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AUTOMORPHISMS OF $P(V)_G$

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ABSTRACT. This paper gives a proof of the theorem that for Coxeter groups the algebra of coinvariants is isomorphic to the normalizer of the Coxeter group in the linear group of a vector space over $\mathbb{R}$. I have tried to give a relatively elementary proof which requires only elementary algebra and a little knowledge of the theory of invariants.

1. PRELIMINARIES

This paper is based on a Staatsexamensarbeit from 1990 with the title *Automorphismen von Ringen von Koinvarianten.* The theme and much help was given to me by Professor L. Smith. In addition, I thank Dr. D. Notbohm for his support in developing the proof of the main theorem. Some of the ideas of that proof are based on an article of Papadima [7]. Finally I thank E. Kurlanda for improving my English.

Before discussing the automorphisms of coinvariants we need some remarks and notation. In the following let G always be a Coxeter group (i.e., a finite effective reflection group) acting on a vector space $V = \mathbb{R}^n$ via the representation $\rho: G \to \mathrm{GL}(\mathbb{R})$ [4]. For this group we get a root system $\Delta = \{\pm n_i | i = 1, \dots, N\} \subset V$. Thus $n_i^\perp$ is the reflecting hyperplane of the reflection $s_i \in G$. Since G is effective, there are no fixed points of G in V and Δ contains a basis of V. With a representation $\rho(g): p(v) \mapsto p(g^{-1}v)$, $g \in G$, G acts on the polynomial algebra $P(V)$.

By the theorem of Shepard and Todd we know that the subring of invariant polynomials $P(V)^G$ is isomorphic to a polynomial algebra $P(V)^G = \mathbb{R}[I_1, \dots, I_n]$. $I_1, \dots, I_n$ are algebraically independent homogeneous invariants generating the ideal of $\mathfrak{I} := (I_1, \dots, I_n)$. Under these conditions $P(V)_G := P(V)/\mathfrak{I}$ defines the algebra of coinvariants. This is a graded finite-dimensional vector space [4]. Let $(P(V)_G)_d$ be the subspace of all cosets with homogeneous representatives of degree d. So we have a grading by degree for a homogeneous basis of the vector space $P(V)_G$. The subspace of highest degree has only one homogeneous basis vector called $\overline{J}$, with $J = \det(\partial I_i/\partial x_j)$ by Smith [11].

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2. Injectivity of the homomorphism

In this section we give a proof of the existence of a homomorphism of groups from the normalizer of G in $\mathrm{GL}(V)$, $N_{\mathrm{GL}(V)}(G)$, [1] to $\mathrm{Aut}(P(V)_G)$, called Φ^*, which is injective if G is a Coxeter group. A crucial fact is

Theorem 2.1. *For a Coxeter group G, the vector space V is isomorphic to $(P(V)_G)_1$.*

Proof. For every vector space we get $V = \operatorname{span}\Delta \oplus \Delta^\perp$ where $\Delta^\perp$ denotes the orthogonal complement of $\operatorname{span}\Delta$, that is, the intersection of all reflecting hyperplanes of G. Here G is a Coxeter group, so we have $V = \operatorname{span}\Delta$ since Coxeter groups are effective (see §1). With the isomorphism

$$\begin{aligned} \Psi\colon V &\to P(V)_1 \\ v &\mapsto p, \qquad p(x) := \langle v|x\rangle, \end{aligned}$$

it follows that V is isomorphic to $P(V)_1$. For the roots of G we define $l_i := \langle n_i|\cdot\rangle$ and $L := \{l_1, \dots, l_N\}$. So $l_i(x) = 0$ is the equation of the hyperplane of the reflection $s_i \in G$. Thus we get $\operatorname{span}\Delta \cong \operatorname{span}(l_1, \dots, l_N)$ and $\operatorname{span} L = P(V)_1$ if G is a Coxeter group because of the isomorphism Ψ. In the case of Coxeter groups we know that $P(V)_1^G = 0$. From this we can deduce $\operatorname{span}\overline{L} \cong \operatorname{span}\Delta$ which implies $(P(V)_G)_1 \cong V$. □

Now we will show that we can reduce our studies to the automorphisms of $(P(V)_G)_1$ instead of those of $P(V)_G$. For any monomial $cx_1^{k_1}\cdots x_n^{k_n}$ we get: $x_i = \sum_{j=1}^N \lambda_{ij} l_j(x)$, since $\operatorname{span} L = P(V)_1$. So it follows that

$$x_1^{k_1}\cdots x_n^{k_n} = \prod_{i=1}^n \left(\sum_{j=1}^N \lambda_{ij} l_j(x)\right)^{k_i} \tag{2.1}$$

$$\equiv \prod_{i=1}^n \left(\sum_{j=1}^N \lambda_{ij} l_j(x)\right)^{k_i} \qquad \bmod \mathfrak{I}, \tag{2.2}$$

and thus

$$\overline{x_1^{k_1}\cdots x_n^{k_n}} = \prod_{i=1}^n \left(\sum_{j=1}^N \lambda_{ij} \overline{l_j(x)}\right)^{k_i}. \tag{2.3}$$

From all this we deduce that every automorphism of $P(V)_G$ is uniquely determined by its image of $(P(V)_G)_1$. Now we can easily define the homomorphism Φ^*.

Definition 2.2. Define $\Phi^*\colon N_{\mathrm{GL}(V)}(G) \to \mathrm{Aut}(P(V)_G)$ by $\Phi^*(A)\colon p(v) \mapsto p(A^{-1}v)$ with $A \in N_{\mathrm{GL}(V)}(G)$ and $p \in P(V)$.

The main aim of this section is:

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[1] The normalizer is defined by $N_{\mathrm{GL}(V)}(G) := \{A \in \mathrm{GL}(V) | A(G) = (G)A\}$.

Theorem 2.3. *For Coxeter groups G, Φ^* defines an injective homomorphism of groups.*

Proof. First we show that Φ^* is a homomorphism. By $A\colon p(v) \mapsto p(A^{-1}v)$, $p \in P(V)$, every $A \in G$ obviously induces a $P(V)$-automorphism. Under these automorphisms $P(V)^G$ is $N_{\mathrm{GL}(V)}(G)$-invariant. To prove this, let q be in $P(V)^G$, $g \in G$, $A \in N_{\mathrm{GL}(V)}(G)$. So we get

$$\begin{aligned} gAq &= AA^{-1}gAq \\ &= A(A^{-1}gA)q = Ag'q \quad (\text{with } g' \in G) \\ &= Aq. \end{aligned}$$

Thus we get $AP(V)^G \subset P(V)^G$. This yields $A\colon f + \mathfrak{I} \mapsto Af + \mathfrak{I}$, for $f \in P(V)$ and so the definition $A\overline{p} := \overline{Ap}$ makes sense. With this definition A is an automorphism of $P(V)_G$, since the homomorphism is bijective. Suppose it is not injective. Then there must be a $\overline{p} \neq 0$ with $\overline{Ap} = 0$. Therefore, $Ap \in \mathfrak{I}$. And by $A\mathfrak{I} \subset \mathfrak{I}\ \forall A \in N_{\mathrm{GL}(V)}(G)$, we have $A^{-1}(Ap) = p \in \mathfrak{I}$, which is a contradiction to the assumption that $\overline{p} \neq 0$. Thus we get $\operatorname{Ker} A = \{0\}$ on $P(V)_G$, that is, the injectivity of A. A also is surjective on $P(V)_G$, for it is surjective on $P(V)$. Consequently Φ^* is a homomorphism of groups.

Finally we give a proof of the injectivity of this homomorphism Φ^*. Because $P(V)_1 \cong V$, $\Phi\colon N_{\mathrm{GL}(V)}(G) \to \operatorname{Aut}(P(V))$ with $\Phi(A)\colon p(v) \mapsto p(A^{-1}v)$ is injective, for only the identity transformation on V induces the identity on $P(V)_1$ and on $P(V)$ as well. So there must exist a $p \in P(V)$ for each $A \neq \mathrm{id}$ with $Ap \neq p$, because $\operatorname{Ker}\Phi = \{\mathrm{id}\}$.

Since G is a Coxeter group, we get $(P(V)_G)_1 \cong P(V)_1$. Hence with $Ap \neq p$ we get $\overline{Ap} \neq \overline{p}$, $p \in P(V)_1$. Thus $\operatorname{Ker}\Phi^* = \{\mathrm{id}\}$, the injectivity of Φ^* follows. □

Our main result, which is Theorem 3.7, is given as follows.

Theorem. *Let G be a Coxeter group. Then $\Phi^*\colon N_{\mathrm{GL}(V)}(G) \to \operatorname{Aut}(P(V)_G)$ is an isomorphism of groups.*

3. Surjectivity of Φ^*

The main subject of this section is the proof of

Theorem 3.1. *Since $P(V)_1 \cong (P(V)_G)_1 \cong V$, each $P(V)_G$-automorphism A induces an element of the linear group of V. Each of these are in $N_{\mathrm{GL}(V)}(G)$. Thus Φ^* is surjective.*

We will show that every element of the orthogonal group $O(V)$, which is induced by a $P(V)_G$-automorphism A, is an element of $N_{\mathrm{GL}(V)}(G)$. The orthogonality is no loss of generality, since we have Lemmata 3.2 and 3.3.

Lemma 3.2. *Let $A \in \operatorname{Aut}(P(V)_G)$ with A inducing a V-automorphism. Then there exists a transformation $K \in Z_{\mathrm{GL}(V)}(G)$, the centralizer of G, so that $AK \in \mathrm{GL}(V)$ is orthogonal.*

A proof is given by Papadima [7]. Because of $Z_{\mathrm{GL}(V)}(G) \subset N_{\mathrm{GL}(V)}(G)$, K must induce an element of $\operatorname{Aut}(P(V)_G)$, since Φ^* is injective by the last section. Hence $AK \in O(V)$ and $AK \in (\Phi^*)^{-1}(\mathrm{Aut}(P(V)_G))$, where $(\Phi^*)^{-1}(\mathrm{Aut}(P(V)_G))$ denotes the set of all elements from $\mathrm{GL}(V)$ which induce elements of $\mathrm{Aut}(P(V)_G)$. Therefore, it follows

Lemma 3.3. *If all orthogonal elements of* $(\Phi^*)^{-1}(\mathrm{Aut}(P(V)_G))$ *are elements of* $N_{\mathrm{GL}(V)}(G)$, *then*

$$(\Phi^*)^{-1}(\mathrm{Aut}(P(V)_G)) \subseteq N_{\mathrm{GL}(V)}(G).$$

Proof of 3.3. Let $A \in (\Phi^*)^{-1}(\mathrm{Aut}(P(V)_G))$. Then we obtain from Lemma 3.2 that there exists a $K \in Z_{\mathrm{GL}(V)}(G)$, with AK is orthogonal. As mentioned AK is an element of $(\Phi^*)^{-1}(\mathrm{Aut}(P(V)_G))$. It follows by assumption that $AK \in N_{\mathrm{GL}(V)}(G)$. Then for any $g \in G$ we get

$$\begin{aligned} A^{-1}gA &= KK^{-1}A^{-1}gAKK^{-1} = K((AK)^{-1}g(AK))K^{-1} \\ &= Kg'K^{-1} \quad (\text{with: } g' \in G, \ \text{ for } AK \in N_{\mathrm{GL}(V)}(G)). \end{aligned}$$

Furthermore, $Kg'K^{-1} = g'$, since: $K \in Z_{\mathrm{GL}(V)}(G)$. It finally follows that

$$A^{-1}gA \in G. \quad \square$$

It remains to show that every orthogonal automorphism of the set $(\Phi^*)^{-1}\,\mathrm{Aut}(P(V)_G)$ is in $N_{\mathrm{GL}(V)}(G)$. Before starting I prove some properties of the Jacobi-determinant and a special polynomial denoted p, since these properties are needed in a crucial way in the proof of Theorem 3.1.

Lemma 3.4. *The Jacobi-determinant*:

$$J := \frac{\partial(I_1, \dots, I_n)}{\partial(x_1, \dots, x_n)}$$

is antiinvariant; *that is,* $gJ = (\det g)J$, $\forall g \in G$ *with* G *acting on* $P(V)_G$ *as usual.*

A proof for this fact is given by Hiller [5, Chapter 2, §4] and in Bourbaki [1]. Now we have to define a new transformation, that is, the raising to the mth power:

$$P(v)_1 \to P(V)_m, \quad p \mapsto p^m \qquad \text{for } p \in P(V)_1.$$

We need the isomorphism $\Psi\colon V \to P(V)_1$ from §2. Let N be the number of reflections in G ($2N$ is the order of the root system). By [4], N is the degree of the homogeneous subspace of $P(V)_G$ of the maximal degree $(P(V)_G)_N$ [11]. Hence $\overline{\Psi(x)}^N$ and $\overline{J}$ are linearly dependent. So there must exist a $\nu \in \mathbb{R}$ with $\overline{\Psi(x)}^N = \nu\overline{J}$ where $\nu \in \mathbb{R}$ obviously depends on x. Thus ν is a real-valued function of $x \in V$, namely, $\nu =: p(x)$. The properties of ν and p are the subject of the following two lemmata.

Lemma 3.5. p *is a polynomial function of degree* N.

Proof of 3.5. By definition we get

$$\Psi(x)^N(y) = \left(\sum_{i=1}^{n} x_i y_i\right)^N.$$

This is a polynomial in the y_i with polynomials in the x_i as coefficients. Such a polynomial has a direct decomposition: $\Psi(x)^N(y) = p(x)J(y) + q(y)$. With

$q \in \mathfrak{J}$ and $p(x)$, J is a polynomial function of y with polynomials in the x_i's of degree N as coefficients. J does not depend on x, so we get

$$p \in \mathbb{R}[x_1, \dots, x_n] \quad \text{and} \quad \deg p \leq N. \quad \square$$

Lemma 3.6. *For p defined as above the following statements are valid.*

(a) *p is anti-invariant; that is, $gp = (\det g)p$, $\forall g \in G$.*
(b) *p is of the form $p = \gamma \prod_{s \in G} l_s$, $\gamma \in \mathbb{R}$.*
(c) *$\gamma \neq 0$ for the γ of* (b).

Proof of 3.6. (a) (I) Because $g \in G$ is an algebra automorphism, we get

$$\begin{aligned}\overline{(g(\Psi(x)))}^N &= g(p(x)\overline{J}) = p(x)g\overline{J} \quad (\text{for } p(x) = \text{const}) \\ &= p(x)(\det g)\overline{J}.\end{aligned}$$

The last equation is valid since J is anti-invariant.

(II) Furthermore,

$$g(\Psi(x)) = \langle x | g^{-1} \cdot \rangle = \langle (g^{-1})^{\mathrm{T}} x | \cdot \rangle = \Psi((g^{-1})^{\mathrm{T}} x).$$

Hence, we get $\overline{g(\Psi(x)))}^N = \overline{\Psi((g^{-1})^{\mathrm{T}} x)}^N$.

Finally, it follows that $\overline{\Psi((g^{-1})^{\mathrm{T}} x)}^N = p((g^{-1})^{\mathrm{T}} x)\overline{J} = (g^{\mathrm{T}} p(x))\overline{J}$. So with (I) and (II) we deduce $g^{\mathrm{T}} p(x) = (\det g)p(x)$. This is anti-invariance, since G is a Coxeter group; thus $g^{\mathrm{T}} = g^{-1}$ and $g^{\mathrm{T}} \in G$. This yields $\deg g = \det g^{\mathrm{T}}$, from which we finally get $gp = (\det g)p$.

(b) If $s \in G$ is a reflection, $\det s = -1$ and $s^{-1}x = x$ for all $x \in H_s$ (H_s is the reflecting hyperplane of s). Thus

$$p(x) = p(s^{-1}x) = sp(x) = (\det s)p(x) = -p(x) \Rightarrow p(x) = 0$$

for all $x \in H_s = \{x \in V | l_s(x) = 0\}$. Then the linear form l_s divides p. In addition, $\deg p = N$, from which we obtain $p = \gamma \prod_{s \in G} l_s$.

(c) Each polynomial of degree m is a linear combination of mth powers of homogeneous polynomials of $P(V)_1$ (see Humphreys [6, (S.134)]). Since $\Psi: V \to P(V)_1$ is bijective, there must be $x_1, \dots, x_r \in V$ for each $q \in P(V)_m$, so that

$$q = \sum_{i=1}^{r} \lambda_i (\Psi(x_i))^m.$$

Because of the surjectivity of the canonical homomorphism $P(V) \to P(V)_G$, there exist such $x_1, \dots, x_r \in V$ for each $\overline{q} \in P(V)_G$ so that

$$\overline{q} = \sum_{i=1}^{m} \lambda_i (\overline{\Psi(x_i)})^m.$$

Since $\overline{J}$ is in $(P(V)_G)_N$, we obtain $(P(V)_G)_N \neq \{0\}$. Thus there exists a $x \in V$ with $\overline{\Psi(x)}^N \neq 0$, since there exist $x_1, \dots, x_k$ so that $0 \neq \overline{J} = \sum_i \lambda_i \overline{\Psi(x_i)}^N$. We know that

$$\overline{\psi(x)}^N = p(x)\overline{J} \quad \forall x \in V$$

and $\overline{\Psi(x)}^N$ is not generally 0. Hence $p \neq 0$ and consequently is $\gamma \neq 0$ because of $p = \gamma \prod_{s \in G} l_s$. $\square$

Now we have finished the preliminaries and arrive at the proof of the main result in this section.

Proof of 3.1. Without loss of generality (Lemma 3.3) I suppose A to be an orthogonal automorphism of V. Since G is a finite reflection group, it suffices to show that for each reflection $s \in G$ is $s' := AsA^{-1} \in G$. After having shown that A transforms reflecting hyperplanes of G again in reflecting hyperplanes of G, it will be easy to show that conjugation with A transforms reflections of G into reflections of G.

From above we know (Lemma 3.6)

$$\overline{\Psi((A^{-1})^{\mathrm{T}}x)}^N = A\overline{\Psi(x)}^N = A(p(x))\overline{J} = p(x)A\overline{J} = p(x)c\overline{J}$$

with $c \in \mathbb{R}\backslash\{0\}$. So we have $A \in \mathrm{Aut}(P(V)_G)$, and it transforms the homogeneous subspace $(P(V)_G)_N = \mathrm{span}(\overline{J})$ into itself. Thus $\overline{J}$ will be transformed in $c\overline{J}$. In addition, we get

$$\overline{\Psi((A^{-1})^{\mathrm{T}}x)}^N = p((A^{-1})^{\mathrm{T}}x)\overline{J} = (A^{\mathrm{T}}p(x))\overline{J}.$$

Altogether, this yields $A^{\mathrm{T}}p(x) = cp(x)$. By assumption A is orthogonal, so is $A^{\mathrm{T}} = A^{-1}$. Thus it follows that

$$A^{\mathrm{T}}p(x) = p(Ax) = cp(x).$$

When $\mathrm{Null}(p)$ denotes the set of all $x \in V$ with $p(x)=0$, we obtain $A(\mathrm{Null}(p)) \subseteq \mathrm{Null}(p)$. This yields $A(\bigcup_{s\in G} H_s) \subseteq \bigcup_{s\in G} H_s$ since $p = \prod_{s\in G} l_s$ with $s \in G$ is a reflection. From the surjectivity of $A \in \mathrm{GL}(V)$ we get

$$A\left(\bigcup_{s\in G} H_s\right) = \bigcup_{s\in G} H_s.$$

By linearity, it follows that: $AH_s = H_{s'}$, s' is a reflection of G. Thus A permutes the reflecting hyperplanes. Let Δ be the root system of G and $A \in O(V)$ with $An_s \in \Delta \forall n_s \in \Delta$, where n_s is the root of the reflection $s \in G$. By [4] conjugation with A transforms reflections of G into reflections of G. This statement remains valid if we demand $An_s = \lambda n_{s'}$, $n_{s'} \in \Delta$, $\lambda \in \mathbb{R}\backslash\{0\}$, instead of $An_s = n_{s'}$. We have shown that $AH_s = H_{s'}$. Therefore, $AH_s^{\perp} = H_{s'}^{\perp}$ since A is orthogonal. Hence, $An_s = \lambda n_{s'}$, $\lambda \in \mathbb{R}\backslash\{0\}$. From this we deduce that $AsA^{-1} \in G$ for all reflections $s \in G$ with [4]. □

Now we have proved the surjectivity of Φ^*. Using §2, we arrive at:

Theorem 3.7 (main theorem). *Let G be a Coxeter group; then*

$$N_{\mathrm{GL}(V)}(G) \cong \mathrm{Aut}(P(V)_G).$$

Corollary 3.8. *From the last proof we can see that for $A \in \mathrm{Aut}(P(V)_G)$ it must be*

$$An_s = r(A, s)n_{s'}, \qquad A\overline{l_s} = r(A, s)\overline{l_{s'}}$$

with $r(A, s) \in \mathbb{R}\backslash\{0\}$ for the element $A \in N_{\mathrm{GL}(V)}(G)$ induced by the $P(V)_G$-automorphism A.

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