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# INVARIANT MEASURES OF SYMMETRIC LÉVY PROCESSES

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ABSTRACT. If $\pi = \{\pi_t: t > 0\}$ is a symmetric convolution semigroup with the Lévy exponent $\phi$, then $\operatorname{supp} \pi_t$ is a group determined by $\phi$, and $\pi$ has a unique Radon invariant measure if and only if $\phi$ has a unique zero at $0$.

## 1. INTRODUCTION

Let $X = (X_t, P^x)$ be a Lévy process on $R^d$ (i.e., a process with stationary independent increments), with transition semigroup $(P_t)$. Let $m$ denote Lebesgue measure on $R^d$. For any probability measure $\mu$ on $R^d$, define the characteristic function of $\mu$ as

$$\hat{\mu}(x) := \int_{R^d} e^{i(x, y)} \mu(dy). \tag{1.1}$$

Let $\pi_t := P_t(0, \cdot)$. Then $\pi = \{\pi_t\}$ is a convolution semigroup with Lévy exponent $\phi$; i.e., $\hat{\pi}_t(x) = e^{-t\phi(x)}$. By the Lévy-Khinchin formula,

$$\phi(x) = i(a, x) + \frac{1}{2}(Sx, x) + \int_{R^d} \left(1 - e^{i(x, y)} + \frac{i(x, y)}{1 + |y|^2}\right) J(dy) \tag{1.2}$$

with $a \in R^d$, $S$ a $d \times d$ nonnegative definite matrix, and $J$ a positive measure carried by $R^d - \{0\}$ such that

$$\int_{R^d} (1 \wedge |x|^2) J(dx) < \infty. \tag{1.3}$$

The measure $J$ is called the Lévy measure of $X$. The process $X$ is called symmetric if each $\pi_t$ is invariant under the map $x \mapsto -x$. In this case, $\phi$ has the form

$$\phi(x) = \tfrac{1}{2}(Sx, x) + \int_{R^d} (1 - \cos(x, y)) J(dy) \tag{1.4}$$

where $J$ is symmetric.

A $\sigma$-finite measure $\mu$ on $R^d$ is called an invariant measure for $X$ if $\mu * \pi_t = \mu$ for all $t > 0$. Denote by Inv the set of all invariant measures for $X$. Clearly

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$m \in \text{Inv}$. We say that $X$ has a unique invariant measure if $\mu \in \text{Inv}$ implies that $\mu$ is a multiple of $m$ and that $X$ has a unique Radon invariant measure if the only Radon measures in Inv are multiplies of $m$.

In this paper, two results concerning symmetric convolution semigroups will be presented. In §2 we prove that the support of each measure $\pi_t$ is the group generated by the sum of the range of $\sqrt{S}$ and the support of the Lévy measure. In §3 we prove that $X$ has a unique Radon invariant measure if and only if its Lévy exponent has a unique zero at $0$. Finally, a few examples are given as an application and explanation of our results.

## 2. Support of convolution semigroup

The translation operators $\{\gamma_x : x \in R^d\}$ on $R^d$ are defined by $\gamma_x(y) = x+y$. For any function $f$ on $R^d$, let

$$\text{per} f = \{x \in R^d : f \circ \gamma_x = f\}. \tag{2.1}$$

Clearly per $f$ is a closed subgroup of $R^d$ if $f$ is continuous. For any $A \subset R^d$, let $G(A)$ denote the closed subgroup generated by $A$ in $R^d$ and

$$A^\perp = \{x \in R^d : e^{i(x,y)} = 1 \ \forall y \in A\}. \tag{2.2}$$

We list some facts here which will be used later. The first four are easy to check and the last one is [BF, Proposition 6.3].

(a) $A^\perp$ is a closed subgroup of $R^d$.
(b) $(A^\perp)^\perp = G(A)$.
(c) $G(A+B) = \overline{G(A) + G(B)}$.
(d) $G(A) = G(\overline{A})$.
(e) For any probability measure $\mu$ on $R^d$, $\text{per}\,\hat{\mu} = (\text{supp}\,\mu)^\perp$, where $\text{supp}\,\mu$ denotes the support of $\mu$.

In the remainder of this section, we will assume $X$ is symmetric. Then $\phi$ has the representation (1.4). Let $\pi^c$ and $\pi^d$ be the convolution semigroups corresponding to the Lévy exponents $\frac{1}{2}(Sx, x)$ and $\int(1-\cos(x,y))J(dy)$, respectively. Then $\pi_t = \pi_t^c * \pi_t^d$ for all $t > 0$ and

$$\text{supp}\,\pi_t = \overline{\text{supp}\,\pi_t^c + \text{sup}\,\pi_t^d}. \tag{2.3}$$

The main result in this section is the following.

**Theorem 1.** *For any* $t > 0$,

$$\text{supp}\,\pi_t = \overline{\sqrt{S}(R^d) + G(\text{supp}\,J)} = G(\sqrt{S}(R^d) + \text{supp}\,J). \tag{2.4}$$

*Proof*. Clearly $\text{supp}\,\pi_t^c = \sqrt{S}(R^d)$. By (2.3), it suffices to show that $\text{supp}\,\pi_t = G(\text{supp}\,J)$ if $S = 0$. Now assume $S = 0$. We will finish the proof in several steps.

1. For any $t > 0$, $G(\text{supp}\,\pi_t) = G(\text{supp}\,J)$. In fact, for any fixed $t > 0$, by facts (b) and (e), we only need to show that $\text{per}\,\hat{\pi}_t = (\text{supp}\,J)^\perp$. Let $\text{zer}\,f$ denote the set of zeros of a function $f : R^d \to R$. Since

$$\phi(x) = \int_{R^d} (1 - \cos(x, y)) J(dy),$$

$x \in \operatorname{zer}\phi$ if and only if $J$ is carried by $\{x\}^{\perp} = \{y\colon \cos(x, y) = 1\}$ or equivalently $x \in (\operatorname{supp} J)^{\perp}$. Thus $\operatorname{zer}\phi = (\operatorname{supp} J)^{\perp}$. On the other hand, it is easy to see that $\operatorname{per}\hat{\pi}_t \subset \operatorname{zer}\phi$. Conversely, if $x \in \operatorname{zer}\phi$, then $\operatorname{supp} J \subset \{x\}^{\perp}$. Thus $\cos(x, \cdot)J = J$ and $\sin(x, \cdot)J = 0$, and it follows that

$$\phi(x+y) = \int (1 - \cos(x+y, z))J(dz) = \phi(y).$$

Therefore, $x \in \operatorname{per}\phi \subset \operatorname{per}\hat{\pi}_t$.

2. $\operatorname{supp}\pi_t$ is a group and $\operatorname{supp}\pi_t = G(\operatorname{supp} J)$ for any $t > 0$. By symmetry of $\pi_t$, it is clear that if $x \in \operatorname{supp}\pi_t$ then $-x \in \operatorname{supp}\pi_t$. We will prove $0 \in \operatorname{supp}\pi_t$ by contradiction. Suppose that $0 \notin \operatorname{supp}(\pi_t)$. Then there exists $\delta > 0$ such that $\pi_t(B_\delta) = 0$ where $B_\delta = \{x\colon |x| < \delta\}$. Let

$$f(t) = \frac{1}{a}\left(1 - \frac{|t|}{a}\right) 1_{[-a,a]}(t), \qquad t \in R, \tag{2.5}$$

and

$$g(x_1, \dots, x_d) = f(x_1)\cdots f(x_d).$$

Then $g$ is a continuous function on $R^d$ with compact support and

$$\hat{g}(x) = \left(\frac{2}{a^2}\right)^d \frac{(1-\cos ax_1)\cdots(1-\cos ax_d)}{x_1^2\cdots x_d^2} \geq 0. \tag{2.6}$$

Now choose $a > 0$ such that $[-a, a]^d \subset B_\delta$. Then

$$0 = \langle \pi_t, g\rangle = \frac{1}{(2\pi)^{d/2}}\langle \hat{\pi}_t, \hat{g}\rangle = \frac{1}{(2\pi)^{d/2}}\int_{R^d} e^{-t\phi(x)}\hat{g}(x)\,dx.$$

But $e^{-t\phi(x)} > 0$ everywhere. This implies that $\hat{g} = 0$, which is a contradiction. Now assume that $J(R^d) < \infty$. Then

$$\pi_t = e^{-t\beta}\sum_{n\geq 0}\frac{t^n}{n!}J^{*n} \tag{2.7}$$

where $\beta = J(R^d)$ and $J^{*n}$ denotes the $n$-fold convolution of $J$. Thus, immediately, we have

$$\operatorname{supp}\pi_t = \overline{\bigcup_{n\geq 0}\operatorname{supp} J^{*n}}.$$

By the result in step 1,

$$G(\operatorname{supp} J) = G(\operatorname{supp}\pi_t) \supset \operatorname{supp}\pi_t = \overline{\bigcup_{n\geq 0}\operatorname{supp} J^{*n}}.$$

It is easy to check that $\bigcup \operatorname{supp} J^{*n}$ is a group. Hence,

$$\overline{\bigcup_{n>0}\operatorname{supp} J^{*n}} = G(\operatorname{supp} J) = \operatorname{supp}\pi_t,$$

since $\bigcup \operatorname{supp} J^{*n} \supset \operatorname{supp} J$.

In general, let $J_n$ be the restriction of $J$ to $B^c_{1/n}$, which is finite and symmetric. Let

$$\begin{aligned}\phi_1(x) &= \int_{R^d}(1-\cos(x, y))J_n(dy),\\ \phi_2(x) &= \int_{R^d}(1-\cos(x, y))(J - J_n)(dy) = \phi(x) - \phi_1(x).\end{aligned} \tag{2.8}$$

Then there exist two convolution semigroups $\xi^n = \{\xi_t^n\}$ and $\mu^n = \{\mu_t^n\}$ with Lévy exponents $\phi_1$ and $\phi_2$, respectively. Clearly, $\xi_t^n * \mu_t^n = \pi_t$ and $\operatorname{supp}\xi_t^n = G(\operatorname{supp} J_n)$ since $J_n$ is finite.

Now $\operatorname{supp} J\backslash\{0\} = \bigcup_{n\geq 1} \operatorname{supp} J_n$ and

$$G(\operatorname{supp} J) = G(\operatorname{supp} J\backslash\{0\}) = G\left(\bigcup \operatorname{supp} J_n\right) \supset G(\operatorname{supp} J_n),$$

or $G(\operatorname{supp} J) \supset \bigcup_{n\geq 1} G(\operatorname{supp} J_n)$. Since $\{G(\operatorname{supp} J_n)\}_{n\geq 1}$ is an increasing sequence of subgroups, $\bigcup_{n\geq 1} G(\operatorname{supp} J_n)$ is also a subgroup. Thus

$$\overline{\bigcup_{n>1} G(\operatorname{supp} J_n)} \supset G\left(\bigcup_{n\geq 1} \operatorname{supp} J_n\right) = G(\operatorname{supp} J).$$

That is, $\overline{\bigcup_{n\geq 1} G(\operatorname{supp} J_n)} = G(\operatorname{supp} J)$.

Next,

$$\operatorname{supp} \pi_t \supset \operatorname{supp} \xi_t^n + \operatorname{supp} \mu_t^n \supset \operatorname{supp} \xi_t^n$$

since $0 \in \operatorname{supp} \mu_t^n$. Then

$$\operatorname{supp} \pi_t \supset \overline{\bigcup_{n\geq 1} \operatorname{supp} \xi_t^n} = \overline{\bigcup_{n\geq 1} G(\operatorname{supp} J_n)} = G(\operatorname{supp} J) = G(\operatorname{supp} \pi_t).$$

Therefore, $\operatorname{supp} \pi_t = G(\operatorname{supp} J) = G(\operatorname{supp} \pi_t)$. This completes the proof. Q.E.D.

## 3. Radon invariant measures

Choquet and Deny [CD] investigated and solved the convolution equation

$$\sigma * \mu = \mu \qquad (\sigma \geq 0 \text{ a given Radon measure}) \tag{3.1}$$

on a locally compact Abelian group. Here we will state their result only for $R^d$.

Let $G_\sigma = G(\operatorname{supp} \sigma)$, and let $\omega$ be the trivial extension to $R^d$ of Haar measure on $G_\sigma$. It is known that there exists a Borel set $\Gamma_\sigma \subset R^d$ consisting of exactly one representative from each equivalence class of the quotient group $R^d/G_\sigma$. Let

$$E_\sigma := \left\{x \in R^d \colon \int_{R^d} e^{-(x,y)} \sigma(dy) = 1\right\}.$$

**Theorem** (Choquet-Deny). *Any Radon measure $\mu$ satisfying the equation* (3.1) *is of the form*

$$\mu = \int_{R^d \times R^d} (e^{(x,\cdot)}\omega) * \delta_y \xi(dx, dy) \tag{3.2}$$

*where $\xi$ is a positive Radon measure on $R^d \times R^d$ carried by $E_\sigma \times \Gamma_\sigma$ and $\delta_y$ is the Dirac measure at $y$.*

Now for the convolution semigroup $\pi$, let

$$\sigma = \int_0^\infty e^{-t} \pi_t \, dt. \tag{3.3}$$

Then $\sigma$ is a probability measure $R^d$ and $\mu \in \mathrm{Inv}$ if and only if $\mu$ satisfies (3.1). Clearly

$$\operatorname{supp}\sigma = \overline{\bigcup_{t>0} \operatorname{supp}\pi_t}.$$

Thus $G_\sigma = G(\bigcup_{t>0}\operatorname{supp}\pi_t)$. By [BF, Proposition 8.27], $G_\sigma = (\operatorname{zer}\phi)^\perp$ and $\operatorname{zer}\phi = G_\sigma^\perp$. On the other hand, $x \in E_\sigma$ if and only if $\int_{R^d} e^{-(x,y)}\pi_t(dy) = 1$ for all $t > 0$. Obviously, there exists a $(-\infty, +\infty]$-valued function $\psi$ on $R^d$ such that

$$\int_{R^d} e^{-(x,y)}\pi_t(dy) = e^{t\psi(x)}. \tag{3.4}$$

Hence $E_\sigma = \operatorname{zer}\psi$. It is easy to see that $0 \in \operatorname{zer}\phi$ and $0 \in \operatorname{zer}\psi$. The main result of this section is

**Theorem 2.** (a) *The process* $X$ *has a unique Radon invariant measure if and only if* $\operatorname{zer}\phi = \operatorname{zer}\psi = \{0\}$.

(b) *If* $X$ *is symmetric, then* $X$ *has a unique Radon invariant measure if and only if* $\operatorname{zer}\phi = \{0\}$.

*Proof.* (a) follows immediately from the theorem of Choquet and Deny since $E_\sigma = \operatorname{zer}\psi$, and $\Gamma_\sigma = \{0\}$ if and only if $\operatorname{zer}\phi = \{0\}$.

(b) By (a), it suffices to show that $\operatorname{zer}\phi = \{0\}$ implies that $\operatorname{zer}\psi = \{0\}$ if $X$ is symmetric. By way of contradiction, suppose that $a \neq 0$ and $a \in E_\sigma$. Then $-a \in E_\sigma$ by the symmetry of $\sigma$, and for any $b \in (-a, a) := \{ta : t \in (-1, 1)\}$ there exist numbers $0 < p, q < 1$ with $p + q = 1$ such that $b = (-a)p + aq$. Applying Hölder's inequality, we have

$$\begin{aligned}\int_{R^d} e^{-(b,x)}\sigma(dx) &= \int [e^{-(-a,x)}]^p \cdot [e^{-(a,x)}]^q \sigma(dx) \\ &\le \left[\int e^{(a,x)}\sigma(dx)\right]^p \left[\int e^{-(a,x)}\sigma(dx)\right]^q = 1.\end{aligned}$$

Again by symmetry and Hölder's inequality, $\int e^{(b,x)}\sigma(dx) \le 1$ and

$$\begin{aligned}1 = \int \sigma(dx) &= \int e^{-(b,x)/2+(b,x)/2}\sigma(dx) \\ &\le \left[\int e^{-(b,x)}\sigma(dx)\right]^{1/2} \cdot \left[\int e^{(b,x)}\sigma(dx)\right]^{1/2} \le 1.\end{aligned}$$

Thus $\int e^{-(b,x)}\sigma(dx) = \int e^{(b,x)}\sigma(dx) = 1$; i.e., $(-a, a) \subset E_\sigma$.

Now, for any $t \in (0, 1)$,

$$\begin{aligned}1 &= \int e^{(ta,x)}\sigma(dx) \\ &= \int_{(a,x)<0} + \int_{(a,x)=0} + \int_{(a,x)>0} e^{(ta,x)}\sigma(dx) \\ &= c + \int_{(a,x)>0} (e^{(ta,x)} + e^{-(ta,x)})\sigma(dx)\end{aligned}$$

where $c = \sigma(\{x : (a, x) = 0\})$. Let $f(t, x) = e^{(ta,x)} + e^{-(ta,x)}$. Then

$$\frac{df}{dt}(t, x) = (a, x)[e^{(ta,x)} - e^{-(ta,x)}] > 0$$

for any $t \in (0, 1)$ and $(a, x) > 0$. But $\int_{(a,x)>0} f(t, x)\sigma(dx) = 1 - c$ does not depend on $t$. Thus $\sigma(\{x: (a, x) > 0\}) = 0$; i.e., $\sigma$ is carried by the subspace $\{x: (a, x) = 0\}$. This contradicts the assumption $a \in E_\sigma$ and completes the proof. Q.E.D.

The uniqueness results above for Radon invariant measures are not true in general. (See the example below.) But it is known that if $m$ is a reference measure for $X$, i.e., $\sigma \ll m$, then each invariant measure of $X$ is a Radon measure [BG, VI(2.3)]. In this case, $G_\sigma = R^d$ and $\Gamma_\sigma = \{0\}$. Thus, the following result is a simple consequence of Theorem 2.

**Corollary.** *Assume* $\sigma \ll m$. *Then*:

(a) $X$ *has a unique invariant measure if and only if* $\mathrm{zer}\, \psi = \{0\}$.
(b) *If* $X$ *is symmetric, then* $X$ *has a unique invariant measure.*

Now we will construct an example such that any Radon invariant measure of $X$ is a multiple of Lebesgue measure but $X$ has an invariant measure which is not Radon. For the case in which $J$ is finite, we have (2.7), and it is easy to check that $\mu \in \mathrm{Inv}$ if and only if $\frac{1}{\beta}\mu * J = \mu$.

**Example 1.** Let $J = \frac{1}{4}(\delta_1 + \delta_{-1} + \delta_{\sqrt{2}} + \delta_{-\sqrt{2}})$ be defined on $R$ and $\pi = \{\pi_t\}_{t>0}$ the corresponding symmetric convolution semigroup; i.e., $\hat{\pi}_t(x) = e^{-t\phi(x)}$ with

$$\phi(x) = \int (1 - \cos xy) J(dy) = \tfrac{1}{2}(1 - \cos x) + \tfrac{1}{2}(1 - \cos \sqrt{2}x).$$

Clearly $\phi$ has a unique zero. Thus the only Radon invariant measure of $\pi$ is Lebesgue measure. But let

$$N = \{n + m\sqrt{2}: n, m \text{ are integers}\} \quad \text{and} \quad \mu = \sum_{x \in N} \delta_x .$$

Then $\mu$ is $\sigma$-finite and satisfies $\mu * J = \mu$. Therefore, $\mu$ is an invariant measure for $X$.

**Example 2. Brownian motion and symmetric stable processes.** Both are special types of symmetric Lévy processes, where the Lévy exponent $\phi(x) = \frac{1}{2}|x|^2$ for Brownian motion and $\phi(x) = |x|^\alpha$, $\alpha \in (0, 2)$, for symmetric stable processes. Clearly both functions have unique zeros. By Corollary 3, both processes have unique invariant measures.

**Example 3. Brownian motion with drift on $R^d$.** In this case,

$$\phi(x) = i(x, \alpha) + \tfrac{1}{2}|x|^2, \qquad \psi(x) = -(x, \alpha) - \tfrac{1}{2}|x|^2,$$

where $\alpha \in R^d$. Then $E_\sigma = \{0, -\alpha\}$ and $\mathrm{zer}\, \phi = \{0\}$. Thus invariant measures are of the form $\mu = c_1 m + c_2 e^{-(\alpha, \cdot)} m$, where $c_1$ and $c_2$ are any positive constants.

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