PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 120, Number 1, January 1994

CLASSIFYING PL 5-MANIFOLDS UP TO REGULAR GENUS SEVEN

MARIA RITA CASALI AND CARLO GAGLIARDI

(Communicated by James West)

ABSTRACT. In the present paper, we show that the only closed orientable 5-manifolds of regular genus less or equal than seven are the 5-sphere $\mathbb{S}^5$ and the connected sums of m copies of $\mathbb{S}^1 \times \mathbb{S}^4$, with $m \leq 7$. As a consequence, the genus of $\mathbb{S}^3 \times \mathbb{S}^2$ is proved to be eight. This suggests a possible approach to the (3-dimensional) Poincaré Conjecture, via the well-known classification of simply connected 5-manifolds, obtained by Smale and Barden.

1. STATEMENTS

The possibility of representing a large class of polyhedra, including all PL-manifolds of dimension n, by means of $(n+1)$-coloured graphs is well known (see, as general references, [FGG, BM, V]).

Within this representation theory, a nonnegative combinatorial invariant $\mathscr{G}(M^n)$, called the regular genus of M^n, has been introduced for each n-manifold M^n. It extends to dimension n the classical notions of genus of a surface and of Heegaard genus of a 3-manifold [G2]; it further characterizes the n-sphere $\mathbb{S}^n$ among all closed n-manifolds, as follows [FG]:

$$\mathscr{G}(M^n) = 0 \quad \text{iff} \quad M^n \cong \mathbb{S}^n$$

(here "$\cong$" means PL-homeomorphism).

In the present paper, we obtain the classification of all closed connected orientable 5-dimensional PL-manifolds with regular genus less than or equal to 7:

Main Theorem. *Let M^5 be a closed connected orientable PL 5-manifold. Then*

$$1 \leq \mathscr{G}(M^5) = m \leq 7 \quad \textit{iff} \quad M^5 \cong \mathop{\#}_m(\mathbb{S}^1 \times \mathbb{S}^4)$$

(*where* $\#_m(\mathbb{S}^1 \times \mathbb{S}^4)$ *denotes the connected sum of* m *copies of* $\mathbb{S}^1 \times \mathbb{S}^4$).

As a direct consequence, we have

Received by the editors May 10, 1991; the contents of this paper have been presented by the first author to XIV Congresso U.M.I., Catania, Italy, September 19–25, 1991.

1991 *Mathematics Subject Classification.* Primary 57N15, 57Q15; Secondary 05C10.

Key words and phrases. 5-Manifolds, regular genus, classification, products of spheres and manifolds, handlebodies, Poincaré Conjecture, edge-coloured graphs, 2-cell embeddings.

This work was performed under the auspices of the G.N.S.A.G.A.-C.N.R., and within the Project "Geometria Reale e Complessa", supported by M.U.R.S.T. of Italy.

Corollary I. (a) $\mathcal{G}(\mathbb{S}^3 \times \mathbb{S}^2) = 8$.
(b) *For every closed orientable PL 3-manifold* M^3, $\mathcal{G}(M^3 \times \mathbb{S}^2) \geq 8$.

The proof of the above results will be presented in §3.

Remark A. By [B], if Σ^3 is a homotopy 3-sphere, then $\Sigma^3 \times \mathbb{S}^2 \cong \mathbb{S}^3 \times \mathbb{S}^2$; therefore, by Corollary I(a), $\mathcal{G}(\Sigma^3 \times \mathbb{S}^2) = 8$. We believe that the above relation characterizes homotopy 3-spheres among orientable 3-manifolds (i.e., *that* $\mathcal{G}(M^3 \times \mathbb{S}^2) = 8$ *iff* M^3 *is a homotopy* 3*-sphere*). On the other hand, since Corollary I gives a positive answer to parts (a) and (b′) of Conjecture I of [GG] (see also [G3]), we can state the following stronger

Conjecture I′. *Let* M^3 *be any closed orientable* 3*-manifold. Then*

$$\mathcal{G}(M^3 \times \mathbb{S}^2) = 8 \quad \textit{iff} \quad M^3 \cong \mathbb{S}^3.$$

Conjecture I′ seems, of course, of hard solution: in fact, we recall that it actually implies the (3-dimensional) Poincaré Conjecture, by the classification of simply connected 5-manifolds, obtained in [S, B]. Let us propose the following general problem about the genus of product manifolds, presented as a sequence of conjectures, for the sake of conciseness.

Conjecture II_n. *For every closed connected orientable* 3*-manifold* M^3, $\mathcal{G}(M^3 \times \mathbb{S}^n) \geq \mathcal{G}(M^3 \times \mathbb{D}^n)$ (*where* $\mathbb{D}^n$ *denotes the* n*-disk*).

Remark B. Conjectures II_1 and II_2 together imply the Poincaré Conjecture for all closed 3-manifolds M^3 of Heegaard genus $\mathcal{H}(M^3) \geq 5$.

In fact, by [CP], $\mathcal{G}(M^h) \geq \mathcal{G}(\partial M^h)$, where M^h is any h-manifold with boundary ∂M^h and $\mathcal{G}(\partial M^h)$ denotes the sum of the genera of its components. Hence, assuming Conjecture II_2, $\mathcal{G}(M^3 \times \mathbb{S}^2) \geq \mathcal{G}(M^3 \times \mathbb{D}^2) \geq \mathcal{G}(M^3 \times \mathbb{S}^1)$; furthermore, assuming Conjecture II_1, $\mathcal{G}(M^3 \times \mathbb{S}^1) \geq \mathcal{G}(M^3 \times \mathbb{D}^1) \geq 2\mathcal{G}(M^3) = 2\mathcal{H}(M^3)$.

This proves that, if $\mathcal{H}(M^3) \geq 5$, then $\mathcal{G}(M^3 \times \mathbb{S}^2) \geq 2\mathcal{H}(M^3) \geq 10$, and therefore M^3 is not a homotopy sphere, since for each homotopy 3-sphere Σ^3 we have $\mathcal{G}(\Sigma^3 \times \mathbb{S}^2) = \mathcal{G}(\mathbb{S}^3 \times \mathbb{S}^2) = 8$, by Corollary I.

2. Preliminaries and notations

In this paper, we shall work with manifolds (closed, connected, and orientable, unless otherwise stated) and maps in the PL-category, for which we refer to [Gl] or [RS]; we will often identify a ball-complex K with its associated polyhedron and every homeomorphic space. For graph theory, see [W].

An $(n+1)$*-coloured graph* is a pair (Γ, γ), where $\Gamma = (V(\Gamma), E(\Gamma))$ is a multigraph (i.e., multiple edges are allowed, but loops are forbidden), regular of degree $n+1$, and $\gamma: E(\Gamma) \to \Delta_n = \{i \in \mathbb{Z} | 0 \leq i \leq n\}$ is a proper edge-colouring on Γ (i.e., $\gamma(\mathbf{e}) \neq \gamma(\mathbf{f})$ for any two adjacent edges $\mathbf{e}$, $\mathbf{f} \in E(\Gamma)$. For the sake of conciseness, we shall often denote the $(n+1)$-coloured graph (Γ, γ) simply by the symbol Γ of its underlying multigraph.

For each $\mathcal{B} \subseteq \Delta_n$, set $\Gamma_{\mathcal{B}} = (V(\Gamma), \gamma^{-1}(\mathcal{B}))$; each connected component of $\Gamma_{\mathcal{B}}$ is called a $\mathcal{B}$*-residue*, or an m*-residue* if the cardinality $\#\mathcal{B}$ of $\mathcal{B}$ is m. For each $i \in \Delta_n$, set $\hat{i} = \Delta_n - \{i\}$. The symbol $\mathbf{g}(\Gamma)$ (resp. $\mathbf{g}_{\mathcal{B}} = \mathbf{g}_{\mathcal{B}}(\Gamma)$) will denote the number of components of Γ (resp. of $\mathcal{B}$-residues of Γ); if $\mathcal{B} = \{i, j\}$ (resp. $\mathcal{B} = \{i, j, k\}$), we shall write $\mathbf{g}_{ij}$ (resp. $\mathbf{g}_{ijk}$) instead of $\mathbf{g}_{\mathcal{B}}$ and $\mathbf{g}_{\hat{i}\hat{j}}$ (resp. $\mathbf{g}_{\hat{i}\hat{j}\hat{k}}$) instead of $\mathbf{g}_{\Delta_n - B}$.

An n-dimensional labelled pseudocomplex $K(\Gamma)$ (see [HW, BM]) can be associated to every $(n+1)$-coloured graph (Γ, γ); the construction, described in [FGG], works as follows:

(i) for each $\mathbf{v} \in V(\Gamma)$, take an n-simplex $\sigma(v)$, with its vertices labelled $0, 1, \dots, n$;
(ii) for each $\mathbf{e} \in E(\Gamma)$ with end points $\mathbf{v}, \mathbf{w} \in V(\Gamma)$ and colour $\gamma(\mathbf{e}) = i$, identify the $(n-1)$-faces of $\sigma(v)$ and $\sigma(w)$ opposite of the vertex labelled i.

We say that (Γ, γ) represents $K(\Gamma)$ and every homeomorphic polyhedron. It is easy to check that, for each $\mathscr{B} \in \Delta_n$, with $\#\mathscr{B} = h \le n$, there is a bijection between the set of $\mathscr{B}$-residues of Γ and the set of $(n-h)$-simplices of $K(\Gamma)$, whose vertices are labelled $\Delta_n - \mathscr{B}$. Moreover, it is known that:

- $K(\Gamma)$ is orientable iff Γ is a bipartite graph;
- $K(\Gamma)$ is a manifold iff, for every n-residue Θ of Γ, $K(\Theta)$ is homeomorphic to the $(n-1)$-sphere $\mathbb{S}^{n-1}$.

A *crystallization* of a closed connected n-manifold M^n is any $(n+1)$-coloured graph (Γ, γ) representing M^n, such that $\Gamma_{\hat{c}}$ is connected for each colour $c \in \Delta_n$. The basic results of [P1, P2] ensure the existence of crystallizations for every closed n-manifold. An alternative proof is contained in [LM].

In [G2] the existence is proved, for each connected bipartite $(n+1)$-coloured graph (Γ, γ) of order $\# V(\Gamma) = \mathbf{p}$ and for each cyclic permutation $\varepsilon = (\varepsilon_0, \varepsilon_1, \dots, \varepsilon_n)$ of Δ_n, of a particular 2-cell embedding (see [W]) $\imath: \Gamma \to F_\varepsilon$, where F_ε denotes the orientable closed surface of genus $\rho(F_\varepsilon) = \rho_\varepsilon(\Gamma) = 1 - \chi_\varepsilon(\Gamma)/2$, where

$$\chi_\varepsilon(\Gamma) = \sum_{c \in \mathbb{Z}_{n+1}} \mathbf{g}_{\varepsilon_c \varepsilon_{c+1}} + (1-n)\mathbf{p}/2.$$

The nonnegative integers $\rho(\Gamma) = \min_\varepsilon\{\rho_\varepsilon(\Gamma)\}$ and $\mathscr{G}(M^n) = \min\{\rho(\Gamma) | \Gamma$ is a crystallization of $M^n\}$ are respectively called the *regular genus* of the graph Γ and of the manifold M^n.

From now on, (Γ, γ) is assumed to be a bipartite $(n+1)$-coloured graph representing a (possibly disconnected) closed orientable n-manifold. If Γ (resp. $\Gamma_{\mathscr{B}}$, with $\mathscr{B} \subseteq \Delta_n$) has $\mathbf{g} = \mathbf{g}(\Gamma) \ge 1$ (resp. $\mathbf{g}_{\mathscr{B}} = \mathbf{g}_{\mathscr{B}}(\Gamma) \ge 1$) connected components $\Gamma^{(1)}, \Gamma^{(2)}, \dots, \Gamma^{(\mathbf{g})}$ (resp. $\Xi^{(1)}, \Xi^{(2)}, \dots, \Xi^{(\mathbf{g}_{\mathscr{B}})}$), we define $\rho = \rho_\varepsilon(\Gamma) = \sum_r \rho_\varepsilon(\Gamma^{(r)})$ (resp. $\rho_{\mathscr{B}} = \rho_{\varepsilon'}(\Gamma_{\mathscr{B}}) = \sum_r \rho_{\varepsilon'}(\Xi^{(r)})$), where $\rho_\varepsilon(\Gamma^{(r)})$ (resp. $\rho_{\varepsilon'}(\Xi^{(r)})$) denotes the genus of $\Gamma^{(r)}$ (resp. of $\Xi^{(r)}$), with respect to the cyclic permutation $\varepsilon = (\varepsilon_0, \varepsilon_1, \dots, \varepsilon_n)$ of Δ_n (resp. to the cyclic permutation ε' induced by ε on the subset $\mathscr{B}$ of Δ_n). For the sake of notational simplicity, we shall always write "i" instead of "ε_i", for all $i \in \Delta_n$. Hence, the following formulae, written for the fundamental cyclic permutation $(0, 1, \dots, n)$, actually hold for every cyclic permutation $(\varepsilon_0, \varepsilon_1, \dots, \varepsilon_n)$ of Δ_n.

Now, the relation $\chi_\varepsilon(\Gamma) = 2\mathbf{g}(\Gamma) - 2\rho_\varepsilon(\Gamma) = 2\mathbf{g} - 2\rho$ gives

$$(1) \qquad \sum_{c \in \mathbb{Z}_{n+1}} \mathbf{g}_{c,c+1} + (1-n)\mathbf{p}/2 = 2\mathbf{g} - 2\rho.$$

For each subgraph $\Gamma_{\hat{i}}$, with $i \in \Delta_n$, the following analogous formula holds:

$$(1_i) \qquad \mathbf{g}_{i-1,i+1} + \sum_{c \in \mathbb{Z}_{n+1}^{(i)}} \mathbf{g}_{c,c+1} + (2-n)\mathbf{p}/2 = 2\mathbf{g}_{\hat{i}} - 2\rho_{\hat{i}},$$

where $\mathbb{Z}_{n+1}^{(i)}$ denotes the set $\Delta_n - \{i-1, i\}$ and the sum is mod $n+1$.

By substituting (1_i) into (1) and by making use of the relation

$$(2) \qquad 2\mathbf{g}_{ijk} = \mathbf{g}_{ij} + \mathbf{g}_{jk} + \mathbf{g}_{ki} - \mathbf{p}/2$$

(which is a direct consequence of $\chi(K(\Theta)) = \chi(\mathbb{S}^2) = 2$, for each $\{i, j, k\}$-residue Θ), we obtain (for details, see [GG, Lemmas 6.1 and 6.2])

$$(3_i) \qquad \mathbf{g}_{i-1,i+1} = \mathbf{g}_{i-1,i,i+1} + \rho - \rho_{\hat{i}} + \mathbf{g}_{\hat{i}} - \mathbf{g}.$$

The following formula is simply obtained by taking the difference between relation (3_i), applied to the graph (Γ, γ), and the same relation, applied to the partial graph $\Gamma_{\hat{j}}$, with $j \neq i \pm 1$:

$$(4) \qquad \mathbf{g}_{\hat{i}\hat{j}} = \mathbf{g}_{\hat{i}} + \mathbf{g}_{\hat{j}} - \mathbf{g} + \rho - \rho_{\hat{i}} - \rho_{\hat{j}} + \rho_{\hat{i}\hat{j}} \qquad (j \neq i \pm 1)$$

where we have set $\rho_{\hat{i}\hat{j}} = \rho_{\Delta_n - \{i,j\}}$.

Remark C. Since $\mathbf{g} - \mathbf{g}_{\hat{i}} \geq \mathbf{g}_{\hat{j}} - \mathbf{g}_{\hat{i}\hat{j}}$ (as it is easy to check), formula (4) implies that, for every i, j nonconsecutive in ε, $\rho - \rho_{\hat{i}} \geq \rho_{\hat{j}} - \rho_{\hat{i}\hat{j}}$.

Let us now restrict our attention to the case of (Γ, γ) being a crystallization of a closed connected orientable 5-manifold M^5. So, $\mathbf{g} = \mathbf{g}_{\hat{c}} = 1$ for every $c \in \Delta_5$; moreover, if we set $\rho_{\hat{i}\hat{j}\hat{k}} = \rho_{\Delta_n - \{i,j,k\}}$, then $\rho_{\hat{i}\hat{j}\hat{k}} = 0$ for every i, j, $k \in \Delta_5$, since each 3-residue of Γ represents a 2-sphere. Hence, relation (4), applied to the graph Γ and to the subgroup $\Gamma_{\hat{k}}$ respectively, gives

$$(5) \qquad \mathbf{g}_{\hat{i}\hat{j}} = 1 + \rho - \rho_{\hat{i}} - \rho_{\hat{j}} + \rho_{\hat{i}\hat{j}}, \qquad \text{for every } i,\ j \text{ nonconsecutive in } \varepsilon$$

and

$$(6) \qquad \mathbf{g}_{\hat{i}\hat{j}\hat{k}} = \mathbf{g}_{\hat{i}\hat{k}} + \mathbf{g}_{\hat{j}\hat{k}} - 1 + \rho_{\hat{k}} - \rho_{\hat{i}\hat{k}} - \rho_{\hat{j}\hat{k}}, \qquad \text{for every } i, j \text{ nonconsecutive in the cyclic permutation } \varepsilon_{\hat{k}} \text{ induced by } \varepsilon \text{ on } \Delta_5 - \{k\}.$$

Remark D. Since, as in Remark C, $\mathbf{g}_{\hat{k}} - \mathbf{g}_{\hat{i}\hat{k}} = 1 - \mathbf{g}_{\hat{i}\hat{k}} \geq \mathbf{g}_{\hat{j}\hat{k}} - \mathbf{g}_{\hat{i}\hat{j}\hat{k}}$, formula (6) actually implies that, for every i, j nonconsecutive in $\varepsilon_{\hat{k}}$, $\rho_{\hat{k}} - \rho_{\hat{i}\hat{k}} - \rho_{\hat{j}\hat{k}} \geq 0$.

By adding formulae (3_i) for $i \in \Delta_5$, i even (resp. i odd), and by making use of relation (2), the following formula (7) (resp. (8)) is obtained:

$$(7) \qquad \mathbf{g}_{\hat{0}\hat{2}\hat{4}} = 1 + 2\rho - (\rho_{\hat{0}} + \rho_{\hat{2}} + \rho_{\hat{4}}),$$

$$(8) \qquad \mathbf{g}_{\hat{1}\hat{3}\hat{5}} = 1 + 2\rho - (\rho_{\hat{1}} + \rho_{\hat{3}} + \rho_{\hat{5}}).$$

Moreover, since the subgraph $\Gamma_{\hat{i}}$ represents a four-dimensional sphere, the direct computation of $\chi(K(\Gamma_{\hat{i}}))$ by means of formulae (1), (1_i), and (2) gives

$$(9) \qquad \rho_{\hat{i}} = \left(\sum_{j \neq i} \rho_{\hat{i}\hat{j}} \right) /2$$

(see [GG, Proposition 6.3] for a detailed proof, in a more general setting).

3. Classifying 5-manifolds of genus ≤ 7

In the present section, (Γ, γ) is assumed to be a crystallization of a closed connected orientable 5-manifold M^5. If $K = K(\Gamma)$ denotes the triangulation of

M^5 associated to Γ and $\{v_c | c \in \Delta_5\}$ is the vertex-set of K, then we may suppose that v_c corresponds to $\Gamma_{\hat{c}}$, for each colour $c \in \Delta_5$. For each subset $\{i, j\}$ (resp. $\{i, j, k\}$) of distinct elements of Δ_5, let $K(i, j)$ (resp. $K(i, j, k)$) be the subcomplex of K generated by the vertices v_i and v_j (resp. v_i, v_j, and v_k), and let $K(\hat{i}, \hat{j})$ (resp. $K(\hat{i}, \hat{j}, \hat{k})$) be the subcomplex of K generated by the vertex-set $\{v_c | c \in \Delta_5 - \{i, j\}\}$ (resp. $\{v_d | d \in \Delta_5 - \{i, j, k\}\}$). Let further $N(i, j)$, $N(i, j, k)$, $N(\hat{i}, \hat{j})$, and $N(\hat{i}, \hat{j}, \hat{k})$ be the regular neighbourhood in K of $K(i, j)$, $K(i, j, k)$, $K(\hat{i}, \hat{j})$, and $K(\hat{i}, \hat{j}, \hat{k})$ respectively.

The following general results are needed in the proofs of the Main Theorem and of Corollary 1.

Lemma 1a [C]. *Let H be a two-dimensional pseudocomplex which contains exactly three vertices $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$. For each pair $\mathbf{v}$, $\mathbf{w}$ of distinct vertices of H, let $H(\mathbf{v}, \mathbf{w})$ be the subcomplex of H, generated by the vertex-set $\{\mathbf{v}, \mathbf{w}\}$. Let further $[\mathbf{v}, \mathbf{w}]$ (resp. $[\mathbf{a}, \mathbf{b}, \mathbf{c}]$) denote the number of edges of $H(\mathbf{v}, \mathbf{w})$ (resp. the number of triangles of H). If H satisfies the following properties:*

(i) $1 \leq [\mathbf{a}, \mathbf{c}] \leq [\mathbf{a}, \mathbf{b}] - 1$;
(ii) $[\mathbf{a}, \mathbf{b}, \mathbf{c}] = [\mathbf{a}, \mathbf{b}] + [\mathbf{a}, \mathbf{c}] - 1$;
(iii) *the inclusion $j: H(\mathbf{b}, \mathbf{c}) \to H$ induces an epimorphism $j_*: \pi_1(H(\mathbf{b}, \mathbf{c})) \to \pi_1(H)$,*

then H collapses to $H(\mathbf{b}, \mathbf{c})$.

Proof. First of all, we note that property (ii) implies the following property:
(iv) all edges of $H(\mathbf{a}, \mathbf{b})$ are faces of at least one triangle of H.

In fact, if an edge $\mathbf{e}$ of $H(\mathbf{a}, \mathbf{b})$ is not a face of any triangle of H, then any edge-loop in H containing $\mathbf{e}$ could not be homotopic to an edge-loop of $H(\mathbf{b}, \mathbf{c})$; this obviously contradicts (iii), and hence (iv) holds, too. As a consequence of properties (i), (ii), (iv), only $r \leq [\mathbf{a}, \mathbf{c}] - 1 \leq [\mathbf{a}, \mathbf{b}] - 2$ edges $(\mathbf{a}, \mathbf{b})$ are faces of more than one triangle of H; then, some triangles of H collapse from their edges "$(\mathbf{a}, \mathbf{b})$" on their edges "$(\mathbf{a}, \mathbf{c})$" and "$(\mathbf{b}, \mathbf{c})$". Hence, H collapses to a subcomplex H' such that $H'(\mathbf{b}, \mathbf{c}) = H(\mathbf{b}, \mathbf{c})$. If $r = 0$, it is easy to see that $[\mathbf{a}, \mathbf{c}]$ must be equal to 1; that is, H' is a graph collapsing to $H'(\mathbf{b}, \mathbf{c})$. On the other hand, if $r \geq 1$, then H' satisfies again the properties (i), (ii), and (iii), with the vertices $\mathbf{b}$ and $\mathbf{c}$ exchanged of meaning. Thus, the process may be iterated, until $r = 0$, leaving $H(\mathbf{b}, \mathbf{c})$ fixed. □

Lemma 1b. *Let H be a two-dimensional pseudocomplex which contains exactly three vertices $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$. If H satisfies the following properties:*

(i) $[\mathbf{a}, \mathbf{b}] = [\mathbf{a}, \mathbf{c}] = 3$;
(ii) $[\mathbf{a}, \mathbf{b}, \mathbf{c}] = [\mathbf{b}, \mathbf{c}] + 3$;
(iii) *for $\mathbf{z} \in \{\mathbf{b}, \mathbf{c}\}$, the inclusion $j_{\mathbf{z}}: H(\mathbf{a}, \mathbf{z}) \to H$ induces an epimorphism $j_{\mathbf{z}*}: \pi_1(H(\mathbf{a}, \mathbf{z})) \to \pi_1(H)$;*
(iv) *$H_1(H; \mathbb{Z})$ is free and $H_2(H; \mathbb{Z}) = 0$,*

then H collapses to a graph.

Proof. First of all we note, as in the proof of Lemma 1a, that property (iii) implies the following property, for every pair $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ of vertices of H:

($\mathrm{v}_{\boldsymbol{\alpha}\boldsymbol{\beta}}$) all edges of $H(\boldsymbol{\alpha}, \boldsymbol{\beta})$ are faces of at least one triangle of H.

Let now $\mathbf{e}_1, \ldots, \mathbf{e}_h$ $(1 \leq h \leq 3)$ be the h edges of $H(\mathbf{b}, \mathbf{c})$, which are faces of at least two triangles of H, and let $T_1, \ldots, T_{3+h}$ be the $3+h$ triangles which have an edge $\mathbf{e}_r$ as face, for $r = 1, \ldots, h$. By property $(\mathrm{v}_{\mathbf{b},\mathbf{c}})$, H collapses to the subcomplex $H' = H(\mathbf{a}, \mathbf{b}) \cup H(\mathbf{a}, \mathbf{c}) \cup \{\mathbf{e}_1, \ldots, \mathbf{e}_h\} \cup \{T_1, \ldots, T_{3+h}\}$, with $1 \leq h \leq 3$, which of course satisfies all conditions (i), (ii), (iii), (iv) and, therefore, also $(\mathrm{v}_{\boldsymbol{\alpha\beta}})$, for every $\boldsymbol{\alpha}$, $\boldsymbol{\beta} \in \{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$. Moreover, property (iv) avoids the existence of any closed surface as a subcomplex of H'; hence, the edges of $H'(\mathbf{a}, \mathbf{b})$ and $H'(\mathbf{a}, \mathbf{c})$ cannot be faces of exactly two triangles each. Since $[\mathbf{a}, \mathbf{b}] = [\mathbf{a}, \mathbf{c}] = 3 \geq (3+h)/2$, some triangles of H' collapse from their edges "$(\mathbf{a}, \mathbf{x})$", where either $\mathbf{x} = \mathbf{b}$ or $\mathbf{x} = \mathbf{c}$; w.l.o.g., $\mathbf{x} = \mathbf{b}$ may be assumed.

Note that every sequence of elementary collapses of triangles from their edges "$(\mathbf{a}, \mathbf{b})$" or "$(\mathbf{b}, \mathbf{c})$" produces a new complex H'', satisfying property (iv) (since H' collapses to H'') and property (iii), only for $\mathbf{z} = \mathbf{c}$ (since $H''(\mathbf{a}, \mathbf{c}) = H'(\mathbf{a}, \mathbf{c})$); hence H'' also satisfies properties $(\mathrm{v}_{\mathbf{ab}})$ and $(\mathrm{v}_{\mathbf{bc}})$.

By looking in details to the three possible cases for the number h of edges of $H'(\mathbf{b}, \mathbf{c})$, we can easily obtain such a complex H'', with s edges "$(\mathbf{a}, \mathbf{b})$", s edges "$(\mathbf{b}, \mathbf{c})$", and $2s$ triangles, for $s = 1, 2$. By property (iv), H'' actually collapses to a graph. □

By 5-dimensional handlebody of genus $m \geq 0$ we mean the orientable 5-manifold Y_m, which admits a decomposition with exactly the 0-handle and m 1-handles. Note that $Y_0 \cong \mathbb{D}^5$ (the 5-disk) and that, for $m \geq 1$, $\partial Y_m \cong \#_m(\mathbb{S}^1 \times \mathbb{S}^3)$.

Moreover, a 5-manifold with boundary M^5 is a handlebody (of genus m) iff it collapses to a graph Ω (with $\pi_1(\Omega) \cong \underset{m}{*}\mathbb{Z}$).

Lemma 2. *Let (Γ, γ) be a crystallization of a closed connected orientable 5-manifold M^5, and let $\varepsilon = (\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_5)$ be a fixed permutation of Δ_5. If there exists $\{r, s, t\} \subset \Delta_5$, with s, t nonconsecutive in $\varepsilon_{\hat{r}}$, such that*

$$(*) \qquad \rho_{\hat{r}} = \rho_{\hat{r}\hat{s}} + \rho_{\hat{r}\hat{t}},$$

then $N(r, s, t)$ is a 5-dimensional handlebody Y_m of genus m, with $\mathrm{rk}(\pi_1(M^5)) \leq m \leq \rho$.

Proof. Set $\boldsymbol{\alpha} = \mathbf{g}_{\hat{r}\hat{s}\hat{t}} - \mathbf{g}_{\hat{r}\hat{s}} \geq 0$; this means that there are α triangles in $K(r, s, t)$ more than edges in $K(r, s)$, i.e., $h(1 \leq h \leq \alpha)$ edges $\mathbf{e}_1, \ldots, \mathbf{e}_h$ of $K(r, s)$ are faces of at least two triangles of $K(r, s, t)$. Let $T_1, \ldots, T_{\alpha+h}$ be the $\alpha + h$ triangles of $K(r, s, t)$, which have an edge $\mathbf{e}_i \in K(r, s)$ as face; it is easy to check that $K(r, s, t)$ collapses to the subcomplex $\overline{K} = K(r, t) \cup K(s, t) \cup \{\mathbf{e}_1, \ldots, \mathbf{e}_h\} \cup \{T_1, \ldots, T_{\alpha+h}\}$. By using $(*)$ into formula (6), $\mathbf{g}_{\hat{r}\hat{s}\hat{t}} = \mathbf{g}_{\hat{r}\hat{s}} + \mathbf{g}_{\hat{r}\hat{t}} - 1$ is obtained.

Thus, the two-dimensional pseudocomplex $\overline{K}$ has: $[\mathbf{v}_r, \mathbf{v}_t] = 1 + \alpha$, $[\mathbf{v}_r, \mathbf{v}_s] = h \leq \alpha$, $[\mathbf{v}_r, \mathbf{v}_s, \mathbf{v}_t] = \alpha + h$ (for the notation, see Lemma 1a). Moreover, since $K(\Gamma)$ is a pseudocomplex triangulating a 5-manifold and having exactly six vertices, the inclusion $j\colon K(s, t) \to K(r, s, t)$ induces an epimorphism $j_*\colon \pi_1(K(r, s)) \to \pi_1(K(r, s, t))$ [G1, Proposition 3]. The same property holds for $\overline{K}$ and $\overline{K}(r, s)$, too, since $\overline{K}$ is obtained from $K(r, s, t)$ by elementary collapses fixing $K(r, s)$. Hence, $\overline{K}$ satisfies all the hypothesis of Lemma 1a and collapses to $K(s, t)$. This obviously ensures that $N(r, s, t)$ is a 5-dimensional handlebody, of genus $m = \mathbf{g}_{\hat{s}\hat{t}} - 1 \geq 0$. Moreover, since s and t are surely nonconsecutive in ε, [CP, Corollary 4.4] implies that $m = \mathbf{g}_{\hat{s}\hat{t}} - 1 \leq \rho - \max\{\rho_{\hat{s}}, \rho_{\hat{t}}\} \leq \rho$.

The proof is now completed by recalling that the $(\Delta_5 - \{s, t\})$-residues of Γ, but one arbitrarily chosen, are in bijection with the generators of a suitable presentation of $\pi_1(M^5)$ [G1]. □

Proposition 3. *Let M^5, (Γ, γ) and ε be as in Lemma 2. If there exists $\{i, j\} \subset \Delta_5$, such that $\rho_{\hat{i}\hat{j}} = 0$, then either $M^5 \cong \mathbb{S}^5$ or $M^5 \cong \#_m((\mathbb{S}^1 \times \mathbb{S}^4))$, $1 \le m \le \rho$.*

Proof. Let $\sigma = (\sigma_0, \sigma_1, \sigma_2, \sigma_3)$ be the cyclic permutation induced by ε on $\Delta_5 - \{i, j\}$. Formula (9) directly implies $\rho_{\hat{i}} = \rho_{\hat{i}\hat{\sigma}_0} + \rho_{\hat{i}\hat{\sigma}_2}$ $(= \rho_{\hat{i}\hat{\sigma}_1} + \rho_{\hat{i}\hat{\sigma}_3})$ and $\rho_{\hat{j}} = \rho_{\hat{j}\hat{\sigma}_1} + \rho_{\hat{j}\hat{\sigma}_3}$ $(= \rho_{\hat{j}\hat{\sigma}_0} + \rho_{\hat{j}\hat{\sigma}_2})$. Since σ_0, σ_2 (resp. σ_1, σ_3) are nonconsecutive in $\varepsilon_{\hat{i}}$ (resp. $\varepsilon_{\hat{j}}$), then Lemma 2 ensures that both $N(i, \sigma_0, \sigma_2)$ and $N(j, \sigma_1, \sigma_3)$ are 5-dimensional handlebodies, obviously of the same genus m, with $0 \le m \le \rho$. If $m = 0$, then $M^5 \cong \mathbb{S}^5$ directly follows. If $1 \le m \le \rho$, then $\partial N(i, \sigma_0, \sigma_2) \cong \partial N(j, \sigma_1, \sigma_3) \cong \#_m(\mathbb{S}^1 \times \mathbb{S}^3)$; so, the result is a consequence of Corollary 4.4 and Theorem 2.1 of [CH]. □

Corollary 4. *Let M^5, (Γ, γ) and ε be as in Lemma 2. If there exists $i \in \Delta_5$, such that $\rho_{\hat{i}} \le 2$, then either $M^5 \cong \mathbb{S}^5$ or $M^5 \cong \#_m(\mathbb{S}^1 \times \mathbb{S}^4)$, $1 \le m \le \rho$.*

Proof. As a consequence of formula (9) and Remark D, $\rho_{\hat{i}} \le 2$ implies the existence of $j \in \Delta_5 - \{i\}$, such that $\rho_{\hat{i}\hat{j}} = 0$. Thus, the thesis follows from Proposition 3. □

Lemma 5. *Let M^5, (Γ, γ) and ε be as in Lemma 2. If M^5 is simply connected and there exists $\{r, s, t\} \subset \Delta_5$, such that $N(r, s, t) \cong \mathbb{D}^5$, then $M^5 \cong \mathbb{S}^5$.*

Proof. The Mayer-Vietoris sequence of the pair $(N(r, s, t), N(\hat{r}, \hat{s}, \hat{t}))$ (with coefficients in $\mathbb{Z}$), together with Poincaré duality, easily gives $H_4(M^5) = H_3(M^5) = H_1(M^5) = 0$ and $\operatorname{Tor} H_2(M^5) \cong H_2(M^5) \cong H_2(N(\hat{r}, \hat{s}, \hat{t}))$. On the other hand, $H_2(N(\hat{r}, \hat{s}, \hat{t}))$ is torsion-free ($K(\hat{r}, \hat{s}, \hat{t})$ being a two-dimensional pseudocomplex); hence, $H_2(M^5) = H_2(N(\hat{r}, \hat{s}, \hat{t})) = 0$. Now, since $H_*(M^5) = H_*(\mathbb{S}^5)$, the well-known Whitehead Theorem ensures that M^5 has the same homotopy type as $\mathbb{S}^5$; thus, the statement follows by the Generalized Poincaré Theorem. □

Lemma 6. *Let M^5, (Γ, γ), and ε be as in Lemma 2. If there exists $\{r, s, t\} \subset \Delta_5$, such that $N(r, s, t) \cong Y_m$ where Y_m denotes a 5-dimensional handlebody of genus $m \ge 1$, then*

$$H_*(N(\hat{r}, \hat{s}, \hat{t}); \mathbb{Z}) \cong H_*(Y_m; \mathbb{Z}) \quad \text{and} \quad H_*(M^5; \mathbb{Z}) \cong H_*\left(\#_m(\mathbb{S}^1 \times \mathbb{S}^4); \mathbb{Z}\right).$$

Proof. Again, the Mayer-Vietoris sequence of the pair $(N(r, s, t), N(\hat{r}, \hat{s}, \hat{t}))$ (with coefficients in $\mathbb{Z}$), together with Poincaré duality easily gives: $H_5(M^5) \cong \mathbb{Z}$, $H_4(M^5) \cong H_1(M^5) \cong \bigoplus_m \mathbb{Z}$, $H_3(M^5) = 0$, and $\operatorname{Tor} H_2(M^5) \cong H_2(M^5) \cong H_2(N(\hat{r}, \hat{s}, \hat{t}))$. As in the proof of the preceding lemma, $H_2(N(\hat{r}, \hat{s}, \hat{t}))$ is torsion-free; hence, $H_2(M^5) = H_2(N(\hat{r}, \hat{s}, \hat{t})) = 0$. Finally, $H_1(N(\hat{r}, \hat{s}, \hat{t}))$ is computed by means of the splitting sequence

$$0 \to \bigoplus_m \mathbb{Z} \to H_1(N(\hat{r}, \hat{s}, \hat{t})) \oplus \left(\bigoplus_m \mathbb{Z}\right) \to \bigoplus_m \mathbb{Z} \to 0. \quad \square$$

Proposition 7. *Let (Γ, γ) be a crystallization of a closed connected orientable 5-manifold M^5. If there exists a cyclic permutation ε of Δ_5, such that $\rho_\varepsilon(\Gamma) = \rho \le 7$, then either $M^5 \cong \mathbb{S}^5$ or $M^5 \cong \#_m(\mathbb{S}^1 \times \mathbb{S}^4)$, $1 \le m \le \rho$.*

Proof. Set $\rho^{**} = \min\{\rho_{\hat{i}\hat{j}} | i, j \in \Delta_5, i \neq j\}$. By Proposition 3, the case $\rho^{**} = 0$ directly implies the thesis; hence, $\rho^{**} \geq 1$ may be supposed. Let us further set $\rho^* = \min\{\rho_{\hat{i}} | i \in \Delta_5\}$. By Corollary 4, $\rho^* \geq 3$ may also be supposed. Now, formulae (7) and (8) imply that, if $\rho(\Gamma) \leq 7$, then $\rho^* \leq 4$ (since $\mathbf{g}_{\mathscr{B}} \geq 1$, for every $\mathscr{B} \subseteq \Delta_5$). If $r \in \Delta_5$ is such that $\rho_{\hat{r}} = \rho^*$, then the only possible cases for formula (9) are:

(a) $\rho_{\hat{r}} = 3 = (1+1+1+1+2)/2$;
(b) $\rho_{\hat{r}} = 4 = (1+1+2+2+2)/2$;
(c) $\rho_{\hat{r}} = 4 = (1+1+1+2+3)/2$.

Note that the further case $\rho_{\hat{r}} = 4 = (1+1+1+1+4)/2$ cannot hold by Remark C.

Anyway, in all the previous three cases (a), (b), (c), there exist s, t nonconsecutive in $\varepsilon_{\hat{r}}$ such that $\rho_{\hat{r}} = \rho_{\hat{r}\hat{s}} + \rho_{\hat{r}\hat{t}}$; thus, Lemma 2 implies that $N(r, s, t)$ is homeomorphic to a 5-dimensional handlebody Y_m of genus m, with $\mathrm{rk}(\pi_1(M^5)) \leq m \leq \rho$.

If $m = 0$, i.e., if $N(r, s, t) \cong \mathbb{D}^5$, then M^5 is simply connected and Lemma 5 concludes the proof. Let us assume $N(r, s, t) \cong Y_m$, $1 \leq m \leq \rho$. Set $\{u, v, w\} = \Delta_5 - \{r, s, t\}$; by the hypothesis on $\{r, s, t\}$, u may be supposed to be nonconsecutive both to v and to w (in ε). If $\rho_{\hat{v}} = \rho_{\hat{u}\hat{v}} + \rho_{\hat{v}\hat{w}}$ (or $\rho_{\hat{w}} = \rho_{\hat{u}\hat{w}} + \rho_{\hat{v}\hat{w}}$), then the thesis directly follows from Lemma 2 by making use of Corollary 4.4 and Theorem 2.1 of [CH].

Let us assume

$$(\$) \qquad \rho_{\hat{v}} - \rho_{\hat{u}\hat{v}} - \rho_{\hat{v}\hat{w}} \geq 1 \quad \text{and} \quad \rho_{\hat{w}} - \rho_{\hat{u}\hat{w}} - \rho_{\hat{v}\hat{w}} \geq 1.$$

Formula (6) gives

$$\mathbf{g}_{\hat{u}\hat{v}\hat{w}} = \mathbf{g}_{\hat{u}\hat{v}} + \mathbf{g}_{\hat{v}\hat{w}} - 1 + (\rho_{\hat{v}} - \rho_{\hat{u}\hat{v}} - \rho_{\hat{v}\hat{w}}) \geq \mathbf{g}_{\hat{u}\hat{v}} + \mathbf{g}_{\hat{v}\hat{w}},$$
$$\mathbf{g}_{\hat{u}\hat{v}\hat{w}} = \mathbf{g}_{\hat{u}\hat{w}} + \mathbf{g}_{\hat{v}\hat{w}} - 1 + (\rho_{\hat{w}} - \rho_{\hat{u}\hat{w}} - \rho_{\hat{v}\hat{w}}) \geq \mathbf{g}_{\hat{u}\hat{w}} + \mathbf{g}_{\hat{v}\hat{w}};$$

i.e.,

$$(+) \qquad \mathbf{g}_{\hat{u}\hat{w}} \leq \mathbf{g}_{\hat{u}\hat{v}\hat{w}} - \mathbf{g}_{\hat{v}\hat{w}} \quad \text{and} \quad \mathbf{g}_{\hat{u}\hat{w}} \leq \mathbf{g}_{\hat{u}\hat{v}\hat{w}} - \mathbf{g}_{\hat{v}\hat{w}}.$$

By Lemma 6, $\chi(K(u, v, w)) = 1 - m \leq 0$; on the other hand,

$$\chi(K(u, v, w)) = 3 - (\mathbf{g}_{\hat{u}\hat{w}} + \mathbf{g}_{\hat{u}\hat{w}} + \mathbf{g}_{\hat{v}\hat{w}}) + \mathbf{g}_{\hat{u}\hat{v}\hat{w}}.$$

Hence, $0 \leq \mathbf{g}_{\hat{u}\hat{v}\hat{w}} - \mathbf{g}_{\hat{v}\hat{w}} \leq \mathbf{g}_{\hat{u}\hat{v}} + \mathbf{g}_{\hat{u}\hat{w}} - 3$, and, by $(+)$, $\mathbf{g}_{\hat{u}\hat{v}} \geq 3$, $\mathbf{g}_{\hat{u}\hat{w}} \geq 3$.

Moreover, by making use of $\rho \leq 7$, $\rho^{**} \geq 1$, and relations ($), we obtain from formula (5)

$$\mathbf{g}_{\hat{u}\hat{v}} = 1 + \rho - \rho_{\hat{u}} - (\rho_{\hat{v}} - \rho_{\hat{u}\hat{v}}) \leq 3,$$
$$\mathbf{g}_{\hat{u}\hat{w}} = 1 + \rho - \rho_{\hat{u}} - (\rho_{\hat{w}} - \rho_{\hat{u}\hat{w}}) \leq 3.$$

Hence, $\mathbf{g}_{\hat{u}\hat{v}} = \mathbf{g}_{\hat{u}\hat{w}} = 3$ and $\mathbf{g}_{\hat{u}\hat{v}\hat{w}} = \mathbf{g}_{\hat{v}\hat{w}} + 3$. This means that the two-dimensional complex $K(u, v, w)$ satisfies the hypotheses (i) and (ii) of Lemma 1b; moreover, hypothesis (iii) and (iv) hold for [G1, Proposition 3] and by Lemma 6 respectively. As a consequence, $K(u, v, w)$ collapses to a graph, and therefore $N(u, v, w) = N(\hat{r}, \hat{s}, \hat{t}) \cong Y_n$. Since $N(r, s, t) \cong Y_m$ is assumed, $m = n$ obviously holds; the thesis again follows from Corollary 4.4 and Theorem 2.1 of [CH]. □

Proof of the main theorem and of Corollary I. The proof of the main theorem is a direct consequence of Proposition 7. On the other hand, the genus of $\mathbb{S}^3 \times \mathbb{S}^2$ is proved to be ≤ 8 in [GG]. Hence Corollary I again follows from Proposition 7. □

REFERENCES

[B] D. Barden, *Simply connected five-manifolds*, Ann. of Math. (2) **82** (1965), 365–385.

[BM] J. Bracho and L. Montejano, *The combinatorics of colored triangulations of manifolds*, Geom. Dedicata **22** (1987), 303–328.

[C] M. R. Casali, *A combinatorial characterization of* 4*-dimensional handlebodies*, Forum Math. **4** (1992), 123–134.

[CH] A. Cavicchioli and F. Hegenbarth, *On the determination of PL-manifolds by handles of lower dimension*, Topology Appl. (to appear).

[CP] R. Chiavacci and G. Pareschi, *Some bounds for the regular genus of closed PL manifolds*, Discrete Math. **82** (1990), 165–180.

[FG] M. Ferri and C. Gagliardi, *The only genus zero n-manifold in* $\mathbb{S}^n$, Proc. Amer. Math. Soc. **85** (1982), 638–642.

[FGG] M. Ferri, C. Gagliardi, and L. Grasselli, *A graph-theoretical representation of PL manifolds—A survey on crystallizations*, Aequationes Math. **31** (1986), 121–141.

[G1] C. Gagliardi, *How to deduce the fundamental group of a closed n-manifold from a contracted triangulation*, J. Combin. Inform. System Sci. **4** (1979), 237–252.

[G2] ——, *Extending the concept of genus to dimension n*, Proc. Amer. Math. Soc. **81** (1981), 473–481.

[G3] ——, *Surface maps and n-dimensional manifolds*, Atti IV Convegno di Topologia, Sorrento (Na)–September 1988, Rend. Circ. Mat. Palermo **24** (1990), 97–126.

[GG] C. Gagliardi and L. Grasselli, *Representing products of polyhedra by products of edge-coloured graphs*, J. Graph Theory (to appear).

[Gl] L. C. Glaser, *Geometrical combinatorial topology*, Van Nostrand Reinhold Math. Stud., Van Nostrand Reinhold, New York, 1970.

[HW] P. J. Hilton and S. Wylie, *An introduction to algebraic topology—Homology theory*, Cambridge Univ. Press, Cambridge, 1960.

[LM] S. Lins and A. Mandel, *Graph-encoded* 3*-manifolds*, Discrete Math. **57** (1985), 261–284.

[P1] M. Pezzana, *Sulla struttura topologica delle varietá compatte*, Atti Sem. Mat. Fis. Univ. Modena **23** (1975), 269–277.

[P2] ——, *Diagrammi di Heegaard e triangolazione contratta*, Boll. Un. Mat. Ital. **12** (1975), 98–105.

[RS] C. Rourke and B. Sanderson, *Introduction to piecewise-linear topology*, Springer Verlag, New York and Heidelberg, 1972.

[S] S. Smale, *On the structure of* 5*-manifolds*, Ann. of Math. (2) **82** (1962), 38–46.

[V] A. Vince, *n-graphs*, Discrete Math. **72** (1988), 367–380.

[W] A. T. White, *Graphs, groups and surfaces*, North-Holland, Amsterdam, 1973.

DIPARTIMENTO DI MATEMATICA, VIA CAMPI 213/B, I-41100 MODENA, ITALIA
E-mail address: GAGLIARDI@IMOVX2.UNIMO.IT