

## AUTOMORPHISMS OF $P(V)_G$

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**ABSTRACT.** This paper gives a proof of the theorem that for Coxeter groups the algebra of coinvariants is isomorphic to the normalizer of the Coxeter group in the linear group of a vector space over  $\mathbb{R}$ . I have tried to give a relatively elementary proof which requires only elementary algebra and a little knowledge of the theory of invariants.

### 1. PRELIMINARIES

This paper is based on a Staatsexamensarbeit from 1990 with the title *Automorphismen von Ringen von Koinvarianten*. The theme and much help was given to me by Professor L. Smith. In addition, I thank Dr. D. Notbohm for his support in developing the proof of the main theorem. Some of the ideas of that proof are based on an article of Papadima [7]. Finally I thank E. Kurlanda for improving my English.

Before discussing the automorphisms of coinvariants we need some remarks and notation. In the following let  $G$  always be a Coxeter group (i.e., a finite effective reflection group) acting on a vector space  $V = \mathbb{R}^n$  via the representation  $\rho: G \rightarrow \text{GL}(\mathbb{R})$  [4]. For this group we get a root system  $\Delta = \{\pm n_i | i = 1, \dots, N\} \subset V$ . Thus  $n_i^\perp$  is the reflecting hyperplane of the reflection  $s_i \in G$ . Since  $G$  is effective, there are no fixed points of  $G$  in  $V$  and  $\Delta$  contains a basis of  $V$ . With a representation  $\rho(g): p(v) \mapsto p(g^{-1}v)$ ,  $g \in G$ ,  $G$  acts on the polynomial algebra  $P(V)$ .

By the theorem of Shepard and Todd we know that the subring of invariant polynomials  $P(V)^G$  is isomorphic to a polynomial algebra  $P(V)^G = \mathbb{R}[I_1, \dots, I_n]$ .  $I_1, \dots, I_n$  are algebraically independent homogeneous invariants generating the ideal of  $\mathfrak{J} := (I_1, \dots, I_n)$ . Under these conditions  $P(V)_G := P(V)/\mathfrak{J}$  defines the algebra of coinvariants. This is a graded finite-dimensional vector space [4]. Let  $(P(V)_G)_d$  be the subspace of all cosets with homogeneous representatives of degree  $d$ . So we have a grading by degree for a homogeneous basis of the vector space  $P(V)_G$ . The subspace of highest degree has only one homogeneous basis vector called  $\bar{J}$ , with  $J = \det(\partial I_i / \partial x_j)$  by Smith [11].

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## 2. INJECTIVITY OF THE HOMOMORPHISM

In this section we give a proof of the existence of a homomorphism of groups from the normalizer of  $G$  in  $\mathrm{GL}(V)$ ,  $N_{\mathrm{GL}(V)}(G)$ ,<sup>1</sup> to  $\mathrm{Aut}(P(V)_G)$ , called  $\Phi^*$ , which is injective if  $G$  is a Coxeter group. A crucial fact is

**Theorem 2.1.** *For a Coxeter group  $G$ , the vector space  $V$  is isomorphic to  $(P(V)_G)_1$ .*

*Proof.* For every vector space we get  $V = \mathrm{span} \Delta \oplus \Delta^\perp$  where  $\Delta^\perp$  denotes the orthogonal complement of  $\mathrm{span} \Delta$ , that is, the intersection of all reflecting hyperplanes of  $G$ . Here  $G$  is a Coxeter group, so we have  $V = \mathrm{span} \Delta$  since Coxeter groups are effective (see §1). With the isomorphism

$$\begin{aligned} \Psi: V &\rightarrow P(V)_1 \\ v &\mapsto p, \quad p(x) := \langle v|x \rangle, \end{aligned}$$

it follows that  $V$  is isomorphic to  $P(V)_1$ . For the roots of  $G$  we define  $l_i := \langle n_i|\cdot \rangle$  and  $L := \{l_1, \dots, l_N\}$ . So  $l_i(x) = 0$  is the equation of the hyperplane of the reflection  $s_i \in G$ . Thus we get  $\mathrm{span} \Delta \cong \mathrm{span}(l_1, \dots, l_N)$  and  $\mathrm{span} L = P(V)_1$  if  $G$  is a Coxeter group because of the isomorphism  $\Psi$ . In the case of Coxeter groups we know that  $P(V)_1^G = 0$ . From this we can deduce  $\mathrm{span} \bar{L} \cong \mathrm{span} \Delta$  which implies  $(P(V)_G)_1 \cong V$ .  $\square$

Now we will show that we can reduce our studies to the automorphisms of  $(P(V)_G)_1$  instead of those of  $P(V)_G$ . For any monomial  $cx_1^{k_1} \dots x_n^{k_n}$  we get:  $x_i = \sum_{j=1}^N \lambda_{ij} l_j(x)$ , since  $\mathrm{span} L = P(V)_1$ . So it follows that

$$(2.1) \quad x_1^{k_1} \dots x_n^{k_n} = \prod_{i=1}^n \left( \sum_{j=1}^N \lambda_{ij} l_j(x) \right)^{k_i}$$

$$(2.2) \quad \equiv \prod_{i=1}^n \left( \sum_{j=1}^N \lambda_{ij} l_j(x) \right)^{k_i} \pmod{\mathfrak{I}},$$

and thus

$$(2.3) \quad \overline{x_1^{k_1} \dots x_n^{k_n}} = \prod_{i=1}^n \left( \sum_{j=1}^N \lambda_{ij} \overline{l_j(x)} \right)^{k_i}.$$

From all this we deduce that every automorphism of  $P(V)_G$  is uniquely determined by its image of  $(P(V)_G)_1$ . Now we can easily define the homomorphism  $\Phi^*$ .

**Definition 2.2.** Define  $\Phi^*: N_{\mathrm{GL}(V)}(G) \rightarrow \mathrm{Aut}(P(V)_G)$  by  $\Phi^*(A): p(v) \mapsto p(A^{-1}v)$  with  $A \in N_{\mathrm{GL}(V)}(G)$  and  $p \in P(V)$ .

The main aim of this section is:

<sup>1</sup> The normalizer is defined by  $N_{\mathrm{GL}(V)}(G) := \{A \in \mathrm{GL}(V) | A(G) = (G)A\}$ .

**Theorem 2.3.** *For Coxeter groups  $G$ ,  $\Phi^*$  defines an injective homomorphism of groups.*

*Proof.* First we show that  $\Phi^*$  is a homomorphism. By  $A: p(v) \mapsto p(A^{-1}v)$ ,  $p \in P(V)$ , every  $A \in G$  obviously induces a  $P(V)$ -automorphism. Under these automorphisms  $P(V)^G$  is  $N_{\text{GL}(V)}(G)$ -invariant. To prove this, let  $q$  be in  $P(V)^G$ ,  $g \in G$ ,  $A \in N_{\text{GL}(V)}(G)$ . So we get

$$\begin{aligned} gAq &= AA^{-1}gAq \\ &= A(A^{-1}gA)q = Ag'q \quad (\text{with } g' \in G) \\ &= Aq. \end{aligned}$$

Thus we get  $AP(V)^G \subset P(V)^G$ . This yields  $A: f + \mathcal{J} \mapsto Af + \mathcal{J}$ , for  $f \in P(V)$  and so the definition  $A\bar{p} := \overline{Ap}$  makes sense. With this definition  $A$  is an automorphism of  $P(V)_G$ , since the homomorphism is bijective. Suppose it is not injective. Then there must be a  $\bar{p} \neq 0$  with  $A\bar{p} = 0$ . Therefore,  $Ap \in \mathcal{J}$ . And by  $A\mathcal{J} \subset \mathcal{J} \forall A \in N_{\text{GL}(V)}(G)$ , we have  $A^{-1}(Ap) = p \in \mathcal{J}$ , which is a contradiction to the assumption that  $\bar{p} \neq 0$ . Thus we get  $\text{Ker } A = \{0\}$  on  $P(V)_G$ , that is, the injectivity of  $A$ .  $A$  also is surjective on  $P(V)_G$ , for it is surjective on  $P(V)$ . Consequently  $\Phi^*$  is a homomorphism of groups.

Finally we give a proof of the injectivity of this homomorphism  $\Phi^*$ . Because  $P(V)_1 \cong V$ ,  $\Phi: N_{\text{GL}(V)}(G) \rightarrow \text{Aut}(P(V))$  with  $\Phi(A): p(v) \mapsto p(A^{-1}v)$  is injective, for only the identity transformation on  $V$  induces the identity on  $P(V)_1$  and on  $P(V)$  as well. So there must exist a  $p \in P(V)$  for each  $A \neq \text{id}$  with  $Ap \neq p$ , because  $\text{Ker } \Phi = \{\text{id}\}$ .

Since  $G$  is a Coxeter group, we get  $(P(V)_G)_1 \cong P(V)_1$ . Hence with  $Ap \neq p$  we get  $A\bar{p} \neq \bar{p}$ ,  $p \in P(V)_1$ . Thus  $\text{Ker } \Phi^* = \{\text{id}\}$ , the injectivity of  $\Phi^*$  follows.  $\square$

Our main result, which is Theorem 3.7, is given as follows.

**Theorem.** *Let  $G$  be a Coxeter group. Then  $\Phi^*: N_{\text{GL}(V)}(G) \rightarrow \text{Aut}(P(V)_G)$  is an isomorphism of groups.*

### 3. SURJECTIVITY OF $\Phi^*$

The main subject of this section is the proof of

**Theorem 3.1.** *Since  $P(V)_1 \cong (P(V)_G)_1 \cong V$ , each  $P(V)_G$ -automorphism  $A$  induces an element of the linear group of  $V$ . Each of these are in  $N_{\text{GL}(V)}(G)$ . Thus  $\Phi^*$  is surjective.*

We will show that every element of the orthogonal group  $O(V)$ , which is induced by a  $P(V)_G$ -automorphism  $A$ , is an element of  $N_{\text{GL}(V)}(G)$ . The orthogonality is no loss of generality, since we have Lemmata 3.2 and 3.3.

**Lemma 3.2.** *Let  $A \in \text{Aut}(P(V)_G)$  with  $A$  inducing a  $V$ -automorphism. Then there exists a transformation  $K \in Z_{\text{GL}(V)}(G)$ , the centralizer of  $G$ , so that  $AK \in \text{GL}(V)$  is orthogonal.*

A proof is given by Papadima [7]. Because of  $Z_{\text{GL}(V)}(G) \subset N_{\text{GL}(V)}(G)$ ,  $K$  must induce an element of  $\text{Aut}(P(V)_G)$ , since  $\Phi^*$  is injective by the

last section. Hence  $AK \in O(V)$  and  $AK \in (\Phi^*)^{-1}(\text{Aut}(P(V)_G))$ , where  $(\Phi^*)^{-1}(\text{Aut}(P(V)_G))$  denotes the set of all elements from  $\text{GL}(V)$  which induce elements of  $\text{Aut}(P(V)_G)$ . Therefore, it follows

**Lemma 3.3.** *If all orthogonal elements of  $(\Phi^*)^{-1}(\text{Aut}(P(V)_G))$  are elements of  $N_{\text{GL}(V)}(G)$ , then*

$$(\Phi^*)^{-1}(\text{Aut}(P(V)_G)) \subseteq N_{\text{GL}(V)}(G).$$

*Proof of 3.3.* Let  $A \in (\Phi^*)^{-1}(\text{Aut}(P(V)_G))$ . Then we obtain from Lemma 3.2 that there exists a  $K \in Z_{\text{GL}(V)}(G)$ , with  $AK$  is orthogonal. As mentioned  $AK$  is an element of  $(\Phi^*)^{-1}(\text{Aut}(P(V)_G))$ . It follows by assumption that  $AK \in N_{\text{GL}(V)}(G)$ . Then for any  $g \in G$  we get

$$\begin{aligned} A^{-1}gA &= KK^{-1}A^{-1}gAKK^{-1} = K((AK)^{-1}g(AK))K^{-1} \\ &= Kg'K^{-1} \quad (\text{with: } g' \in G, \text{ for } AK \in N_{\text{GL}(V)}(G)). \end{aligned}$$

Furthermore,  $Kg'K^{-1} = g'$ , since:  $K \in Z_{\text{GL}(V)}(G)$ . It finally follows that

$$A^{-1}gA \in G. \quad \square$$

It remains to show that every orthogonal automorphism of the set  $(\Phi^*)^{-1} \text{Aut}(P(V)_G)$  is in  $N_{\text{GL}(V)}(G)$ . Before starting I prove some properties of the Jacobi-determinant and a special polynomial denoted  $p$ , since these properties are needed in a crucial way in the proof of Theorem 3.1.

**Lemma 3.4.** *The Jacobi-determinant:*

$$J := \frac{\partial(I_1, \dots, I_n)}{\partial(x_1, \dots, x_n)}$$

is antiinvariant; that is,  $gJ = (\det g)J$ ,  $\forall g \in G$  with  $G$  acting on  $P(V)_G$  as usual.

A proof for this fact is given by Hiller [5, Chapter 2, §4] and in Bourbaki [1]. Now we have to define a new transformation, that is, the raising to the  $m$ th power:

$$P(v)_1 \rightarrow P(V)_m, \quad p \mapsto p^m \quad \text{for } p \in P(V)_1.$$

We need the isomorphism  $\Psi: V \rightarrow P(V)_1$  from §2. Let  $N$  be the number of reflections in  $G$  ( $2N$  is the order of the root system). By [4],  $N$  is the degree of the homogeneous subspace of  $P(V)_G$  of the maximal degree  $(P(V)_G)_N$  [11]. Hence  $\overline{\Psi(x)}^N$  and  $\overline{J}$  are linearly dependent. So there must exist a  $\nu \in \mathbb{R}$  with  $\overline{\Psi(x)}^N = \nu \overline{J}$  where  $\nu \in \mathbb{R}$  obviously depends on  $x$ . Thus  $\nu$  is a real-valued function of  $x \in V$ , namely,  $\nu =: p(x)$ . The properties of  $\nu$  and  $p$  are the subject of the following two lemmata.

**Lemma 3.5.**  *$p$  is a polynomial function of degree  $N$ .*

*Proof of 3.5.* By definition we get

$$\Psi(x)^N(y) = \left( \sum_{i=1}^n x_i y_i \right)^N.$$

This is a polynomial in the  $y_i$  with polynomials in the  $x_i$  as coefficients. Such a polynomial has a direct decomposition:  $\Psi(x)^N(y) = p(x)J(y) + q(y)$ . With

$q \in \mathfrak{J}$  and  $p(x)$ ,  $J$  is a polynomial function of  $y$  with polynomials in the  $x_i$ 's of degree  $N$  as coefficients.  $J$  does not depend on  $x$ , so we get

$$p \in \mathbb{R}[x_1, \dots, x_n] \text{ and } \deg p \leq N. \quad \square$$

**Lemma 3.6.** *For  $p$  defined as above the following statements are valid.*

- (a)  $p$  is anti-invariant; that is,  $gp = (\det g)p$ ,  $\forall g \in G$ .
- (b)  $p$  is of the form  $p = \gamma \prod_{s \in G} l_s$ ,  $\gamma \in \mathbb{R}$ .
- (c)  $\gamma \neq 0$  for the  $\gamma$  of (b).

*Proof of 3.6.* (a) (I) Because  $g \in G$  is an algebra automorphism, we get

$$\begin{aligned} \overline{(g(\Psi(x)))^N} &= g(p(x)\bar{J}) = p(x)g\bar{J} \quad (\text{for } p(x) = \text{const}) \\ &= p(x)(\det g)\bar{J}. \end{aligned}$$

The last equation is valid since  $J$  is anti-invariant.

(II) Furthermore,

$$g(\Psi(x)) = \langle x | g^{-1} \cdot \rangle = \langle (g^{-1})^T x | \cdot \rangle = \Psi((g^{-1})^T x).$$

Hence, we get  $\overline{g(\Psi(x))}^N = \overline{\Psi((g^{-1})^T x)}^N$ .

Finally, it follows that  $\overline{\Psi((g^{-1})^T x)}^N = p((g^{-1})^T x)\bar{J} = (g^T p(x))\bar{J}$ . So with (I) and (II) we deduce  $g^T p(x) = (\det g)p(x)$ . This is anti-invariance, since  $G$  is a Coxeter group; thus  $g^T = g^{-1}$  and  $g^T \in G$ . This yields  $\deg g = \det g^T$ , from which we finally get  $gp = (\det g)p$ .

(b) If  $s \in G$  is a reflection,  $\det s = -1$  and  $s^{-1}x = x$  for all  $x \in H_s$  ( $H_s$  is the reflecting hyperplane of  $s$ ). Thus

$$p(x) = p(s^{-1}x) = sp(x) = (\det s)p(x) = -p(x) \Rightarrow p(x) = 0$$

for all  $x \in H_s = \{x \in V | l_s(x) = 0\}$ . Then the linear form  $l_s$  divides  $p$ . In addition,  $\deg p = N$ , from which we obtain  $p = \gamma \prod_{s \in G} l_s$ .

(c) Each polynomial of degree  $m$  is a linear combination of  $m$ th powers of homogeneous polynomials of  $P(V)_1$  (see Humphreys [6, (S.134)]). Since  $\Psi: V \rightarrow P(V)_1$  is bijective, there must be  $x_1, \dots, x_r \in V$  for each  $q \in P(V)_m$ , so that

$$q = \sum_{i=1}^r \lambda_i (\Psi(x_i))^m.$$

Because of the surjectivity of the canonical homomorphism  $P(V) \rightarrow P(V)_G$ , there exist such  $x_1, \dots, x_r \in V$  for each  $\bar{q} \in P(V)_G$  so that

$$\bar{q} = \sum_{i=1}^m \lambda_i \overline{\Psi(x_i)}^m.$$

Since  $\bar{J}$  is in  $(P(V)_G)_N$ , we obtain  $(P(V)_G)_N \neq \{0\}$ . Thus there exists a  $x \in V$  with  $\overline{\Psi(x)}^N \neq 0$ , since there exist  $x_1, \dots, x_k$  so that  $0 \neq \bar{J} = \sum_i \lambda_i \overline{\Psi(x_i)}^N$ . We know that

$$\overline{\psi(x)}^N = p(x)\bar{J} \quad \forall x \in V$$

and  $\overline{\Psi(x)}^N$  is not generally 0. Hence  $p \neq 0$  and consequently is  $\gamma \neq 0$  because of  $p = \gamma \prod_{s \in G} l_s$ .  $\square$

Now we have finished the preliminaries and arrive at the proof of the main result in this section.

*Proof of 3.1.* Without loss of generality (Lemma 3.3) I suppose  $A$  to be an orthogonal automorphism of  $V$ . Since  $G$  is a finite reflection group, it suffices to show that for each reflection  $s \in G$  is  $s' := AsA^{-1} \in G$ . After having shown that  $A$  transforms reflecting hyperplanes of  $G$  again in reflecting hyperplanes of  $G$ , it will be easy to show that conjugation with  $A$  transforms reflections of  $G$  into reflections of  $G$ .

From above we know (Lemma 3.6)

$$\overline{\Psi((A^{-1})^T x)}^N = A\overline{\Psi(x)}^N = A(p(x))\overline{J} = p(x)A\overline{J} = p(x)c\overline{J}$$

with  $c \in \mathbb{R} \setminus \{0\}$ . So we have  $A \in \text{Aut}(P(V)_G)$ , and it transforms the homogeneous subspace  $(P(V)_G)_N = \text{span}(\overline{J})$  into itself. Thus  $\overline{J}$  will be transformed in  $c\overline{J}$ . In addition, we get

$$\overline{\Psi((A^{-1})^T x)}^N = p((A^{-1})^T x)\overline{J} = (A^T p(x))\overline{J}.$$

Altogether, this yields  $A^T p(x) = cp(x)$ . By assumption  $A$  is orthogonal, so is  $A^T = A^{-1}$ . Thus it follows that

$$A^T p(x) = p(Ax) = cp(x).$$

When  $\text{Null}(p)$  denotes the set of all  $x \in V$  with  $p(x) = 0$ , we obtain  $A(\text{Null}(p)) \subseteq \text{Null}(p)$ . This yields  $A(\bigcup_{s \in G} H_s) \subseteq \bigcup_{s \in G} H_s$  since  $p = \prod_{s \in G} l_s$  with  $s \in G$  is a reflection. From the surjectivity of  $A \in \text{GL}(V)$  we get

$$A\left(\bigcup_{s \in G} H_s\right) = \bigcup_{s \in G} H_s.$$

By linearity, it follows that:  $AH_s = H_{s'}$ ,  $s'$  is a reflection of  $G$ . Thus  $A$  permutes the reflecting hyperplanes. Let  $\Delta$  be the root system of  $G$  and  $A \in O(V)$  with  $An_s \in \Delta \forall n_s \in \Delta$ , where  $n_s$  is the root of the reflection  $s \in G$ . By [4] conjugation with  $A$  transforms reflections of  $G$  into reflections of  $G$ . This statement remains valid if we demand  $An_s = \lambda n_{s'}$ ,  $n_{s'} \in \Delta$ ,  $\lambda \in \mathbb{R} \setminus \{0\}$ , instead of  $An_s = n_{s'}$ . We have shown that  $AH_s = H_{s'}$ . Therefore,  $AH_s^\perp = H_{s'}^\perp$  since  $A$  is orthogonal. Hence,  $An_s = \lambda n_{s'}$ ,  $\lambda \in \mathbb{R} \setminus \{0\}$ . From this we deduce that  $AsA^{-1} \in G$  for all reflections  $s \in G$  with [4].  $\square$

Now we have proved the surjectivity of  $\Phi^*$ . Using §2, we arrive at:

**Theorem 3.7** (main theorem). *Let  $G$  be a Coxeter group; then*

$$N_{\text{GL}(V)}(G) \cong \text{Aut}(P(V)_G).$$

**Corollary 3.8.** *From the last proof we can see that for  $A \in \text{Aut}(P(V)_G)$  it must be*

$$An_s = r(A, s)n_{s'}, \quad A\overline{l}_s = r(A, s)\overline{l}_{s'}$$

*with  $r(A, s) \in \mathbb{R} \setminus \{0\}$  for the element  $A \in N_{\text{GL}(V)}(G)$  induced by the  $P(V)_G$ -automorphism  $A$ .*

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