ are homotopic. The result follows from Lemma 1.5. \square

We recall the uniqueness part of Williams's [Wil] Poincaré embedding theorem.

Theorem 1.7. *Let (M, A) be an oriented, finite, m -dimensional Poincaré pair, with $\pi_1(A) = \pi_1(M) = 0$ and $m \geq 6$. Suppose M is n -dimensional as a CW-complex, and let $q = m - n - 1$. If $m < 3q$, then any two Poincaré embeddings of (M, A) in S^m whose unstable normal invariants are equal are concordant.*

Proof of Theorem B. Let $\Sigma^n \hookrightarrow S^{n+2}$ be a $(q - 1)$ -simple knot, $n + 3 \leq 3q$, with knot complement X^{n+2} , and attaching map $\alpha: S^n \times S^1 \rightarrow X$. Let M^{n+1} be a Seifert surface with resulting Poincaré embedding $(M \times I, A) \hookrightarrow S^{n+2}$, with attaching map $f: A \rightarrow W$. By a theorem of Levine [Le3] we can assume that M^{n+1} is $(q - 1)$ -connected. By Poincaré duality of the pair $(M, \partial M)^{n+1}$, M is then $(n + 1 - q)$ -dimensional. Theorems 1.6 and 1.7 imply the Poincaré embeddings with attaching maps $f \cdot \gamma, f: A \rightarrow W$ are concordant. Let $\xi: W \xrightarrow{\cong} W$ be a concordance, so $\xi \cdot f \simeq f \cdot \gamma: A \rightarrow W$. Since the geometric map f is a cofibration, we may assume that $\xi \cdot f = f \cdot \gamma: A \rightarrow W$. By Lemma 1.4 we have $\xi \cdot f_+ = f_+$ and $\xi \cdot f_- = f_- \cdot \delta$. The knot complement X is the pushout $X = \varinjlim (M \times I \xleftarrow{\iota} M \times \{-1, 1\} \xrightarrow{f_{\pm}} W)$, so we can define a self-map $\zeta: X \rightarrow X$ to be the identity on $M \times I$ and ξ on W . But the attaching map $\alpha: S^n \times S^1 \rightarrow X$ and the composite $\alpha \cdot \tau: S^n \times S^1 \rightarrow X$ are the induced maps of colimits of the following strictly commutative diagrams:

$$\begin{CD} M \times I @<\iota<< M \times \{\pm 1\} @>f_{\pm}>> W \\ @V\omega \times \text{id}VV @Vw \times \text{id}VV @Vf_{\pm}VV \\ S^n \times I @<\iota<< S^n \times \{\pm 1\} @<\iota<< S^n \times I \end{CD}$$

$$\begin{array}{ccccc}
 M \times I & \xleftarrow{\iota} & M \times \{\pm 1\} & \xrightarrow{f_+} & W \\
 \uparrow \omega \times \text{id} & & \omega \times \text{id} \uparrow & & \uparrow f_- \cdot \delta \\
 S^n \times I & \xleftarrow{\iota} & S^n \times \{\pm 1\} & \xrightarrow{\iota} & S^n \times I
 \end{array}$$

Using Definition 1.2, we have $\zeta \cdot \alpha = \alpha \cdot \tau: S^n \times S^1 \rightarrow X$, and thus diagram (1) commutes. By Lemma 1.1 $\zeta: X \rightarrow X$ is a homotopy equivalence. \square

2. PROOF OF THEOREM A; SMOOTH KNOTS AND SURGERY

Lashof and Shaneson [LS, Theorem 2.1] show that any self-homotopy equivalence of a knot complement pair $(X, \partial X)^{n+2}$ is homotopic to a diffeomorphism, if $n \geq 5$ and $\pi_1(X) \cong \mathbb{Z}$. This follows from the Sullivan-Wall surgery exact sequence [Wa2, §10]

$$0 = L_{n+3}(\mathbb{Z}[\mathbb{Z}] \xrightarrow{\text{id}} \mathbb{Z}[\mathbb{Z}]) \rightarrow \mathcal{S}^0(X_1) \rightarrow [X, G/O] = 0.$$

Let $\phi: X_1 \xrightarrow{\cong} X_2$ be a diffeomorphism between the knot complements of two smooth $(q-1)$ -simple knots $\Sigma_i^n \hookrightarrow S^{n+2}$ with $n+3 \leq 3q$, $n \geq 5$, for $i = 1, 2$. The homotopy equivalence $\zeta: X_1 \xrightarrow{\cong} X_1$ of Theorem B is thus homotopic to a diffeomorphism $\theta: X_1 \xrightarrow{\cong} X_1$. Following Browder [Br1, Corollary 2], we have an exact sequence

$$(3) \quad \Gamma^{n+1} \oplus \Gamma^{n+2} \xrightarrow{\iota} \text{Diff}(\Sigma_1^n \times S^1) \xrightarrow{\mathcal{F}} \mathcal{E}(S^n \times S^1) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2,$$

involving the pseudoisotopy and homotopy automorphism groups. The first two $\mathbb{Z}/2$ summands are given by the degree -1 maps of S^n and S^1 , and the third summand is given by the self-map τ , which is detected by the Hopf construction (cf. Lemma 1.3). We have to modify Browder’s argument slightly, since \mathcal{F} will not be surjective if the exotic sphere Σ_1^n does not possess an orientation-preserving diffeomorphism (cf. [KM]).

Let $\beta_i: \Sigma_i^n \times S^1 \xrightarrow{\cong} \partial X_i$ be the preferred diffeomorphisms, for $i = 1, 2$. The restriction of ϕ to the boundary gives a diffeomorphism $\partial\phi: \Sigma_1^n \times S^1 \xrightarrow{\cong} \Sigma_2^n \times S^1$. If the Hopf construction of the composite $\pi_2 \cdot \partial\phi: \Sigma_1^n \times S^1 \rightarrow \Sigma_2^n$ is $\eta \in \pi_{n+2}(S^{n+1}) \cong \mathbb{Z}/2$, then replace the diffeomorphism ϕ by the composite $\phi \cdot \theta$. By Browder’s application of the Browder-Levine fibering theorem [Br1, Lemma 2], we can assume that $\partial\phi$ restricts to a diffeomorphism $\phi_0: \Sigma_1^n \xrightarrow{\cong} \Sigma_2^n$. Let $\varepsilon = \pm 1$ be the degree of $\partial\phi$ on the S^1 factor. Consider the diffeomorphism $\psi = (\phi_0 \times \varepsilon)^{-1} \cdot \partial\phi \in \text{Diff}(\Sigma_1^n \times S^1)$, which induces the identity in homology. Since its Hopf construction is zero, $\psi \simeq \text{id}$, so $\mathcal{F}(\psi) = \text{id}$, and (3) shows that $\partial\phi = (\phi_0 \times \varepsilon) \cdot \psi$ is pseudoisotopic to the composite

$$\Sigma_1^n \times S^1 \xrightarrow{e} \Sigma_1^n \times S^1 \xrightarrow{d \times 1} \Sigma_1^n \times S^1 \xrightarrow{\phi_0 \times \varepsilon} \Sigma_2^n \times S^1,$$

where $d \in \Gamma^{n+1}$ is a diffeomorphism of Σ_1^n , and $e \in \Gamma^{n+2}$ is obtained from the identity map of $\Sigma_1^n \times S^1$ by “connecting sum” with a diffeomorphism of an $(n+1)$ -disk. We claim that $\partial\phi$ extends to a diffeomorphism of $\widetilde{\partial\phi}: \Sigma_1^n \times D^2 \xrightarrow{\cong} \Sigma_2^n \times D^2$. Certainly $(\phi_0 \cdot d) \times \varepsilon$ extends. But e must be pseudoisotopic to the

identity, otherwise we could glue in the tubular neighborhoods to get a diffeomorphism between the standard sphere S^{n+2} and the exotic sphere represented by $e \in \Gamma^{n+2}$. By gluing together ϕ with the extension $\widetilde{\partial\phi}$, we have an equivalence of the two knots.

3. APPENDIX

Farber [Fa2] shows that PL $(q - 1)$ -simple knots with $n + 3 < 3q$ are classified up to isotopy by a stable homotopy Seifert pairing, which amounts to the homotopy class $v_+ \in [M, W]$, which lies in a stable group by the Freudenthal suspension theorem. Using our S-duality map $\Delta: S^{n+3} \rightarrow \Sigma W \wedge \Sigma M$ [Ri] we have a bijection $[M, W] \xrightarrow{\Delta \cap \cdot} \pi_{n+3}^s((\Sigma W)^{[2]})$ in Farber’s range. We note that Farber uses a dual S-duality map $M \wedge W \rightarrow S^{n+1}$. We show that Farber’s stable homotopy invariant is essentially the second Hopf invariant $\lambda_2(\rho)$ of our unstable normal invariant.

Theorem 3.1. *The second Hopf invariant $\lambda_2(\rho)$ of the unstable normal invariant $\rho: S^{n+2} \rightarrow \Sigma(M/\partial M)$ is the S-dual of the map $v_+: M \rightarrow W$:*

$$\lambda_2(\rho) = [\Sigma l \cdot (\Sigma v_- - \Sigma v_+)^{-1}]^{[2]}(\text{id} \wedge v_+) \cdot \Sigma \Delta \in \pi_{n+3}^s((\Sigma(M/\partial M))^{[2]}),$$

using the isomorphisms

$$[M, W] \xrightarrow[\cong]{\Delta \cap \cdot} \pi_{n+3}^s((\Sigma W)^{[2]}) \xrightarrow[\cong]{[\Sigma l \cdot (\Sigma v_- - \Sigma v_+)^{-1}]^{[2]}} \pi_{n+3}^s((\Sigma(M/\partial M))^{[2]}).$$

Consider the general case of a Poincaré embedding $(M, A) \hookrightarrow S^m$, with complement W and attaching map $f: A \rightarrow W$ as in [Ri]. The boundary map $\partial: M/A \rightarrow \Sigma A$ is defined to be the homotopy class making the diagram

$$\begin{array}{ccc} M \cup_l CA & \xrightarrow{\text{pinch}} & \Sigma A \\ \text{pinch} \downarrow \cong & \nearrow \partial & \\ M/A & & \end{array} \quad \left(\begin{array}{l} CA = A \times [0, 1]/A \times 0, \\ \Sigma A = CA/A = A \wedge ([0, 1]/\{0, 1\}) \end{array} \right)$$

commute up to homotopy. Let $\rho_0 = \partial \cdot \rho: S^m \rightarrow \Sigma A$ be the composite. Extending the splitting (2), from Williams’s work [Wi2] we have a homotopy equivalence

$$\Pi = x \cdot \Sigma f + y \cdot \text{pinch} + z \cdot \Sigma l: \Sigma A \xrightarrow{\cong} \Sigma W \vee S^m \vee \Sigma M,$$

where x, y , and z are the inclusions of the three factors, and $\text{pinch}: \Sigma A \rightarrow S^m$ is the unique degree one homotopy class. The two maps $\Pi, y: \Sigma A \rightarrow \Sigma W \vee S^m \vee \Sigma M$ are equalized up to homotopy by the collapse map $M \cup_{A \times 1} A \times [0, 1] \cup_f W \rightarrow \Sigma A$: the first and third maps $x \cdot \Sigma f$ and $z \cdot \Sigma l$ can be nullhomotoped when restricted to $M \cup_{A \times 1} A \times [0, 1] \cup_f W$ since the ends M and W are “free” (as in the proof of the equivalence of Whitehead products and Samelson products

[Wh]). Thus the diagram

$$\begin{array}{ccccc}
 M/A & \xleftarrow{\rho} & S^m & \xrightarrow{y} & \Sigma W \vee S^m \vee \Sigma M \\
 \uparrow \text{pinch} & & \downarrow \cong & \searrow \rho_0 & \cong \uparrow \Pi \\
 M \cup_i CA & \xleftarrow{\text{pinch}} & M \cup_{A \times 1} A \times [0, 1] \cup_f W & \xrightarrow{\text{pinch}} & \Sigma A
 \end{array}$$

is homotopy commutative. Now apply Boardman and Steer’s Cartan formula and composition formula [BS] to the equation $\Pi \cdot \rho_0 \simeq y: S^m \rightarrow \Sigma W \vee S^m \vee \Sigma M$. We obtain $\Pi \wedge \Pi \cdot \lambda_2(\rho_0) + \lambda_2(\Pi) \cdot \Sigma \rho_0 = 0$ and $\lambda_2(\Pi) = x\Sigma f \smile z\Sigma i$, which implies

$$\Pi \wedge \Pi \cdot \lambda_2(\rho_0) = -(x\Sigma f \smile z\Sigma i) \cdot \Sigma \rho_0 \in \pi_{m+1}((\Sigma W \vee S^m \vee \Sigma M)^{[2]}).$$

Proof of Theorem 3.1. Now consider a Seifert surface $(M \times I, A) \hookrightarrow S^{n+2}$, with complement W and attaching map $f: A \rightarrow W$. Let $h: A \rightarrow M/\partial M$ be defined by collapsing the subspace $M \times 1 \cup \partial M \times I$. Then we see that Σh is a homotopy retraction of $\partial: \Sigma(M/\partial M) \rightarrow \Sigma A$. Thus $\rho \simeq \Sigma h \cdot \rho_0: S^m \rightarrow \Sigma(M, \partial M)$, and $\lambda_2 \rho = (\Sigma h)^{[2]} \cdot \lambda_2 \rho_0$.

Let us factor $\Sigma h: \Sigma A \rightarrow \Sigma(M/\partial M)$ through the homotopy equivalence Π , by a map $\alpha \vee \beta \vee \gamma: \Sigma W \vee S^{n+2} \vee \Sigma M \rightarrow \Sigma(M/\partial M)$. The homotopy commutative diagram

$$\begin{array}{ccccc}
 \Sigma M \vee \Sigma M \vee S^{n+2} & & & & \\
 \downarrow * \vee \Sigma i \vee \rho & \searrow \cong & & \xrightarrow{(x\Sigma v_+ + z) \vee (x\Sigma v_- + z) \vee y} & \\
 \Sigma i_+ \vee \Sigma i_- \vee \rho & & \Sigma A & \xrightarrow[\cong]{\Pi} & \Sigma W \vee S^{n+2} \vee \Sigma M \\
 \downarrow \Sigma h & \swarrow \Sigma h & \swarrow \alpha \vee \beta \vee \gamma & & \\
 \Sigma(M/\partial M) & & & &
 \end{array}$$

yields $\alpha = \Sigma i \cdot (\Sigma v_- - \Sigma v_+)^{-1} \in [\Sigma W, \Sigma(M/\partial M)]$, $\gamma = -\Sigma i \cdot (\Sigma v_- - \Sigma v_+)^{-1} \cdot \Sigma v_+ \in [\Sigma M, \Sigma(M/\partial M)]$, and $\beta = \rho$. Recall that $\Pi^{[2]} \cdot \lambda_2(\rho_0) = -(x\Sigma f \smile z\Sigma i) \cdot \Sigma \rho_0$, which is the negative of the suspension of the S-duality map $\Delta: S^{n+2} \rightarrow \Sigma W \wedge M$. Thus,

$$\begin{aligned}
 \Sigma h &= [\Sigma i \cdot (\Sigma v_- - \Sigma v_+)^{-1} \vee \rho \vee -\Sigma i \cdot (\Sigma v_- - \Sigma v_+)^{-1} \cdot \Sigma v_+] \cdot \Pi, \\
 (\Sigma h)^{[2]} \cdot \lambda_2(\rho_0) &= [\Sigma i \cdot (\Sigma v_- - \Sigma v_+)^{-1}]^{[2]}(\text{id} \wedge v_+) \cdot \Sigma \Delta,
 \end{aligned}$$

and we are done. \square

Remark. This provides further evidence for our conjecture: the two Seifert surface Poincaré embeddings that we consider are equivalent under a stronger equivalence relation than Farber’s relation of isotopy.

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DEPARTMENT OF MATHEMATICS, MASSACHUSETTS INSTITUTE OF TECHNOLOGY, CAMBRIDGE, MASSACHUSETTS 02139

Current address: Department of Mathematics, Northwestern University, Evanston, Illinois 60208

E-mail address: richter@math.mit.edu