PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 120, Number 1, January 1994

NONCOMMUTATIVE DECOMPOSITION THEOREMS IN RIESZ SPACES

PAOLO DE LUCIA AND PEDRO MORALES

(Communicated by Andrew Bruckner)

ABSTRACT. We show that an additive function defined on an orthomodular poset and taking its values in the positive cone of a normed Riesz space admits a Lebesgue Decomposition and a Yosida-Hewitt Decomposition.

1. INTRODUCTION

The classical Decomposition Theorems of Lebesgue [13] and Yosida-Hewitt [7] have recieved considerable attention for their applications to a systematic and detailed study of finitely additive set functions (see [3]). In this paper we give twofold extensions of these results; namely, the corresponding additive function is defined on an orthomodular poset and takes its values in the positive cone of a Dedekind complete normed Riesz space with order continuous norm. Our method is a natural refinement of the arguments used by Hewitt and Yosida [17] (see also [8]), and we get, as byproducts, some interesting results established by Aarnes [1], Darst [6], Rüttiman [14], and Schmidt [16].

The paper is organized as follows: In §2 we give some elementary notions of orthomodular posets and Riesz spaces. Section 3 presents the noncommutative version of the Lebesgue Decomposition Theorem, and we deduce some corollaries. In the last section we establish two noncommutative versions of the Yosida-Hewitt Decomposition Theorem and deduce in every detail the important decomposition given by Aarnes [1].

2. PRELIMINARIES

Let $(E, \leq)$ be a partially ordered set, and let D be a nonempty subset of E. If the supremum (resp. the infimum) of D in E exists, it will be denoted by $\bigvee D$ (resp. $\bigwedge D$). In particular, if $D = \{u, v\}$ we write $u \vee v = \bigvee D$ and $u \wedge v = \bigwedge D$.

Received by the editors May 1, 1992.

1991 *Mathematics Subject Classification*. Primary 28B05, 06A10, 06C15.

Key words and phrases. Orthomodular poset, normed Riesz space, Hilbert space, additive function, completely additive function.

The research of the first author was partially supported by Ministero dell'Università e della Ricerca Scientifica e Tecnologica (Italy). The research of the second author was partially supported by a grant from the Natural Sciences and Engineering Research Council of Canada.

Consider the quintuplet $(L, \leq, ', 0, 1)$ where L is a set, $\leq$ is a binary relation on L, $'$ is a function from L into L, and $0, 1$ are two distinct distinguished elements of L. We say that $L = (L, \leq, ', 0, 1)$ is an *orthomodular poset* if the following conditions are satisfied:

(i) $(L, \leq)$ is a partially ordered set.
(ii) 0 is the least element of L and 1 is the greatest element of L.
(iii) $'$ is a decreasing function, $a'' = a$ and $a \wedge a' = 0$ for all $a \in L$.
(iv) If $a, b \in L$ and $a \leq b'$, then $a \vee b$ exists in L.
(v) If $a, b \in L$ and $a \leq b$, then $b = a \vee (a' \wedge b)$.

From these axioms it follows immediately that $0' = 1$, $1' = 0$, and $a \vee a' = 1$ for all $a \in L$.

An *orthomodular lattice* is an orthomodular poset $(L, \leq, ', 0, 1)$ such that $(L, \leq)$ is a lattice. A distributive orthomodular lattice is called a *Boolean algebra.*

The following example of orthomodular poset will be used in §4: Let H be a Hilbert space over R or C with inner product $(\cdot, \cdot)$, let $L(H)$ be the set of all closed subspaces of H, and let $^\perp$ be the function from $L(H)$ into $L(H)$ defined by the formula $M^\perp = \{v \in H: (u, v) = 0 \text{ for all } u \in M\}$. Then $(L(H), \subseteq, ^\perp, 0, H)$ is a complete orthomodular lattice which is not a Boolean algebra if $\mathrm{Dim}(H) \geq 2$ (see [11, 12]).

Let $L = (L, \leq, ', 0, 1)$ be an orthomodular poset. Consider the following binary relation on L: $a \perp b$ if $a \leq b'$. It is clear that the binary relation $\perp$ is symmetric and $a \perp a$ implies $a = 0$ for all $a \in L$. If $a \in L$ and B is a nonempty subset of L, we write $a \perp B$ if $a \perp b$ for every $b \in B$. If $b \perp B \backslash \{b\}$ for all $b \in B$, we say that B is an *orthogonal set.* It follows from axiom (iv) that, if F is a finite orthogonal subset of L, then $\bigvee F$ exists in L.

For more details concerning orthomodular posets or orthomodular lattices we refer to [2, 11, 12].

Throughout this paper, $L = (L, \leq, ', 0, 1)$ is assumed to be an orthomodular poset.

Consider the quadruple $(V, +, ', \leq)$ where $(V, +, \cdot)$ is a vector space over R and $\leq$ is a binary relation on V. We say that $V = (V, +, \cdot, \leq)$ is a *Riesz space* if the following conditions are satisfied:

(i) $(V, \leq)$ is a lattice.
(ii) If $x \leq y$, then $x + z \leq y + z$ for all $z \in V$.
(iii) If $x \leq y$, then $\alpha \cdot x \leq \alpha \cdot y$ whenever $\alpha \in R$ and $\alpha \geq 0$.

Let $V = (V, +, \cdot, \leq)$ be a Riesz space. We write

$$x^+ = x \vee 0, \qquad x^- = (-x) \vee 0, \qquad |x| = x^+ + x^-$$

for any $x \in V$. If A is a nonempty subset of V and $x \in V$, then the following identities are well known:

(a) $\bigwedge A = -(\bigvee(-A))$ if either side exists.
(b) $\bigvee(x + A) = x + \bigvee A$ if either side exists.

If $x, y \in V$ and $x \leq y$, the set $[x, y] = \{z \in V: x \leq z \leq y\}$ is called an *order interval* in V. A nonempty subset D of V is said to be *directed downward*, and we write $D \downarrow$ if for any $x, y \in D$ there exists $z \in D$ such that $z \leq x$ and $z \leq y$. If $D \downarrow$ and $x \in D$, the set $S_x = \{y \in D: y \leq x\}$ is called the *section* of D determined by x; the family $(S_x)_{x \in D}$ is a filter base in V for a filter $\mathscr{F}(D)$ called the *filter of sections* of D. Let $\mathscr{F}$ be a filter on V, and let $x \in V$. We say that $\mathscr{F}$ is *order convergent* to x if $\mathscr{F}$ contains a family of order intervals with intersection $\{x\}$. For example, if $D \downarrow$ and there exists $x = \bigwedge D$ in V, then $\mathscr{F}(D)$ contains the family of order intervals $([x, y])_{y \in D}$ and $\bigcap_{y \in D}[x, y] = \{x\}$. Then $\mathscr{F}(D)$ is order convergent to x.

A Riesz space V is called *Dedekind complete* if, for every nonempty majorized subset B of V, $\bigvee B$ exists in V.

Let V be a Riesz space. A norm $\|\cdot\|$ on V is called a Riesz norm if $|x| \leq |y|$ implies $\|x\| \leq \|y\|$ for all $x, y \in V$. If $\|\cdot\|$ is a Riesz norm on V, the pair $(V, \|\cdot\|)$ is called a *normed Riesz space*; if, further, $(V, \|\cdot\|)$ is norm complete, it is called a *Banach lattice*.

A normed Riesz space is said to have *order continuous norm* if every order-convergent filter in V converges in norm to its order limit. It is well known that every Banach lattice with order-continuous norm is Dedekind complete.

For more details concerning Riesz spaces we refer to [9, 15].

Throughout this paper, $V = (V, +, \cdot, \leq, \|\cdot\|)$ is a Dedekind complete normed Riesz space with order-continuous norm.

Next we list some examples of such Riesz spaces:

(1) The n-dimensional vector space R^n endowed with its canonical order and norm.
(2) The vector subspace $C_0(I)$ of the Dedekind complete normed Riesz space $(l^\infty(I), \|\cdot\|_\infty)$, where I is an infinite index set.
(3) Any reflexive Banach lattice and, in particular, the classical spaces $L^p((\Omega, \Sigma, \mu); R)$ with $1 < p < +\infty$.
(4) Any nonreflexive weakly sequentially complete Banach lattice and, in particular, the classical space $L^1(\mu)$, where μ is the Lebesgue measure on R.
(5) The Orlicz lattices considered in [5].

Now let W be any normed Riesz space, and let L be an orthomodular I. Consider the following binary relation on W^L: $\lambda_1 \leq \lambda_2$ if $\lambda_1(a) \leq \lambda_2(a)$ for all $a \in L$. Clearly the pair $(W^L, \leq)$ is a partially ordered set. An element $\lambda \in W^L$ is said to be positive if $0 \leq \lambda$.

Let $\lambda \in W^L$ be such that $\lambda(0) = 0$. We say that:

(a) λ is additive if, whenever $a, b \in L$ and $a \perp b$, then $\lambda(a \vee b) = \lambda(a) + \lambda(b)$.

(b) λ is countably additive if, for every orthogonal sequence $(a_n)_{n \in N}$ of elements of L such that $\bigvee\{a_n : n \in N\}$ exists in L, we have

$$\lambda\left(\bigvee\{a_n : n \in N\}\right) = \sum_{n=0}^{+\infty} \lambda(a_n)$$

in the norm topology of W.

(c) λ is completely additive if, for every orthogonal family $(a_i)_{i \in I}$ of elements of L such that $\bigvee\{a_i : i \in I\}$ exists in L, the family $(\lambda(a_i))_{i \in I}$ is summable in W and $\lambda(\bigvee\{a_i : i \in I\}) = \sum_{i \in I} \lambda(a_i)$.

The symbols $a(L, W)_+$, $ca(L, W)_+$, and $cca(L, W)_+$ will denote the set of all positive functions from L into W which are additive, countably additive, and completely additive respectively. It is clear that $ca(L, W)_+ \subseteq a(L, W)_+$

and $cca(L, W)_+ \subseteq a(L, W)_+$, and from [4, Chapter 3, §5, no. 7, Proposition 9], it follows that $cca(L, W)_+ \subseteq ca(L, W)_+$.

3. The Lebesgue Decomposition Theorem

Let $\mu \in a(L, V)_+$, let $W = (W, +, \cdot, \leq, \|\cdot\|)$ be a normed Riesz space, and let $\lambda \in a(L, W)$. We say that:

(a) μ is λ-continuous, and we write $\mu << \lambda$, if, for every real number $\varepsilon > 0$, there exists a real number $\delta > 0$ such that $a \in L$ and $\|\lambda(a)\| < \delta$ imply $\|\mu(a)\| < \varepsilon$.

(b) μ is λ-singular, and we write $\mu \perp \lambda$, if, whenever $\gamma \in a(L, V)_+$, $\gamma << \lambda$ and $\gamma \leq \mu$, then $\gamma = 0$.

Remarks 3.1. (1) The above definition of singularity is weaker than the extension of the classical notion. More precisely, $\mu \perp \lambda$ is implied by the condition: There exists $a \in L$ such that $\mu(a) = 0$ and $\lambda(a') = 0$. In fact, let $\gamma \in a(L, V)$ be such that $\gamma << \lambda$ and $\gamma \leq \mu$. Let $\varepsilon \in R$ be such that $\varepsilon > 0$. Then there exists a real number $\delta > 0$ such that $b \in L$ and $\|\lambda(b)\| < \delta$ imply $\|\gamma(b)\| < \varepsilon$. So $\|\gamma(a')\| < \varepsilon$, and therefore $\gamma(a') = 0$. Since $\gamma(1) = \gamma(a) + \gamma(a')$, it follows that $\gamma(1) = 0$, and therefore $\gamma = 0$.

(2) If $W = V$, then $\mu \perp \lambda$ implies that $\mu \wedge \lambda = 0$ in $a(L, V)_+$. In fact, 0 is a minorant of the set $\{\lambda, \mu\}$ in $a(L, V)_+$. Let $\gamma \in a(L, V)_+$ be such that $\gamma \leq \mu$ and $\gamma \leq \lambda$. Then $\gamma << \lambda$, and, since $\mu \perp \lambda$, we get $\gamma = 0$.

(3) If L is a Boolean algebra and $W = V = R$, then $\mu \perp \lambda$ implies that, for every real number $\varepsilon > 0$, there exists $a \in L$ such that $\mu(a) < \varepsilon$ and $\lambda(a') < \varepsilon$. In fact, it suffices to apply Remark (2) and the well-known identity:

$$(\mu \wedge \lambda)(b) = \bigwedge\{\mu(a) + \lambda(a' \wedge b) \colon a \in L \text{ and } a \leq b\} \quad \text{for all } b \in L.$$

(4) If L is a σ-complete Boolean algebra $W = V = R$ and $\mu, \lambda \in ca(L, V)_+$, then $\mu \perp \lambda$ if and only if there exists $a \in L$ such that $\mu(a) = 0$ and $\lambda(a') = 0$. In fact, suppose that $\mu \perp \lambda$. For every $k \in N$, Remark (3) implies that there exists $a_k \in L$ such that $\mu(a_k) < 2^{-k}$ and $\lambda(a'_k) < 2^{-k}$. It is easy to see that

$$a = \bigwedge\{\bigvee\{a_k \colon k \in N \text{ and } k \geq n\} \colon n \in N\}$$

exists in L and satisfies $\mu(a) = 0$ and $\lambda(a') = 0$.

We are in position to establish the first main result of this paper:

Theorem 3.2. *Let $\mu \in a(L, V)_+$, let $W = (W, +, ', \leq, \|\cdot\|)$ be a normed Riesz space, and let $\lambda \in a(L, W)_+$. Then there exist two elements ξ and η in $a(L, V)_+$ such that:*

(i) $\mu = \xi + \eta$,
(ii) $\xi << \lambda$,
(iii) $\eta \perp \lambda$.

Proof. Define $\Gamma = \{\gamma \in a(L, V)_+ \colon \gamma << \lambda$ and $\gamma \leq \mu\}$. Since $0 \in \Gamma$, Γ is nonempty. Let Γ_0 be a totally ordered subset of $(\Gamma, \leq)$. Fix $c \in L$. Since $\mu(c)$ is a majorant of the set $\{\gamma(c) \colon \gamma \in \Gamma_0\}$ in V and V is Dedekind complete, there exists $\bigvee\{\gamma(c) \colon \gamma \in \Gamma_0\}$ in V. Write $\gamma_0(c) = \bigvee\{\gamma(c) \colon \gamma \in \Gamma_0\}$ and $D(c) = \{\gamma_0(c) - \gamma(c) \colon \gamma \in \Gamma_0\}$. Since Γ_0 is totally ordered, the set $D(c)$ is directed downwards. Moreover,

$$0 = \gamma_0(c) - \bigvee\{\gamma(0) : \gamma \in \Gamma_0\} = -(\bigvee\{-\gamma_i(c) + \gamma(c) : \gamma \in \Gamma_0\}) = \bigwedge D(c).$$

Since the norm of V is order continuous, it follows that the filter $\mathscr{F}(D(c))$ of sections of $D(c)$ converges in norm to 0 for all $c \in L$.

Write $x_\gamma = \gamma_0(1) - \gamma(1)$ for all $\gamma \in \Gamma_0$. Then $(x_\gamma)_{\gamma \in \Gamma_0}$ is a net in $D(1)$ such that, for every $\gamma_1 \in \Gamma_0$, $\{x_\gamma : \gamma \in \Gamma_0 \text{ and } \gamma_1 \le \gamma\} \subseteq S_{x_{\gamma_1}}$, where $S_{x_{\gamma_1}}$ is the section of $D(1)$ determined by x_{γ_1}. Let $\varepsilon \in R$ be such that $\varepsilon > 0$. By [4, Chapter 1, §7, no. 1, Proposition 1] there exists $\gamma_1 \in \Gamma$ such that $\|x_\gamma\| < \varepsilon$ whenever $\gamma \in \Gamma_0$ and $\gamma_1 \le \gamma$. This establishes that

$$\lim_{\gamma \in \Gamma_0} (\gamma_0(1) - \gamma(1)) = 0$$

in the norm topology of V. Let $c \in L$. Since

$$0 \le \gamma_0(c) - \gamma(c) = \gamma_0(1) - \gamma(1) - (\gamma_0(c') - \gamma(c')) \le \gamma_0(1) - \gamma(1),$$

it follows that $\|\gamma_0(c) - \gamma(c)\| \le \|\gamma_0(1) - \gamma(1)\|$. This implies that

$$\lim_{\gamma \in \Gamma_0} (\gamma_0(c) - \gamma(c)) = 0 \tag{1}$$

uniformly for $c \in L$.

Clearly $\gamma_0 \in V^L$, γ_0 is positive, and $\gamma_0 \le \mu$. To show that γ_0 is additive, let $a, b \in L$ be such that $a \perp b$. Since

$$\begin{aligned}\|\gamma_0(a \vee b) - \gamma_0(a) - \gamma_0(b)\| \le \|\gamma_0(a \vee b) - \gamma(a \vee b)\| + \|\gamma_0(a) - \gamma(a)\| \\ + \|\gamma_0(b) - \gamma(b)\| \quad \text{for all } \gamma \in \Gamma_0,\end{aligned}$$

it follows from (1) that $\gamma_0(a \vee b) = \gamma_0(a) + \gamma_0(b)$. To establish that $\gamma_0 \in \Gamma$, we must show that $\gamma_0 << \lambda$. Let $\varepsilon \in R$ be such that $\varepsilon > 0$. By (1) there exists $\gamma \in \Gamma_0$ such that $\|\gamma_0(c) - \gamma(c)\| < \varepsilon/2$ for all $c \in L$. Since $\gamma << \lambda$, there exists a real number $\delta > 0$ such that $a \in L$ and $\|\lambda(a)\| < \delta$ imply $\|\gamma(a)\| < \varepsilon/2$. Then $a \in L$ and $\|\lambda(a)\| < \delta$ imply $\|\gamma_0(a)\| < \varepsilon$. Therefore, γ_0 is a majorant of Γ_0 in Γ. It follows from Zorn's Lemma that Γ contains a maximal element ξ. Then $\xi << \lambda$ and $\xi \le \mu$. Put $\eta - \mu - \xi$. Clearly $\eta \in a(L, V)_+$. To finish the proof, it suffices to show that $\eta \perp \lambda$. Let $\gamma \in a(L, V)_+$ be such that $\gamma << \lambda$ and $\gamma = \eta$. Then $\xi + \gamma \in a(L, V)_+$, $\xi + \gamma \le \mu$, and $\xi + \gamma << \lambda$. So $\xi + \gamma \in \Gamma$ and the maximality of ξ implies that $\gamma = 0$.

Lemma 3.3. *Let $\mu \in a(L, V)_+$, let $W = (W, +, ', \le, \|\cdot\|)$ be a normed Riesz space, and let $\lambda \in a(L, W)_+$ be such that $\mu << \lambda$. Then*:

(a) *If $\lambda \in ca(L, W)_+$, then $\mu \in ca(L, V)_+$.*
(b) *If $\lambda \in cca(L, W)_+$, then $\mu \in cca(L, V)_+$.*

Proof. Since (a) is immediate, we give the proof of (b).

Let $(a_i)_{i \in I}$ be an orthogonal family of elements of L such that $a = \bigvee\{a_i : i \in I\}$ exists in L. Let $\varepsilon \in R$ be such that $\varepsilon > 0$. Since $\mu << \lambda$, there exists a real number $\delta > 0$ such that $c \in L$ and $\|\lambda(c)\| < \delta$ imply $\|\mu(c)\| < \varepsilon$. Because $\lambda \in cca(L, W)_+$, the family $(\lambda(a_i))_{i \in I}$ is summable in W and $\lambda(a) = \sum_{i \in I} \lambda(a_i)$. Then there exists a finite subset J_0 of I such that $\|\lambda(a) - \sum_{i \in I} \lambda(a_i)\| < \delta$ whenever J is a finite subset of I containing J_0. Since $\bigvee\{a_i : i \in J\} \le a$, we have $a = (\bigvee\{a_i : i \in J\}) \vee (a \wedge (\bigvee\{a_i : i \in J\})')$. Then $\lambda(a) = \sum_{i \in J} \lambda(a_i) +$

$\lambda(a \wedge (\bigvee\{a_i : i \in J\})')$, and therefore $\|\lambda(a \wedge (\bigvee\{a_i : i \in J\})')\| < \delta$. So $\|\mu(a \wedge (\bigvee\{a_i : i \in J\})')\| < \varepsilon$. Hence, $\|\mu(a) - \sum_{i \in J} \mu(a_i)\| < \varepsilon$ whenever J is a finite subset of I containing J_0. Consequently, μ is completely additive.

A straightforward application of Theorem 3.2 and Lemma 3.3 yields the following corollaries:

Corollary 3.4. *Let $\mu \in ca(L, V)_+$, let W be a normed Riesz space, and let $\lambda \in ca(L, W)$. Then there exist two elements ξ and η in $ca(L, V)$ such that:*

(i) $\mu = \xi , \eta$,
(ii) $\xi << \lambda$,
(iii) $\eta \perp \lambda$.

Corollary 3.5. *Let $\mu \in cca(L, V)_+$, let W be a normed Riesz space, and let $\lambda \in cca(L, W)_+$. Then there exist two elements ξ and η in $cca(L, V)_+$ such that:*

(i) $\mu = \xi + \eta$,
(ii) $\xi << \lambda$,
(iii) $\eta \perp \lambda$.

Remarks 3.6. (1) Taking into account Remark 3.1(2), Corollaries 3.4 and 3.5 contain Corollary 3.5 of [14, p. 119].

(2) If L is a Boolean algebra and $W = V = R$, then the decompositions given by Theorem 3.2 and Corollaries 3.4 and 3.5 are unique. Then taking into account the Jordan decomposition and Remark 3.1(3), it is easy to see that Theorem 3.2 contains Theorem 2.2(1)–(3) of [6, p. 34].

(3) If L is a σ-complete Boolean algebra and $W = V = R$, then by the usual reduction we can deduce the classical Lebesgue Decomposition Theorem [13, Proposition 24, p. 240] from Corollary 3.4, taking into account Remark 3.1(4) and [8, Lemma III, 4.13].

(4) If L is a Boolean algebra, V is a KB-space, and $W = R$, then using Corollary 4.3 and Lemmas 4.10 and 4.11 of [16] we can deduce the improved Lebesgue Decomposition Theorem [16, Theorem 4.12], in this special case, from Theorem 3.2.

4. The Yosida-Hewitt Decomposition Theorems

Let $\mu \in a(L, V)_+$. We say that μ is *purely finitely additive* (resp. *weakly purely finitely additive*) if $\gamma \in ca(L, V)_+$ (resp. $\gamma \in cca(L, V)_+$) and $\gamma \leq \mu$ imply $\gamma = 0$. We are in position to establish the second and third main results of this paper:

Theorem 4.1. *Let $\mu \in a(L, V)_+$. Then there exist two elements ξ and η in $a(L, V)$ such that:*

(i) $\mu = \xi + \eta$,
(ii) ξ *is completely additive,*
(iii) η *is weakly purely finitely additive.*

Proof. Define $\Gamma = \{\gamma \in cca(L, V)_+ : \gamma \leq \mu\}$. Since $0 \in \Gamma$, Γ is nonempty. Let Γ_0 be a totally ordered subset of $(\Gamma, \leq)$. As in the proof of Theorem 3.2, we can define $\gamma_0(c) = \bigvee\{\gamma(c) : \gamma \in \Gamma_0\}$ and prove that

$$\lim_{\gamma \in \Gamma_0} (\gamma_0(c) - \gamma(c)) = 0$$

uniformly for $c \in L$ in the norm topology of V. From this we deduce that $\gamma_0 \in a(L, V)$. We shall show that γ_0 is completely additive. Let $(a_i)_{i \in I}$ be an orthogonal family of elements of L such that $a = \bigvee\{a_i : i \in I\}$ exists in L. Let $\varepsilon \in R$ be such that $\varepsilon > 0$. Then there exists $\gamma \in \Gamma_0$ such that $\|\gamma_0(c) - \gamma(c)\| < (1/2)\varepsilon$ for all $c \in L$. Since $\gamma \in cca(L, V)_+$, there exists a finite subset J_0 of I such that $\|\gamma(a) - \sum_{i \in J} \gamma(a_i)\| < (1/2)\varepsilon$ whenever J is a finite subset of I containing J_0. Let J be a finite subset of I containing J_0. Since

$$a = (\bigvee\{a_i : i \in J\}) \vee (a \wedge (\bigvee\{a_i : i \in J\})'),$$

we have

$$\begin{aligned}\gamma_0(a) - \sum_{i \in J} \gamma_0(a_i) = \gamma(a) - \sum_{i \in J} \gamma(a) \\ + \gamma_0(a \wedge (\bigvee\{a_i : i \in J\})') - \gamma(a \wedge (\bigvee\{a_i : i \in J\})').\end{aligned}$$

Then $\|\gamma_0(a) - \sum_{i \in J} \gamma_0(a_i)\| < \varepsilon$ whenever J is a finite subset of I containing J_0.

Since $\gamma_0 \leq \mu$, γ_0 is a majorant of Γ_0 in Γ. It follows from Zorn's Lemma that Γ contains a maximal element ξ. Then ξ is completely additive and $\xi \subseteq \mu$. Put $\eta = \mu - \xi$. Clearly $\eta \in a(L, V)_+$. To finish the proof, it suffices to show that η is weakly purely finitely additive. Let $\gamma \in cca(L, V)_+$ be such that $\gamma \leq \eta$. Then $\xi + \gamma \in cca(L, V)$ and $\xi + \gamma \leq \mu$. So $\xi + \gamma \in \Gamma$ and the maximality of ξ implies that $\gamma = 0$.

Theorem 4.2. *Let $\mu \in a(L, V)$. Then there exist two elements ξ and η in $a(L, V)_+$, such that:*

(i) $\mu = \xi + \eta$,
(ii) *ξ is countably additive,*
(iii) *η is purely finitely additive.*

Proof. A trivial modification of the proof of Theorem 4.1.

Remarks 4.3. (1) Theorems 4.1 and 4.2 contain Corollary 3.3 of [14, p. 117].

(2) If L is a Boolean algebra and $V = R$, then the decompositions given by Theorems 4.1 and 4.2 are unique. Then Theorem 4.2 contains the classical Yosida-Hewitt Decomposition Theorem [17, Theorem 1.23].

(3) If L is a Boolean algebra and V is a KB-space, then using Corollary 4.3 and Lemmas 4.6 and 4.7 of [16], we can deduce the improved Yosida-Hewitt Decomposition Theorem [16, Theorem 4.8], in this special case from Theorem 4.2.

Let H be a Hilbert space over R or C with inner product $(\cdot, \cdot)$, and let $B(H)$ be the von Neumann algebra of all bounded linear operators acting on H. An element $T \in B(H)$ is said to be a positive operator, and we write $T \geq 0$ if $(Tx, x) \geq 0$ for all $x \in H$. If $S, T \in B(H)$ we write $S \leq T$ if $T - S \geq 0$. Then the binary relation $\leq$ is a partial order on $B(H)$. A nonempty subset $\mathscr{F}$ of $B(H)$ is said to be *directed upwards*, and we write $\mathscr{F} \uparrow$, if for any $S, T \in \mathscr{F}$ there exists $U \in \mathscr{F}$ such that $S \leq U$ and $T \leq U$. If $\mathscr{M}$ is a nonempty subset of $B(H)$ we write $\mathscr{M}^+ = \{T \in \mathscr{M} : T^* = T \text{ and } T \geq 0\}$.

Let ψ be a linear form on $B(H)$. We say that ψ is *positive* if $\psi(T) \geq 0$ for all $T \in B(H)^+$. A positive linear form ψ on $B(H)$ is said to be *normal* if, for every nonempty subset $\mathscr{F}$ of $B(H)^+$ such that $\mathscr{F} \uparrow$ and there exists $S = \bigvee \mathscr{F}$ in $B(H)^+$, we have $\psi(S) = \bigvee\{\psi(T) \colon T \in \mathscr{F}\}$.

A function $\varphi \colon B(H)^+ \to [0, +\infty]$ is called a *trace* on $B(H)^+$ if the following conditions are satisfied:

(i) If $S, T \in B(H)^+$, then $\varphi(S+T) = \varphi(S) + \varphi(T)$.

(ii) If $S \in B(H)^+$ and $\alpha \in R$ with $\alpha \geq 0$, then $\varphi(\alpha S) = \alpha\varphi(S)$ (with the convention $0 \cdot (+\infty) = 0$).

(iii) If $S \in B(H)^+$ and U is a unitary operator of $B(H)$, then $\varphi(USU^{-1}) = \varphi(S)$.

A trace φ on $B(H)^+$ is called *normal* if, for every nonempty subset $\mathscr{F}$ of $B(H)$ such that $\mathscr{F} \uparrow$ and there exists $S = \bigvee \mathscr{F}$ in $B(H)^+$, we have $\varphi(S) = \bigvee\{\varphi(T) \colon T \in \mathscr{F}\}$.

Let $(e_i)_{i\in I}$ be an orthonormal basis of H. For $T \in B(H)$ we write $\varphi(T) = \sum_{i\in I}(Te_i, e_i)$. Then φ is a normal trace on $B(H)$ independent of the choice of the orthonormal basis $(e_i)_{i\in I}$, and we write, as usual, $\varphi(T) = \operatorname{tr}(T)$ (see [7, Théorème 5, p. 94]). An element $T \in B(H)$ is said to be a *trace-class operator* if $\operatorname{tr}(T) < +\infty$. Consider the set $\mathscr{M}_0 = \{T \in B(H)^+ \colon \operatorname{tr}(T) < +\infty\}$. Then $\mathscr{M}_0$ satisfies the hypothesis of [7, Proposition 10, p. 11], and therefore there exists a two-sided ideal $\mathscr{M}$ of $B(H)$ such that $\mathscr{M}_0 = \mathscr{M}^+$. Moreover, there exists a unique linear form ψ on $\mathscr{M}$ which coincides with $\operatorname{tr}(\cdot)$ on $\mathscr{M}^+$ and, for any trace-class operator T, the positive linear form $\psi(S) = \operatorname{tr}(TS)$ on $B(H)$ is normal (see [7, Proposition 1, p. 82]). By [7, Problème 9, p. 68], for any orthogonal family $(P_i)_{i\in I}$ of projections of $B(H)$, we have $\operatorname{tr}(T\sum_{i\in I} P_i) = \sum_{i\in I} \operatorname{tr}(TP_i)$ for all trace-class operators T.

Suppose that $\dim(H) \geq 3$. Let $S = \{x \in H \colon \|x\| = 1\}$ be the unit sphere of H, and let $\alpha \in R$. A real-valued function f on S is called a *frame-function of weight* α if $\sum_{i\in I} f(e_i) = \alpha$ for all orthomodular bases $(e_i)_{i\in I}$ of H.

Lemma 4.4. *Let H be a Hilbert space over R or C of dimensions ≥ 3 with inner product $(\cdot, \cdot)$, let $L = L(H)$, let $V = R$, and let $\mu \in a(L, V)_+$. If μ is weakly purely finitely additive, then μ vanishes on all finite-dimensional subspaces of H.*

Proof. Suppose the contrary. Then there exist an element $\mu_0 \in a(L, V)$ and a subspace M_0 of H such that μ_0 is weakly purely finitely additive, $\dim(M_0) < +\infty$, and $\mu_0(M_0) \geq 0$.

For every $x \in S$, define $f(x) = \mu_0(B_x)$, where B_x is the one-dimensional subspace of H spanned by the unit vector x. Then $0 \leq f(x) \leq \mu_0(H)$ for all $x \in S$. Moreover, the finite additivity of μ_0 implies that, if M is a finite-dimensional subspace of H, then the restriction $f|M$ is a frame-function of weight $\mu_0(M)$ on the unit sphere of M. By Proposition 1 of [1, p. 605] there exists an operator $T \in B(H)^+$ such that $f(x) = (Tx, x)$ for all $x \in S$. With the argument appearing in the proof of Proposition 2 of [1, pp. 609–610], we establish that T is a trace class operator.

From every $M \in L$, define $\gamma(M) = \operatorname{tr}(TP^M)$, where P^M is the projection of H onto M. It is easy to verify that $\gamma \in a(L, V)$ and $\gamma \leq \mu_0$. Let $(M_i)_{i\in I}$ be an orthogonal family in L, and let $M = \bigvee\{M_i \colon i \in I\}$. Since $P^M = \sum_{i\in I} P^{M_i}$ and $(P^{M_i})_{i\in I}$ is an orthogonal family of projections of $B(H)$, we have $\gamma(M) = \operatorname{tr}(T\sum_{i\in I} P^{M_i}) = \sum_{i\in I} \operatorname{tr}(TP^{M_i}) = \sum_{i\in I} \gamma(M_i)$, and therefore γ is completely additive. Since $\gamma(M_0) = \mu_0(M_0) > 0$, this contradicts the hypothesis that μ_0 is weakly purely finitely additive.

Corollary 4.5. [1, Proposition 2, p. 609]. *Let H be a Hilbert space over R or C of dimension ≥ 3, let $L = L(H)$, let $V = R$, and let $\mu \in a(L, V)$. Then there exist two unique elements ξ and η in $a(L, V)$ such that:*

(i) $\mu = \xi + \eta$,
(ii) *ξ is completely additive,*
(iii) *η vanishes on all finite-dimensional subspaces of H.*

Proof. The existence of the decomposition follows from Theorem 4.1 and Lemma 4.4.

To prove the uniqueness, suppose that there exist four elements ξ_1, η_1, ξ_2, and η_2 in $a(L, V)$ such that $\xi_1 + \eta_1 = \mu = \xi_2 + \eta_2$, ξ_1 and ξ_2 are completely additive, and η_1 and η_2 vanish on all finite-dimensional subspaces of H.

Let $N \in L$ be such that $\dim(N) < +\infty$. Then $\eta_1(N) = 0 = \eta_2(N)$, and therefore $\xi_1(N) = \xi_2(N)$. Let M be an arbitrary closed subspace of H, and let $(e_i)_{i\in I}$, be an orthonormal basis of M. Then $(B_{e_i})_{i\in I}$ is an orthogonal family of one-dimensional subspaces of H and $M = \bigvee\{B_{e_i}: i \in I\}$. Since ξ_1 and ξ_2 are completely additive, it follows that

$$\xi_i(M) = \sum_{i\in I} \xi_1(B_{e_i}) = \sum_{i\in I} \xi_2(B_{e_i}) = \xi_2(M).$$

So $\xi_1 = \xi_2$ and therefore $\eta_1 = \eta_2$.

References

1. J. F. Aarnes, *Quasi-states on C^*-algebras*, Trans. Amer. Math. Soc. **149** (1970), 601–625.
2. L. Beran, *Orthomodular lattices. Algebraic approach*, Academia, Praha, and Reidel, Dordrecht, 1984.
3. K. P. S. Bhaskara Rao and M. Bhaskara Rao, *Theory of charges*, Academic Press, London, 1983.
4. N. Bourbaki, *Topologie générale,* Chapitres 1–4, Hermann, Paris, 1971.
5. W. J. Claas and A. C. Zaanen, *Orlicz lattices*, Compositio Math., Tomus specialis in honorem Ladislai Orlicz (1978), 77–93.
6. R. Darst, *A decomposition of finitely additive set functions*, J. Reine Angew. Math. **210** (1962), 31–37.
7. J. Dixmier, *Les algèbres d' opérateurs dans l'espace Hilbertien* (Algèbres de von Neumann), Gauthier-Villars, Paris, 1969.
8. N. Dunford and J. Schwartz, *Linear operators.* I, Interscience, New York, 1958.
9. D. H. Fremlin, *Topological Riesz spaces and measure theory*, Cambridge Univ. Press, Cambridge, 1974.
10. P. Halmos, *Introduction to Hilbert spaces and the theory of spectral multiplicity*, Chelsea, New York, 1951.
11. G. Kalmbach, *Orthomodular lattices*, Academic Press, London, 1983.
12. P. Ptak and S. Pulmannova, *Orthomodular structures as quantum logics*, Kluwer Academic, Dordrecht, 1991.
13. H. L. Royden, *Real analysis*, 2nd ed., MacMillan, New York, 1971.
14. G. T. Rüttimann, *Decomposition of cones of measures*, Atti Sem. Mat. Fis. Univ. Modena **28** (1990), 109–121.
15. H. H. Schaefer, *Banach lattices and positive operators*, Springer-Verlag, Berlin, 1974.

16. K. D. Schmidt, *Decomposition of vector measures in Riesz spaces and Banach lattices*, Proc. Edinburgh Math. Soc. (2) **29** (1986), 23–39.

17. K. Yosida and E. Hewitt, *Finitely additive measures*, Trans. Amer. Math. Soc. **72** (1952), 46–66.

Department of Mathematics, University of Naples, Complesso Monte San Angelo, 80126 Naples, Italy

Department of Mathematics, University of Sherbrooke, Sherbrooke, Quebec, Canada J1K 2R1