

INVARIANT MEASURES OF SYMMETRIC LÉVY PROCESSES

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ABSTRACT. If $\pi = \{\pi_t : t > 0\}$ is a symmetric convolution semigroup with the Lévy exponent ϕ , then $\text{supp } \pi_t$ is a group determined by ϕ , and π has a unique Radon invariant measure if and only if ϕ has a unique zero at 0.

1. INTRODUCTION

Let $X = (X_t, P^x)$ be a Lévy process on R^d (i.e., a process with stationary independent increments), with transition semigroup (P_t) . Let m denote Lebesgue measure on R^d . For any probability measure μ on R^d , define the characteristic function of μ as

$$(1.1) \quad \hat{\mu}(x) := \int_{R^d} e^{i(x,y)} \mu(dy).$$

Let $\pi_t := P_t(0, \cdot)$. Then $\pi = \{\pi_t\}$ is a convolution semigroup with Lévy exponent ϕ ; i.e., $\hat{\pi}_t(x) = e^{-t\phi(x)}$. By the Lévy-Khinchin formula,

$$(1.2) \quad \phi(x) = i(a, x) + \frac{1}{2}(Sx, x) + \int_{R^d} \left(1 - e^{i(x,y)} + \frac{i(x,y)}{1+|y|^2} \right) J(dy)$$

with $a \in R^d$, S a $d \times d$ nonnegative definite matrix, and J a positive measure carried by $R^d - \{0\}$ such that

$$(1.3) \quad \int_{R^d} (1 \wedge |x|^2) J(dx) < \infty.$$

The measure J is called the Lévy measure of X . The process X is called symmetric if each π_t is invariant under the map $x \mapsto -x$. In this case, ϕ has the form

$$(1.4) \quad \phi(x) = \frac{1}{2}(Sx, x) + \int_{R^d} (1 - \cos(x, y)) J(dy)$$

where J is symmetric.

A σ -finite measure μ on R^d is called an invariant measure for X if $\mu * \pi_t = \mu$ for all $t > 0$. Denote by Inv the set of all invariant measures for X . Clearly

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$m \in \text{Inv}$. We say that X has a unique invariant measure if $\mu \in \text{Inv}$ implies that μ is a multiple of m and that X has a unique Radon invariant measure if the only Radon measures in Inv are multiples of m .

In this paper, two results concerning symmetric convolution semigroups will be presented. In §2 we prove that the support of each measure π_t is the group generated by the sum of the range of \sqrt{S} and the support of the Lévy measure. In §3 we prove that X has a unique Radon invariant measure if and only if its Lévy exponent has a unique zero at 0. Finally, a few examples are given as an application and explanation of our results.

2. SUPPORT OF CONVOLUTION SEMIGROUP

The translation operators $\{\gamma_x : x \in R^d\}$ on R^d are defined by $\gamma_x(y) = x + y$. For any function f on R^d , let

$$(2.1) \quad \text{per } f = \{x \in R^d : f \circ \gamma_x = f\}.$$

Clearly $\text{per } f$ is a closed subgroup of R^d if f is continuous. For any $A \subset R^d$, let $G(A)$ denote the closed subgroup generated by A in R^d and

$$(2.2) \quad A^\perp = \{x \in R^d : e^{i(x,y)} = 1 \ \forall y \in A\}.$$

We list some facts here which will be used later. The first four are easy to check and the last one is [BF, Proposition 6.3].

- (a) A^\perp is a closed subgroup of R^d .
- (b) $(A^\perp)^\perp = G(A)$.
- (c) $G(A + B) = \overline{G(A) + G(B)}$.
- (d) $G(A) = G(\overline{A})$.
- (e) For any probability measure μ on R^d , $\text{per } \hat{\mu} = (\text{supp } \mu)^\perp$, where $\text{supp } \mu$ denotes the support of μ .

In the remainder of this section, we will assume X is symmetric. Then ϕ has the representation (1.4). Let π^c and π^d be the convolution semigroups corresponding to the Lévy exponents $\frac{1}{2}(Sx, x)$ and $\int(1 - \cos(x, y))J(dy)$, respectively. Then $\pi_t = \pi_t^c * \pi_t^d$ for all $t > 0$ and

$$(2.3) \quad \text{supp } \pi_t = \overline{\text{supp } \pi_t^c + \text{supp } \pi_t^d}.$$

The main result in this section is the following.

Theorem 1. For any $t > 0$,

$$(2.4) \quad \text{supp } \pi_t = \overline{\sqrt{S}(R^d) + G(\text{supp } J)} = G(\sqrt{S}(R^d) + \text{supp } J).$$

Proof. Clearly $\text{supp } \pi_t^c = \sqrt{S}(R^d)$. By (2.3), it suffices to show that $\text{supp } \pi_t = G(\text{supp } J)$ if $S = 0$. Now assume $S = 0$. We will finish the proof in several steps.

1. For any $t > 0$, $G(\text{supp } \pi_t) = G(\text{supp } J)$. In fact, for any fixed $t > 0$, by facts (b) and (e), we only need to show that $\text{per } \hat{\pi}_t = (\text{supp } J)^\perp$. Let $\text{zer } f$ denote the set of zeros of a function $f: R^d \rightarrow R$. Since

$$\phi(x) = \int_{R^d} (1 - \cos(x, y))J(dy),$$

$x \in \text{zer } \phi$ if and only if J is carried by $\{x\}^\perp = \{y: \cos(x, y) = 1\}$ or equivalently $x \in (\text{supp } J)^\perp$. Thus $\text{zer } \phi = (\text{supp } J)^\perp$. On the other hand, it is easy to see that $\text{per } \hat{\pi}_t \subset \text{zer } \phi$. Conversely, if $x \in \text{zer } \phi$, then $\text{supp } J \subset \{x\}^\perp$. Thus $\cos(x, \cdot)J = J$ and $\sin(x, \cdot)J = 0$, and it follows that

$$\phi(x + y) = \int (1 - \cos(x + y, z))J(dz) = \phi(y).$$

Therefore, $x \in \text{per } \phi \subset \text{per } \hat{\pi}_t$.

2. $\text{supp } \pi_t$ is a group and $\text{supp } \pi_t = G(\text{supp } J)$ for any $t > 0$. By symmetry of π_t , it is clear that if $x \in \text{supp } \pi_t$ then $-x \in \text{supp } \pi_t$. We will prove $0 \in \text{supp } \pi_t$ by contradiction. Suppose that $0 \notin \text{supp } (\pi_t)$. Then there exists $\delta > 0$ such that $\pi_t(B_\delta) = 0$ where $B_\delta = \{x: |x| < \delta\}$. Let

$$(2.5) \quad f(t) = \frac{1}{a} \left(1 - \frac{|t|}{a}\right) 1_{[-a, a]}(t), \quad t \in \mathbb{R},$$

and

$$g(x_1, \dots, x_d) = f(x_1) \cdots f(x_d).$$

Then g is a continuous function on \mathbb{R}^d with compact support and

$$(2.6) \quad \hat{g}(x) = \left(\frac{2}{a^2}\right)^d \frac{(1 - \cos ax_1) \cdots (1 - \cos ax_d)}{x_1^2 \cdots x_d^2} \geq 0.$$

Now choose $a > 0$ such that $[-a, a]^d \subset B_\delta$. Then

$$0 = \langle \pi_t, g \rangle = \frac{1}{(2\pi)^{d/2}} \langle \hat{\pi}_t, \hat{g} \rangle = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-t\phi(x)} \hat{g}(x) dx.$$

But $e^{-t\phi(x)} > 0$ everywhere. This implies that $\hat{g} = 0$, which is a contradiction. Now assume that $J(\mathbb{R}^d) < \infty$. Then

$$(2.7) \quad \pi_t = e^{-t\beta} \sum_{n \geq 0} \frac{t^n}{n!} J^{*n}$$

where $\beta = J(\mathbb{R}^d)$ and J^{*n} denotes the n -fold convolution of J . Thus, immediately, we have

$$\text{supp } \pi_t = \overline{\bigcup_{n \geq 0} \text{supp } J^{*n}}.$$

By the result in step 1,

$$G(\text{supp } J) = G(\text{supp } \pi_t) \supset \text{supp } \pi_t = \overline{\bigcup_{n \geq 0} \text{supp } J^{*n}}.$$

It is easy to check that $\bigcup \text{supp } J^{*n}$ is a group. Hence,

$$\overline{\bigcup_{n > 0} \text{supp } J^{*n}} = G(\text{supp } J) = \text{supp } \pi_t,$$

since $\bigcup \text{supp } J^{*n} \supset \text{supp } J$.

In general, let J_n be the restriction of J to $B_{1/n}^c$, which is finite and symmetric. Let

$$(2.8) \quad \begin{aligned} \phi_1(x) &= \int_{\mathbb{R}^d} (1 - \cos(x, y))J_n(dy), \\ \phi_2(x) &= \int_{\mathbb{R}^d} (1 - \cos(x, y))(J - J_n)(dy) = \phi(x) - \phi_1(x). \end{aligned}$$

Then there exist two convolution semigroups $\xi^n = \{\xi_t^n\}$ and $\mu^n = \{\mu_t^n\}$ with Lévy exponents ϕ_1 and ϕ_2 , respectively. Clearly, $\xi_t^n * \mu_t^n = \pi_t$ and $\text{supp } \xi_t^n = G(\text{supp } J_n)$ since J_n is finite.

Now $\text{supp } J \setminus \{0\} = \bigcup_{n \geq 1} \text{supp } J_n$ and

$$G(\text{supp } J) = G(\text{supp } J \setminus \{0\}) = G\left(\bigcup \text{supp } J_n\right) \supset G(\text{supp } J_n),$$

or $G(\text{supp } J) \supset \bigcup_{n \geq 1} G(\text{supp } J_n)$. Since $\{G(\text{supp } J_n)\}_{n \geq 1}$ is an increasing sequence of subgroups, $\bigcup_{n \geq 1} G(\text{supp } J_n)$ is also a subgroup. Thus

$$\overline{\bigcup_{n \geq 1} G(\text{supp } J_n)} \supset G\left(\bigcup_{n \geq 1} \text{supp } J_n\right) = G(\text{supp } J).$$

That is, $\overline{\bigcup_{n \geq 1} G(\text{supp } J_n)} = G(\text{supp } J)$.

Next,

$$\text{supp } \pi_t \supset \text{supp } \xi_t^n + \text{supp } \mu_t^n \supset \text{supp } \xi_t^n$$

since $0 \in \text{supp } \mu_t^n$. Then

$$\text{supp } \pi_t \supset \overline{\bigcup_{n \geq 1} \text{supp } \xi_t^n} = \overline{\bigcup_{n \geq 1} G(\text{supp } J_n)} = G(\text{supp } J) = G(\text{supp } \pi_t).$$

Therefore, $\text{supp } \pi_t = G(\text{supp } J) = G(\text{supp } \pi_t)$. This completes the proof. Q.E.D.

3. RADON INVARIANT MEASURES

Choquet and Deny [CD] investigated and solved the convolution equation

$$(3.1) \quad \sigma * \mu = \mu \quad (\sigma \geq 0 \text{ a given Radon measure})$$

on a locally compact Abelian group. Here we will state their result only for R^d .

Let $G_\sigma = G(\text{supp } \sigma)$, and let ω be the trivial extension to R^d of Haar measure on G_σ . It is known that there exists a Borel set $\Gamma_\sigma \subset R^d$ consisting of exactly one representative from each equivalence class of the quotient group R^d/G_σ . Let

$$E_\sigma := \left\{ x \in R^d : \int_{R^d} e^{-\langle x, y \rangle} \sigma(dy) = 1 \right\}.$$

Theorem (Choquet-Deny). *Any Radon measure μ satisfying the equation (3.1) is of the form*

$$(3.2) \quad \mu = \int_{R^d \times R^d} (e^{\langle x, \cdot \rangle} \omega) * \delta_y \xi(dx, dy)$$

where ξ is a positive Radon measure on $R^d \times R^d$ carried by $E_\sigma \times \Gamma_\sigma$ and δ_y is the Dirac measure at y .

Now for the convolution semigroup π , let

$$(3.3) \quad \sigma = \int_0^\infty e^{-t} \pi_t dt.$$

Then σ is a probability measure R^d and $\mu \in \text{Inv}$ if and only if μ satisfies (3.1). Clearly

$$\text{supp } \sigma = \overline{\bigcup_{t>0} \text{supp } \pi_t}.$$

Thus $G_\sigma = G(\bigcup_{t>0} \text{supp } \pi_t)$. By [BF, Proposition 8.27], $G_\sigma = (\text{zer } \phi)^\perp$ and $\text{zer } \phi = G_\sigma^\perp$. On the other hand, $x \in E_\sigma$ if and only if $\int_{R^d} e^{-(x,y)} \pi_t(dy) = 1$ for all $t > 0$. Obviously, there exists a $(-\infty, +\infty]$ -valued function ψ on R^d such that

$$(3.4) \quad \int_{R^d} e^{-(x,y)} \pi_t(dy) = e^{t\psi(x)}.$$

Hence $E_\sigma = \text{zer } \psi$. It is easy to see that $0 \in \text{zer } \phi$ and $0 \in \text{zer } \psi$. The main result of this section is

Theorem 2. (a) *The process X has a unique Radon invariant measure if and only if $\text{zer } \phi = \text{zer } \psi = \{0\}$.*

(b) *If X is symmetric, then X has a unique Radon invariant measure if and only if $\text{zer } \phi = \{0\}$.*

Proof. (a) follows immediately from the theorem of Choquet and Deny since $E_\sigma = \text{zer } \psi$, and $\Gamma_\sigma = \{0\}$ if and only if $\text{zer } \phi = \{0\}$.

(b) By (a), it suffices to show that $\text{zer } \phi = \{0\}$ implies that $\text{zer } \psi = \{0\}$ if X is symmetric. By way of contradiction, suppose that $a \neq 0$ and $a \in E_\sigma$. Then $-a \in E_\sigma$ by the symmetry of σ , and for any $b \in (-a, a) := \{ta : t \in (-1, 1)\}$ there exist numbers $0 < p, q < 1$ with $p + q = 1$ such that $b = (-a)p + aq$. Applying Hölder's inequality, we have

$$\begin{aligned} \int_{R^d} e^{-(b,x)} \sigma(dx) &= \int [e^{-(-a,x)}]^p \cdot [e^{-(a,x)}]^q \sigma(dx) \\ &\leq \left[\int e^{(a,x)} \sigma(dx) \right]^p \left[\int e^{-(a,x)} \sigma(dx) \right]^q = 1. \end{aligned}$$

Again by symmetry and Hölder's inequality, $\int e^{(b,x)} \sigma(dx) \leq 1$ and

$$\begin{aligned} 1 &= \int \sigma(dx) = \int e^{-(b,x)/2+(b,x)/2} \sigma(dx) \\ &\leq \left[\int e^{-(b,x)} \sigma(dx) \right]^{1/2} \cdot \left[\int e^{(b,x)} \sigma(dx) \right]^{1/2} \leq 1. \end{aligned}$$

Thus $\int e^{-(b,x)} \sigma(dx) = \int e^{(b,x)} \sigma(dx) = 1$; i.e., $(-a, a) \subset E_\sigma$.

Now, for any $t \in (0, 1)$,

$$\begin{aligned} 1 &= \int e^{(ta,x)} \sigma(dx) \\ &= \int_{(a,x)<0} + \int_{(a,x)=0} + \int_{(a,x)>0} e^{(ta,x)} \sigma(dx) \\ &= c + \int_{(a,x)>0} (e^{(ta,x)} + e^{-(ta,x)}) \sigma(dx) \end{aligned}$$

where $c = \sigma(\{x : (a, x) = 0\})$. Let $f(t, x) = e^{(ta,x)} + e^{-(ta,x)}$. Then

$$\frac{df}{dt}(t, x) = (a, x)[e^{(ta,x)} - e^{-(ta,x)}] > 0$$

for any $t \in (0, 1)$ and $(a, x) > 0$. But $\int_{(a,x)>0} f(t, x)\sigma(dx) = 1 - c$ does not depend on t . Thus $\sigma(\{x: (a, x) > 0\}) = 0$; i.e., σ is carried by the subspace $\{x: (a, x) = 0\}$. This contradicts the assumption $a \in E_\sigma$ and completes the proof. Q.E.D.

The uniqueness results above for Radon invariant measures are not true in general. (See the example below.) But it is known that if m is a reference measure for X , i.e., $\sigma \ll m$, then each invariant measure of X is a Radon measure [BG, VI(2.3)]. In this case, $G_\sigma = R^d$ and $\Gamma_\sigma = \{0\}$. Thus, the following result is a simple consequence of Theorem 2.

Corollary. *Assume $\sigma \ll m$. Then:*

- (a) X has a unique invariant measure if and only if $\text{zer } \psi = \{0\}$.
- (b) If X is symmetric, then X has a unique invariant measure.

Now we will construct an example such that any Radon invariant measure of X is a multiple of Lebesgue measure but X has an invariant measure which is not Radon. For the case in which J is finite, we have (2.7), and it is easy to check that $\mu \in \text{Inv}$ if and only if $\frac{1}{\beta}\mu * J = \mu$.

Example 1. Let $J = \frac{1}{4}(\delta_1 + \delta_{-1} + \delta_{\sqrt{2}} + \delta_{-\sqrt{2}})$ be defined on R and $\pi = \{\pi_t\}_{t>0}$ the corresponding symmetric convolution semigroup; i.e., $\hat{\pi}_t(x) = e^{-t\phi(x)}$ with

$$\phi(x) = \int (1 - \cos xy)J(dy) = \frac{1}{2}(1 - \cos x) + \frac{1}{2}(1 - \cos \sqrt{2}x).$$

Clearly ϕ has a unique zero. Thus the only Radon invariant measure of π is Lebesgue measure. But let

$$N = \{n + m\sqrt{2}: n, m \text{ are integers}\} \quad \text{and} \quad \mu = \sum_{x \in N} \delta_x.$$

Then μ is σ -finite and satisfies $\mu * J = \mu$. Therefore, μ is an invariant measure for X .

Example 2. Brownian motion and symmetric stable processes. Both are special types of symmetric Lévy processes, where the Lévy exponent $\phi(x) = \frac{1}{2}|x|^2$ for Brownian motion and $\phi(x) = |x|^\alpha$, $\alpha \in (0, 2)$, for symmetric stable processes. Clearly both functions have unique zeros. By Corollary 3, both processes have unique invariant measures.

Example 3. Brownian motion with drift on R^d . In this case,

$$\phi(x) = i(x, \alpha) + \frac{1}{2}|x|^2, \quad \psi(x) = -(x, \alpha) - \frac{1}{2}|x|^2,$$

where $\alpha \in R^d$. Then $E_\sigma = \{0, -\alpha\}$ and $\text{zer } \phi = \{0\}$. Thus invariant measures are of the form $\mu = c_1 m + c_2 e^{-(\alpha, \cdot)} m$, where c_1 and c_2 are any positive constants.

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