

INVARIANT SUBSPACES: CONTINUOUS STABILITY IMPLIES SMOOTH STABILITY

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ABSTRACT. The stability of an invariant subspace is defined in the terms of the existence of a certain type of function. The imposition of further conditions on this function leads to different forms of stability. Of these, the equivalence of continuous and smooth stability is proved; two proofs are offered for comparison.

1. INTRODUCTION

Roughly speaking, an invariant subspace M of an operator A on a Hilbert space is stable if invariant subspaces near M exist for all operators near A . By requiring that the nearby subspaces meet other conditions, we can define different types of stability. Some of these forms of stability have been studied by Gohberg et al. [2] and Noakes [6].

The present work concerns proving the equivalence of two types of stability, namely, continuous and smooth stability in finite-dimensional complex Hilbert spaces. We shall provide two proofs: one using elementary linear techniques and a shorter proof using more advanced nonlinear methods.

In §2 we develop the tools we shall need. In the process, the seemingly different approaches in [2, 6] for defining stability are reconciled and a common basis is established for the definition of many types of stability. The above-mentioned proofs are then given in §§3 and 4 respectively.

Throughout, H is assumed to be a finite-dimensional complex Hilbert space. The set of invariant (closed linear) subspaces of an operator A on a Hilbert space will be denoted $\text{Lat } A$. Inner products in H are denoted $\langle \cdot, \cdot \rangle$ and the (orthogonal) projection onto the subspace M is denoted P_M . Given Hilbert spaces K_1 and K_2 , we shall write $\mathcal{L}(K_1, K_2)$ and $\mathcal{E}(K_1)$ respectively for the Banach space of (bounded linear) operators $T: K_1 \rightarrow K_2$ and the set of subspaces of K_1 . For convenience, we write $\mathcal{L}(H)$ in place of $\mathcal{L}(H, H)$. Let ρ denote the metric on $\mathcal{E}(H)$ defined by identifying subspaces with projections and referring to the operator norm; thus, $\rho(N, M) = \|P_N - P_M\|$. It is easily verified that by suitably restricting the domain of ρ , we obtain the corresponding metric on $\mathcal{E}(K)$ for any subspace K of H . We always assume the operator

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norm topology on $\mathcal{L}(H)$ and the metric topology on $\mathcal{E}(H)$. Finally, we shall write $f|_E$ for the restriction of the function f to the set E .

2. PRELIMINARIES

From now on, A is assumed to be an operator on H and M an invariant subspace of A . Stability is defined in [2] as follows: M is *stable* if for every $\varepsilon > 0$ there is a $\delta > 0$ such that every operator B satisfying $\|B - A\| < \delta$ has an invariant subspace N with $\rho(N, M) < \varepsilon$. Equivalently, M is stable if there is a function $G: V \rightarrow \mathcal{E}(H)$ defined on a neighbourhood V of A such that G is continuous at A , $G(A) = M$, and $G(B) \in \text{Lat } M$ for all $B \in V$. On the other hand, the definition in [6] requires the existence of a C^1 function $F: V \rightarrow \mathcal{L}(M, M^\perp)$ defined near A satisfying $F(A) = 0$ and $(1 + F(B))M \in \text{Lat } B$ for all $B \in V$; here, $(1 + T)M = \{x + Tx : x \in M\}$.

Let us make the

Definition 1. An *invariance function* for (A, M) is a function $F: V \rightarrow \mathcal{L}(M, M^\perp)$ such that $F(A) = 0$ and $(1 + F(B))M \in \text{Lat } B$ for all $B \in V$, where V is a neighbourhood of A . We say that M is a *stable A -invariant subspace* if there exists an invariance function F for (A, M) that is continuous at A . Moreover, we say that M is *smoothly stable* (respectively *continuously stable*) if F can be chosen to be C^1 (respectively continuous).

Implicit to the following result is a homeomorphism between $\{N \in \mathcal{E}(H) : \rho(N, M) < 1\}$ and $\mathcal{L}(M, M^\perp)$ which allows us to conclude that M is stable if and only if it is stable in the sense of [2].

Proposition 1 (see [1, Theorem 1]). *Let M and N be subspaces of H . The following are equivalent.*

- (1) $\|P_N - P_M\| < 1$.
- (2) $N = (1 + T)M$ for some $T \in \mathcal{L}(M, M^\perp)$.

If any of the above conditions holds, then T is unique and satisfies

$$\|T\| = \frac{\|P_N - P_M\|}{\sqrt{1 - \|P_N - P_M\|^2}}.$$

Indeed, if we refer to the structure of a C^∞ -manifold for $\mathcal{E}(H)$ given in [1], we may use the alternative

Definition 2. A function $G: V \rightarrow \mathcal{E}(H)$ defined on a neighbourhood of A is called an *invariance function* for (A, M) if $G(A) = M$ and $G(B) \in \text{Lat } B$ for all $B \in V$. We say that M is *stable* if there is an invariance function for (A, M) that is continuous at A . Moreover, M is *continuously stable* (respectively *smoothly stable*) if the invariance function can be chosen to be continuous (respectively C^1).

The invariance function of Definitions 1 and 2 are related by homeomorphism, and we shall use the two definitions interchangeably; the applicable definition can be inferred from the codomain specified.

Smooth stability is clearly stronger than continuous stability, so it suffices to prove

Theorem 1. *Continuous stability implies smooth stability.*

We shall need the following results in the latter sections.

Lemma 1. *Let M be a nontrivial subspace of H , and let $A \in \mathcal{L}(H)$. Suppose that, relative to the decomposition $H = M \oplus M^\perp$, A has the matrix representation*

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

where $A_{11} \in \mathcal{L}(M)$, $A_{12} \in \mathcal{L}(M^\perp, M)$, $A_{21} \in \mathcal{L}(M, M^\perp)$, and $A_{22} \in \mathcal{L}(M^\perp, M^\perp)$. If $T \in \mathcal{L}(M, M^\perp)$, then $(1 + T)M \in \text{Lat } A$ if and only if $A_{21} + A_{22}T = TA_{11} + TA_{12}T$.

Proof. Relative to the decomposition $H = M \oplus M^\perp$, $(1 + T)M$ is the graph of T . Now, the image of (x, Tx) under A is $(A_{11}x + A_{12}Tx, A_{21}x + A_{22}Tx)$, which is in $(1 + T)M$ if and only if $A_{21}x + A_{22}Tx = T(A_{11}x + A_{12}Tx)$. Therefore, $(1 + T)M \in \text{Lat } A$ if and only if $(A_{21} + A_{22}T)x = T(A_{11} + A_{12}T)x$ for all $x \in M$. \square

Before giving the next result, we need a definition. Recall that, if X is a topological space and $\{E_n\}_{n=1}^\infty$ is a sequence of subsets of X , then $\liminf_{n \rightarrow \infty} E_n$ is the set of all limits of sequences $\{x_n\}_{n=1}^\infty$ with $x_n \in E_n$ and $\limsup_{n \rightarrow \infty} E_n$ is the set of all limits of subsequences of sequences $\{x_n\}_{n=1}^\infty$ with $x_n \in E_n$.

Definition 3. We say that a sequence $\{M_n\}_{n=1}^\infty$ of subspaces of H converges *semistrong* to a subspace M if

$$\liminf_{n \rightarrow \infty} M_n = \limsup_{n \rightarrow \infty} M_n = M.$$

Proposition 2. *Metric convergence, strong convergence, and semistrong convergence in $\mathcal{E}(H)$ are equivalent.*

Proof. A result of Halmos [4, Theorem 1] shows that strong convergence and semistrong convergence are equivalent. Since H is finite dimensional, metric convergence is the same as strong convergence. \square

From now on, Ξ denotes the operator on $\mathcal{L}(M, M^\perp)$ defined by

$$\Xi(T) = TA_{11} - A_{22}T$$

for all $T \in \mathcal{L}(M, M^\perp)$.

Theorem 2. *Let H be a complex finite-dimensional Hilbert space, and let M be an invariant subspace of an operator A in $\mathcal{L}(H)$. Then the following are equivalent.*

- (1) M is smoothly stable.
- (2) Ξ is surjective.
- (3) $\sigma(A_{11}) \cap \sigma(A_{22}) = \emptyset$.

Proof. (1) implies (2). This is a special case of [6, Theorem 1].

(2) implies (3). We shall prove the contrapositive. Suppose $\sigma(A_{11}) \cap \sigma(A_{22}) \neq \emptyset$, and let $\lambda \in \sigma(A_{11}) \cap \sigma(A_{22})$. Then $A_{11} - \lambda$ is not injective and $A_{22} - \lambda$ is not surjective. Choose $u \in \ker(A_{11} - \lambda) \setminus \{0\}$ and $v \in M^\perp \setminus \text{ran}(A_{22} - \lambda)$, and define $T \in \mathcal{L}(M, M^\perp)$ by $Tx = \langle x, u \rangle v$. For all $S \in \mathcal{L}(M, M^\perp)$ we have $\Xi(S) = SA_{11} - A_{22}S = S(A_{11} - \lambda) - (A_{22} - \lambda)S$, so $\Xi(S)u = -(A_{22} - \lambda)Su \in \text{ran}(A_{22} - \lambda)$. Therefore, $\Xi(S)u \neq Tu$; whence, Ξ is not surjective.

(3) implies (1). This is a special case of [6, Theorem 2]. \square

If H is a complex Hilbert space, then the spectrum $\sigma(A)$ of an operator A on H is the set of eigenvalues of A . Given $\lambda \in \sigma(A)$, we shall denote the root subspace of A corresponding to the eigenvalue λ by $R_\lambda(A)$, thus,

$$R_\lambda(A) = \bigcup_{k=1}^{\infty} \ker(A - \lambda)^k.$$

A standard result tells us that

$$M = \bigoplus_{\lambda \in \sigma(A)} (M \cap R_\lambda(A))$$

for all A -invariant subspaces M .

Proposition 3 (see [2, Theorem 15.1.4]). *Let $M \in \text{Lat } A$, and let E be an open subset of C such that $\sigma(A|_M) \subseteq E$. If $B \in \mathcal{L}(H)$, $N \in \text{Lat } B$, and $\rho(N, M) + \|B - A\|$ is sufficiently small, then $\sigma(B|_N) \subseteq E$.*

Proposition 4 (see [2, Theorem 15.2.1]). *Let $\lambda_1, \dots, \lambda_p$ be the different eigenvalues of a transformation A in $\mathcal{L}(H)$. A subspace M of H is A -invariant and stable if and only if $M = \bigoplus_{j=1}^p M_j$ where each M_j is an arbitrary A -invariant subspace of $R_{\lambda_j}(A)$ if $\dim \ker(A - \lambda_j) = 1$; if $\dim \ker(A - \lambda_j) \neq 1$ then either $M_j = \{0\}$ or $M_j = R_{\lambda_j}(A)$.*

Proposition 5 (see [2, p. 449]). *A subspace M is a stable invariant subspace of A if and only if it is an isolated A -invariant subspace. Equivalently, M is stable if and only if there exists an $r > 0$ such that $(1 + T)M \notin \text{Lat } A$ for all nonzero $T \in \mathcal{L}(M, M^\perp)$ satisfying $\|T\| < r$.*

3. AN ELEMENTARY PROOF USING LINEAR METHODS

Our first approach to proving Theorem 1 is to first reduce the problem to the case where the operator has only one eigenvalue. The following two lemmas achieve this goal.

Lemma 2. *Suppose H is a direct sum $\bigoplus_{j=1}^p K_j$ of subspaces, and let M be a subspace of H satisfying $M = \bigoplus_{j=1}^p (M \cap K_j)$. Given $\varepsilon > 0$, there exists a $\delta > 0$ such that if N is a subspace of H for which $N = \bigoplus_{j=1}^p (N \cap K_j)$ and if $\rho(N, M) < \delta$ then $\rho(N \cap K_j, M \cap K_j) < \varepsilon$ for $j = 1, \dots, p$.*

Proof. Suppose the assertion is false, so that there is an $\varepsilon > 0$ and a sequence $\{M_n\}_{n=1}^{\infty}$ of subspaces such that $M_n = \bigoplus_{j=1}^p (M_n \cap K_j)$, $M_n \rightarrow M$, and $\rho(M_n \cap K_q, M \cap K_q) \geq \varepsilon$ for some q (possibly depending on n). By extracting an appropriate subsequence if necessary, we can assume without loss of generality that q is fixed and that $\rho(M_n \cap K_q, M \cap K_q) \geq \varepsilon$ for all n .

Now $M = \liminf_{n \rightarrow \infty} M_n$ by Proposition 2, so if $x \in M \cap K_q$, then there is a sequence $\{x_n\}_{n=1}^{\infty}$ with $x_n \in M_n$ such that $x_n \rightarrow x$. Each $x_n = \sum_{j=1}^p x_{nj}$, where $x_{nj} \in M_n \cap K_j$, so applying the projection onto K_q along the direct sum of the remaining K_j yields $x_{nq} \rightarrow x$; whence, $x \in \liminf_{n \rightarrow \infty} M_n \cap K_q$. Therefore,

$$M \cap K_q \subseteq \liminf_{n \rightarrow \infty} M_n \cap K_q.$$

Conversely, $\limsup_{n \rightarrow \infty} M_n \cap K_q \subseteq \limsup_{n \rightarrow \infty} M_n = M$ and $\limsup_{n \rightarrow \infty} M_n \cap K_q \subseteq \limsup_{n \rightarrow \infty} K_q = K_q$; whence,

$$\limsup_{n \rightarrow \infty} M_n \cap K_q \subseteq M \cap K_q.$$

Consequently $\rho(M_n \cap K_q, M \cap K_q) \rightarrow 0$ by Proposition 2, contradicting $\rho(M_n \cap K_q, M \cap K_q) \geq \varepsilon$. \square

Lemma 3. *Suppose M is a continuously stable A -invariant subspace, and let $\lambda \in \sigma(A)$. Then $M \cap R_\lambda(A)$ is a continuously stable $A|_{R_\lambda(A)}$ -invariant subspace.*

Proof. Let $K_1 = R_\lambda(A)$, and let K_2 be the direct sum of the root subspaces of A other than $R_\lambda(A)$. Then $A = A|_{K_1} \oplus A|_{K_2}$ relative to the direct sum $H = K_1 \oplus K_2$. The eigenvalues of A are isolated, so there is an open subset E of C such that $\sigma(A) \cap E = \{\lambda\}$. Since $\sigma(A|_{K_1}) = \{\lambda\} \subseteq E$, there is a neighbourhood V_0 of $A|_{K_1}$ in $\mathcal{L}(K_1)$ such that $\sigma(B) \subseteq E$ for all $B \in V_0$. Note that $\sigma(B) \cap \sigma(A|_{K_2}) = \emptyset$ and $K_1 = \bigoplus_{\mu \in \sigma(B)} R_\mu(B)$ for all $B \in V_0$.

Let $F': V' \rightarrow \mathcal{E}(H)$ be a continuous invariance function for (A, M) , where V' is a neighbourhood of A . Put $W = V' \cap \{B \oplus A|_{K_2} : B \in V_0\}$. Then $F'|_W : W \rightarrow \mathcal{E}(H)$ is continuous. Since K_1 is a spectral subspace of every $B' \in W$, it follows that $F'(B') \cap K_1 \in \text{Lat } B'|_{K_1}$ for all $B' \in W$. Put $V = \{B \in V_0 : B \oplus A|_{K_2} \in V'\}$, and define $F : V \rightarrow \mathcal{E}(R_\lambda(A))$ by $F(B) = F'(B \oplus A|_{K_2}) \cap K_1$. Then $F(B) \in \text{Lat } B$ for all $B \in V$.

If $B \in V$, then any $(B \oplus A|_{K_2})$ -invariant subspace N satisfies $N = (N \cap K_1) \oplus (N \cap K_2)$. Fix $B_0 \in V$; then $\|(B \oplus A|_{K_2}) - (B_0 \oplus A|_{K_2})\| = \|B - B_0\|$. Let $N = F'(B \oplus A|_{K_2})$ and $N_0 = F'(B_0 \oplus A|_{K_2})$. Given $\varepsilon > 0$, Lemma 2 asserts the existence of $\varepsilon' > 0$ such that $\rho(N, N_0) < \varepsilon'$ implies $\rho(N \cap K_1, N_0 \cap K_1) < \varepsilon$. But there is a $\delta > 0$ such that $\|B - B_0\| < \delta$ implies $\rho(N, N_0) < \varepsilon'$; whence, $\rho(F(B), F(B_0)) = \rho(N \cap K_1, N_0 \cap K_1) < \varepsilon$, so F is continuous on V . Observe that $F(A|_{R_\lambda(A)}) = M \cap R_\lambda(A)$, so $M \cap R_\lambda(A)$ is continuously stable. \square

The following result allows us to keep track of eigenvalues systematically.

Lemma 4. *Suppose M is a continuously stable invariant subspace of A , and let $F : V \rightarrow \mathcal{E}(H)$ be a continuous invariance function for (A, M) . Let $\Gamma : [0, 1] \rightarrow V$ be continuous, and let $\lambda_1, \dots, \lambda_n$ be continuous complex-valued functions such that the eigenvalues of $\Gamma(\alpha)$ are $\lambda_1(\alpha), \dots, \lambda_n(\alpha)$ with (algebraic) multiplicities counted, for all $\alpha \in [0, 1]$. For each $\alpha \in [0, 1]$, put*

$$\Omega_\alpha = \{j \in \{1, \dots, n\} : \lambda_j(\alpha) \in \sigma(\Gamma(\alpha)|_{F(\Gamma(\alpha))})\}.$$

If $|\Omega_\alpha|$ is constant for all $\alpha \in [0, 1]$, then the Ω_α are all equal.

Proof. In what follows, α and β are always understood to be elements of $[0, 1]$, and j is always assumed to be in $\{1, \dots, n\}$. Given α , choose disjoint open sets E_α and E'_α such that

$$\sigma(\Gamma(\alpha)|_{F(\Gamma(\alpha))}) \subseteq E_\alpha \quad \text{and} \quad \sigma(\Gamma(\alpha)) \setminus \sigma(\Gamma(\alpha)|_{F(\Gamma(\alpha))}) \subseteq E'_\alpha.$$

Thus $\sigma(\Gamma(\alpha)) \subseteq E_\alpha \cup E'_\alpha$, and $\lambda_j(\alpha) \in E_\alpha$ if and only if $j \in \Omega_\alpha$, for all j .

Now, there exists an $r_\alpha > 0$ such that if $|\beta - \alpha| < r_\alpha$, then $\sigma(\Gamma(\beta)) \subseteq E_\alpha \cup E'_\alpha$. Because E_α and E'_α are disjoint and since $\{\lambda_j(\beta) : \beta \in (\alpha - r_\alpha, \alpha + r_\alpha) \cap [0, 1]\}$ is connected for each j , it follows that $\lambda_j(\beta) \in E_\alpha$ if and only if $j \in \Omega_\alpha$, for all j and for all $\beta \in (\alpha - r_\alpha, \alpha + r_\alpha) \cap [0, 1]$.

The map $\beta \mapsto \rho(F(\Gamma(\beta)), F(\Gamma(\alpha))) + \|\Gamma(\beta) - \Gamma(\alpha)\|$ is continuous, so by Proposition 3, there exists an $s_\alpha > 0$ such that $|\beta - \alpha| < s_\alpha$ implies $\sigma(\Gamma(\beta)|_{F(\Gamma(\beta))}) \subseteq E_\alpha$.

Put $\delta_\alpha = \min\{r_\alpha, s_\alpha\}$, and suppose $|\beta - \alpha| < \delta_\alpha$. If $j \in \Omega_\beta$, then $\lambda_j(\beta) \in \sigma(\Gamma(\beta)|_{F(\Gamma(\beta))}) \subseteq E_\alpha$, so $j \in \Omega_\alpha$; hence, $\Omega_\beta \subseteq \Omega_\alpha$. Since $|\Omega_\beta| = |\Omega_\alpha|$, the two sets are equal.

Letting α vary, we obtain an open cover $\{(\alpha - \delta_\alpha, \alpha + \delta_\alpha) \cap [0, 1] : \alpha \in [0, 1]\}$ for $[0, 1]$. By Lebesgue's Covering Lemma, there exists an $\varepsilon > 0$ such that, for every β , there is an α satisfying $[\beta, \beta + \varepsilon] \cap [0, 1] \subseteq (\alpha - \delta_\alpha, \alpha + \delta_\alpha) \cap [0, 1]$. In particular, if we choose p so large that $1/p < \varepsilon$, then there exist $\alpha_0, \dots, \alpha_{p-1}$ such that

$$[q/p, (q + 1)/p] \subseteq (\alpha - \delta_\alpha, \alpha + \delta_\alpha)$$

for $q = 0, \dots, p - 1$. Consequently, $\Omega_{q/p} = \Omega_{\alpha_q} = \Omega_{(q+1)/p}$; successively, letting $q = 0, \dots, p - 1$, we see that $\Omega_{(q+1)/p} = \Omega_0$ for each q . Finally, every β is in some $[q/p, (q + 1)/p]$, so $\Omega_\beta = \Omega_{\alpha_q} = \Omega_{(q+1)/p} = \Omega_0$. We conclude that the Ω_α are all equal. \square

If the eigenvalues $\lambda_1(\alpha), \dots, \lambda_n(\alpha)$ are distinct for each α and $\dim F(\Gamma(\alpha))$ is constant, then $|\Omega_\alpha|$ is also constant since each subscript j contributes $\dim(F(\Gamma(\alpha)) \cap R_{\lambda_j(\alpha)}(\Gamma(\alpha)))$ elements to Ω_α .

Lemma 5. *Let $A \in \mathcal{L}(H)$, and suppose $\sigma(A) = \{\lambda\}$. Then the only continuously stable A -invariant subspaces are trivial.*

Proof. If $\dim H = 1$, there is nothing to prove, so we assume $\dim H > 1$. Suppose M is a continuously stable A -invariant subspace. Then M is obviously stable. By hypothesis $\sigma(A) = \{\lambda\}$, so $R_\lambda(A) = H$ and, by Proposition 4, M is trivial if $\dim \ker(A - \lambda) \neq 1$.

It remains to consider the case where $\dim \ker(A - \lambda) = 1$. Here, the Jordan decomposition of A consists of exactly one Jordan block, so there is a basis $\{v_1, \dots, v_n\}$ for H relative to which A has the matrix

$$\begin{bmatrix} \lambda & 1 & \cdots & 0 & 0 \\ 0 & \lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda & 1 \\ 0 & 0 & \cdots & 0 & \lambda \end{bmatrix}.$$

Let $F: V \rightarrow \mathcal{C}(H)$ be a continuous invariance function for (A, M) . Define the operator $Z \in \mathcal{L}(H)$ by

$$Zv_j = \begin{cases} v_n & \text{if } j = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mu \in \mathbb{C}$; then $A + \mu Z$ has the matrix

$$\begin{bmatrix} \lambda & 1 & \cdots & 0 & 0 \\ 0 & \lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda & 1 \\ \mu & 0 & \cdots & 0 & \lambda \end{bmatrix}$$

relative to the basis $\{v_1, \dots, v_n\}$.

Put $\mu = \nu^n$; then straightforward calculation shows that $A + \mu Z$ has n distinct eigenvalues, namely, $\lambda + \omega^j \nu$, $j = 1, \dots, n$, with corresponding eigenvectors having coordinates $(1, \omega^j \nu, \dots, (\omega^j \nu)^{n-1})$. Here, $\omega = e^{i2\pi/n}$ is a primitive n th root of unity. Without loss of generality we may assume that $\lambda = 0$ (since $\text{Lat}(B - \lambda) = \text{Lat } B$ for all $B \in \mathcal{L}(H)$), so the eigenvalues of $A + \mu Z$ are $\omega^j \nu$, $j = 1, \dots, n$.

We can choose $r > 0$ small enough so that $A + \mu Z \in V$ and $\dim F(A + \mu Z) = \dim M$, if $|\mu| = r^n$. Define $\Gamma: [0, 1] \rightarrow V$ by $\Gamma(\alpha) = A + r^n e^{i2\pi\alpha} Z$, and let $\lambda_1, \dots, \lambda_n$ be the complex-valued functions defined on $[0, 1]$ by $\lambda_j(\alpha) = r\omega^j e^{i2\pi\alpha/n}$. Then $\lambda_1(\alpha), \dots, \lambda_n(\alpha)$ are the eigenvalues of $\Gamma(\alpha)$. Let Ω_α be defined as in Lemma 4. Then the Ω_α are equal ($|\Omega_\alpha| \equiv \dim M$ since the eigenvalues of $\Gamma(\alpha)$ have multiplicity one each).

It is obvious that $\Gamma(0) = \Gamma(1)$. If $j \in \Omega_0 = \Omega_1$, then $r\omega^j = \lambda_j(0) \in \sigma(\Gamma(0)|_{F(\Gamma(0))})$ and $r\omega^{j+1} = \lambda_j(1) \in \sigma(\Gamma(1)|_{F(\Gamma(1))}) = \sigma(\Gamma(0)|_{F(\Gamma(0))})$; whence, $(j + 1 \bmod n) \in \Omega_0$. Consequently, either $\Omega_0 = \emptyset$ or $\Omega_0 = \{1, \dots, n\}$; that is, either $\dim M = 0$ or $\dim M = n$. Hence M is trivial. \square

We are now ready to prove Theorem 1.

Proof of Theorem 1. Suppose to the contrary that M is a continuously stable but not smoothly stable A -invariant subspace. Then $\sigma(A_{11}) \cap \sigma(A_{22}) \neq \emptyset$ by Theorem 2. Let $\lambda \in \sigma(A_{11}) \cap \sigma(A_{22})$. Then $M \cap R_\lambda(A)$ is a nontrivial subspace of $R_\lambda(A)$. On the other hand, $M \cap R_\lambda(A)$ is a continuously stable $A|_{R_\lambda(A)}$ -invariant subspace by Lemma 3; whence, $M \cap R_\lambda(A) = \{0\}$ or $M \cap R_\lambda(A) = R_\lambda(A)$ by Lemma 5. This is a contradiction. \square

4. A PROOF USING NONLINEAR METHODS

We now present an alternative proof of Theorem 1 which involves the calculation of the topological degree of various maps.

Alternative proof of Theorem 1. Suppose M is continuously stable, and let $F: V_0 \rightarrow \mathcal{L}(M, M^\perp)$ be a continuous invariance function for (A, M) . Suppose A has the matrix representation given in Lemma 1. Note that $A_{21} = 0$ because $M \in \text{Lat } A$. Given $S \in \mathcal{L}(M, M^\perp)$, let $B(S)$ be the operator on H with matrix representation

$$\begin{bmatrix} A_{11} & A_{12} \\ S & A_{22} \end{bmatrix}$$

relative to the decomposition $H = M \oplus M^\perp$. By Lemma 1, $(1+T)M \in \text{Lat } B(S)$ if and only if

$$TA_{12}T + TA_{11} - A_{22}T - S = 0,$$

or equivalently

$$(1) \quad TA_{12}T + TA_{11} - A_{22}T = S.$$

By Proposition 5, we can choose $r > 0$ such that $(1+T)M \notin \text{Lat } A$ for all $T \in \mathcal{L}(M, M^\perp)$ with $0 < \|T\| < r$. In terms of Lemma 1, this means that

$$TA_{12}T + TA_{11} - A_{22}T \neq 0.$$

Put $V = \{T \in V_0: \|T\| < r\}$, and let $W = \{S \in \mathcal{L}(M, M^\perp): B(S) \in V\}$. Define $g: W \setminus \{0\} \rightarrow \mathcal{L}(M, M^\perp) \setminus \{0\}$ by $g(S) = F(B(S))$. If $S \neq 0$, then clearly $M \notin \text{Lat } B(S)$, so $g(S) \neq 0$ and g is well defined.

Let $f: V \setminus \{0\} \rightarrow \mathcal{L}(M, M^\perp) \setminus \{0\}$ be defined by $f(T) = TA_{12}T + TA_{11} - A_{22}T$. Our choice of r ensures that f is well defined. Moreover, $f(g(S)) = S$ by equation (1) since $(1 + g(S))M \in \text{Lat } B(S)$.

Since f and g are continuous functions mapping punctured neighbourhoods of 0 in $\mathcal{L}(M, M^\perp)$ into punctured neighbourhoods of 0 in $\mathcal{L}(M, M^\perp)$, they have well-defined topological degrees $\deg f$ and $\deg g$. We have seen that $f \circ g$ is the identity map, so $\deg f \times \deg g = \deg f \circ g = 1$.

Now assume to the contrary that M is not smoothly stable. The linear terms of $f(T)$ are given by $\Xi(T)$, and by Theorem 2, Ξ is not surjective and hence is singular. If we choose a basis for $\mathcal{L}(M, M^\perp)$, then f reduces to p polynomials π_1, \dots, π_p in p complex variables z_1, \dots, z_p , where $p = mn$, $n = \dim H$, and $m = \dim M$. We can choose a linear change of coordinates so that the coefficients of z_1 in π_1, \dots, π_p are identically zero.

Invoking two results in [3, p. 670] yields $\deg f = \dim \mathcal{O}/I$, where \mathcal{O} is the local ring at the origin and I is the ideal generated by the π_j . Define the homomorphism $\phi: \mathcal{O}/I \rightarrow \mathbb{C}$ by $\phi(\theta + I) = \theta(0)$. Then $\ker \phi = M/I$ where M is the maximal ideal of \mathcal{O} . However, it is clear that $z_1 + I \in M$, so $M \neq I$ and $\dim \ker \phi \geq 1$. Consequently

$$\deg f = \dim \mathcal{O}/I \geq 2,$$

and then $\deg f \times \deg g = 1$ cannot hold—a contradiction. \square

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