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NON-COHEN-MACAULAY SYMBOLIC BLOW-UPS FOR SPACE MONOMIAL CURVES AND COUNTEREXAMPLES TO COWSIK'S QUESTION

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ABSTRACT. Let $A = k[[X, Y, Z]]$ and $k[[T]]$ be formal power series rings over a field k, and let $n \geq 4$ be an integer such that $n \not\equiv 0 \mod 3$. Let $\varphi: A \to k[[T]]$ denote the homomorphism of k-algebras defined by $\varphi(X) = T^{7n-3}$, $\varphi(Y) = T^{(5n-2)n}$, and $\varphi(Z) = T^{8n-3}$. We put $\mathbf{p} = \operatorname{Ker}\varphi$. Then $R_s(\mathbf{p}) = \bigoplus_{i \geq 0} \mathbf{p}^{(i)}$ is a Noetherian ring if and only if $\operatorname{ch} k > 0$. Hence on Cowsik's question there are counterexamples among the prime ideals defining space monomial curves, too.

1. INTRODUCTION

Let $A = k[[X, Y, Z]]$ and $S = k[[T]]$ be formal power series rings over a field k, and let n_1, n_2, and n_3 be positive integers with $\operatorname{GCD}(n_1, n_2, n_3) = 1$. We denote by $\mathbf{p}(n_1, n_2, n_3)$ the kernel of the homomorphism $\varphi: A \to S$ of k-algebras defined by $\varphi(X) = T^{n_1}$, $\varphi(Y) = T^{n_2}$, and $\varphi(Z) = T^{n_3}$. Hence $\mathbf{p} = \mathbf{p}(n_1, n_2, n_3)$ is the extended ideal in the ring A of the defining ideal for the monomial curve $x = t^{n_1}$, $y = t^{n_2}$, and $z = t^{n_3}$ in $\mathbf{A}_k^3$.

A little more generally, let $\mathbf{p}$ be a prime ideal of height 2 in a 3-dimensional regular local ring A. We put $R_s(\mathbf{p}) = \sum_{n \geq 0} \mathbf{p}^{(n)} T^n$ (here T denotes an indeterminate over A) and call it the symbolic Rees algebra of $\mathbf{p}$. Then, as is well known, $R_s(\mathbf{p})$ is a Krull ring with the divisor class group $\mathbf{Z}$, and if $R_s(\mathbf{p})$ is a Noetherian ring, its canonical module is given by $[R_s(\mathbf{p})](-1)$ (cf., e.g., [12, Corollary 3.4]). Consequently $R_s(\mathbf{p})$ is a Gorenstein ring, once it is Cohen-Macaulay. The readers may consult [2, 3] for a criterion of $R_s(\mathbf{p})$ being Cohen-Macaulay, where several examples of prime ideals $\mathbf{p}$ with Gorenstein symbolic Rees algebras are explored, too.

Nevertheless, as was first shown by Morimoto and the first author [8], $R_s(\mathbf{p})$ are not necessarily Cohen-Macaulay even for the space monomial curves $\mathbf{p} = \mathbf{p}(n_1, n_2, n_3)$. This research is a succession to [8] and the aim is to provide the following new examples.

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Theorem 1.1. *Choose an integer* $n \geq 4$ *so that* $n \not\equiv 0 \bmod 3$ *and put* $\mathbf{p} = \mathbf{p}(7n-3,\ (5n-2)n,\ 8n-3)$. *Let* $c = Y^3 - X^n Z^n$ *and assume that* $\operatorname{ch} k = p > 0$. *Then there exists an element* $h \in \mathbf{p}^{(3p)}$ *satisfying the equality*

$$\operatorname{length}(A/(x\,,\,c\,,\,h)) = 3p \cdot \operatorname{length}(A/(x) + \mathbf{p})$$

for any $x \in (X\,,\,Y\,,\,Z)A \backslash \mathbf{p}$. *In particular,* $R_s(\mathbf{p})$ *is a Noetherian ring but not Cohen-Macaulay.*

Here let us note that the ideal $\mathbf{p}$ in the above theorem is generated by the maximal minors of the matrix

$$\begin{pmatrix} X^n & Y^2 & Z^{2n-1} \\ Y & Z^n & X^{2n-1} \end{pmatrix};$$

hence $\mathbf{p}$ is a self-linked ideal in the sense of Herzog and Ulrich [6, Corollary 1.10], that is, $\mathbf{p} = (f\,,\,g):\mathbf{p}$ for some elements $f\,,\,g \in \mathbf{p}$. Roughly speaking, self-linked ideals enjoy more excellent natures than those of the ideals which are not self-linked. For instance, the subrings $A[\mathbf{p}T\,,\,\mathbf{p}^{(2)}T^2]$ of $R_s(\mathbf{p})$ are always Gorenstein, if the corresponding ideals $\mathbf{p}$ are self-linked (see [6, Proof of Theorem 2.1]). From this point of view it seems rather natural to expect that the whole rings $R_s(\mathbf{p})$ are Gorenstein at least for self-linked ideals $\mathbf{p}$; however, the answer is negative as we claim in Theorem 1.1.

In [8] Morimoto and the first author constructed, for each prime number p, space monomial curves $\mathbf{p}$ whose symbolic Rees algebras $R_s(\mathbf{p})$ are Noetherian but not Cohen-Macaulay, if the characteristic $\operatorname{ch} k$ of the ground field k is equal to p. Nevertheless their examples are *not* self-linked and, furthermore, it is not clear for their examples whether $R_s(\mathbf{p})$ are Noetherian or not in the case where the characteristic is different from the given prime number p, while our examples are Noetherian and non-Cohen-Macaulay whenever $\operatorname{ch} k > 0$.

This advantage naturally enables us, passing to the reduction modulo prime numbers, to explore the case of characteristic 0, too. Moreover, somewhat surprisingly, as an immediate consequence of Theorem 1.1 we get the following counterexamples to Cowsik's question [1], that asked whether the symbolic Rees algebra $R_s(\mathbf{p})$ be a Noetherian ring for any prime ideal $\mathbf{p}$ in a regular local ring A:

Corollary 1.2. *Let* $\mathbf{p}$ *be a prime ideal stated in Theorem* 1.1. *Then* $R_s(\mathbf{p})$ *is not a Noetherian ring, if* $\operatorname{ch} k = 0$.

When Cowsik raised the question, he aimed also a possible new approach toward the problem posed by Kronecker, who asked whether any irreducible affine algebraic curve in $\mathbf{A}^n_{\mathbf{C}}$ could be defined by $n-1$ equations. In fact, Cowsik pointed out in [1] that $\mathbf{p}$ is a set-theoretic complete intersection, if $R_s(\mathbf{p})$ is a Noetherian ring and if $\dim A/\mathbf{p} = 1$; however, as is well known, while Kronecker's problem remained open on Cowsick's question there was already given a counterexample by Roberts [9]. Because his first example did not remain prime when the ring was completed, he recently constructed the second counterexamples [10]. They are actually height 3 prime ideals in a formal power series ring with seven variables over a field; now, providing new and simpler counterexamples among the prime ideals in the formal power series ring $\mathbf{Q}[[X\,,\,Y\,,\,Z]]$, our Corollary 1.2 settles Cowsik's question, though it says nothing about Kronecker's problem itself.

The simplest example given by Theorem 1.1 is the prime ideal $\mathbf{p} = \mathbf{p}(25, 72, 29) = (X^{11} - YZ^7,\ Y^3 - X^4Z^4,\ Z^{11} - X^7Y^2)$ in $A = k[[X, Y, Z]]$. The symbolic Rees algebra $R_s(\mathbf{p})$ is a Noetherian ring but not Cohen-Macaulay, if $\operatorname{ch} k > 0$, and $R_s(\mathbf{p})$ is not a Noetherian ring, if $\operatorname{ch} k = 0$. Therefore, the symbolic blow-up $R_s(P) = \sum_{n \geq 0} P^{(n)}T^n$ is not a finitely generated $\mathbf{Q}$-algebra for the prime ideal $P = (X^{11} - YZ^7,\ Y^3 - X^4Z^4,\ Z^{11} - X^7Y^2)$ in the polynomial ring $\mathbf{Q}[X, Y, Z]$, too.

Now let us briefly explain how this paper is organized. The proof of Theorem 1.1 and Corollary 1.2 will be given in §4. Section 3 is devoted to a reduction technique modulo prime numbers. In §2 we shall summarize some preliminary steps that we need to prove Theorem 1.1 and Corollary 1.2.

Otherwise specified, in what follows let $A = k[[X, Y, Z]]$ denote a formal power series ring over a fixed field k. Let $\mathbf{m} = (X, Y, Z)A$ be the maximal ideal of A. We denote by $\ell_A(M)$, for an A-module M, the length of M. For a given prime ideal $\mathbf{p}$ in A we put

$$R_s(\mathbf{p}) = \sum_{n \geq 0} \mathbf{p}^{(n)} T^n \subset A[T],$$

$$R_s'(\mathbf{p}) = \sum_{n \in Z} \mathbf{p}^{(n)} T^n \ (= R_s(\mathbf{p})[T^{-1}]) \subset A[T, T^{-1}],$$

$$G_s(\mathbf{p}) = R_s'(\mathbf{p})/T^{-1}R_s'(\mathbf{p})$$

where T is an indeterminate over A.

2. Preliminaries

First of all let us recall Huneke's criterion [7] for $R_s(\mathbf{p})$ to be a Noetherian ring. For a while let $(A, \mathbf{m})$ denote a regular local ring of $\dim A = 3$ and let $\mathbf{p}$ be a prime ideal of A with $\dim A/\mathbf{p} = 1$.

Theorem 2.1 [7]. *If there exist elements $f \in \mathbf{p}^{(l)}$ and $g \in \mathbf{p}^{(m)}$ with positive integers l and m such that*

$$\ell_A(A/(x, f, g)) = lm \cdot \ell_A(A/(x) + \mathbf{p})$$

for some $x \in \mathbf{m}\backslash\mathbf{p}$, then $R_s(\mathbf{p})$ is a Noetherian ring. When the field $A/\mathbf{m}$ is infinite, the converse is also true.

The next proposition allows us to arbitrarily choose the element x in Theorem 2.1. The result is implicitly found in [7, Proof of Theorem 3.1] and is due to Huneke.

Proposition 2.2. *Let $f \in \mathbf{p}^{(l)}$ and $g \in \mathbf{p}^{(m)}$ $(l, m > 0)$ and assume that*

$$\ell_A(A/(x, f, g)) = lm \cdot \ell_A(A/(x) + \mathbf{p})$$

for some $x \in \mathbf{m}\backslash\mathbf{p}$. Then the above equality holds for any $x \in \mathbf{m}\backslash\mathbf{p}$.

Proof. We may assume $l = m$. Then $(f, g)A$ is a reduction of $\mathbf{p}^{(m)}$; actually, $[\mathbf{p}^{(m)}]^2 = (f, g) \cdot \mathbf{p}^{(m)}$ (cf. [2, Proof of Proposition (3.1)]). Let $R = A_\mathbf{p}$ and $\mathbf{n} = \mathbf{p}A_\mathbf{p}$. Then as the ideal $(f, g)R$ is a reduction of $\mathbf{n}^m$ and as $(R, \mathbf{n})$ is a 2-dimensional regular local ring, we get

$$\ell_R(R/(f, g)R) = e^0_{(f, g)R}(R) = e^0_{\mathbf{n}^m}(R) = m^2$$

where $e^0_{(f,g)R}(R)$ and $e^0_{\mathbf{n}^m}(R)$ denote the multiplicities. Therefore as $\mathbf{p} = \sqrt{(f, g)}$, we have by the additive formula [11, p. 126] of multiplicity that

$$\begin{aligned}\ell_A(A/(x, f, g)) &= \ell_R(R/(f, g)R) \cdot \ell_A(A/(x) + \mathbf{p}) \\ &= m^2 \cdot \ell_A(A/(x) + \mathbf{p})\end{aligned}$$

for any element $x \in \mathbf{m} \backslash \mathbf{p}$.

In [2] Goto, Nishida, and Shimoda gave a criterion for $R_s(\mathbf{p})$ to be a Cohen-Macaulay (hence Gorenstein) ring in terms of the elements f and g in Theorem 2.1. Let us note here their result, too.

Theorem 2.3 [2]. *Let f and g be as in Theorem 2.1. Then the following two conditions are equivalent.*

(1) *$R_s(\mathbf{p})$ is a Gorenstein ring.*

(2) *$A/(f, g) + \mathbf{p}^{(n)}$ are Cohen-Macaulay for all $1 \leq n \leq l + m - 2$.*

When this is the case, the A-algebra $R_s(\mathbf{p})$ is generated by $\{\mathbf{p}^{(n)}T^n\}_{1 \leq n \leq l+m-2}$, fT^l, and gT^m, and the rings $A/(f) + \mathbf{p}^{(n)}$, $A/(g) + \mathbf{p}^{(n)}$, and $A/(f, g) + \mathbf{p}^{(n)}$ are Cohen-Macaulay for all $n \geq 1$.

We now let n_1, n_2, and n_3 be positive integers with $\mathrm{GCD}(n_1, n_2, n_3) = 1$ and take $\mathbf{p} = \mathbf{p}(n_1, n_2, n_3)$ in $A = k[[X, Y, Z]]$ (hence $\ell_A(A/(X) + \mathbf{p}) = n_1$, $\ell_A(A/(Y) + \mathbf{p}) = n_2$, and $\ell_A(A/(Z) + \mathbf{p}) = n_3$). In what follows let us assume that $\mathbf{p}$ is generated by the maximal minors of a matrix of the form

$$\begin{pmatrix} X^\alpha & Y^{\beta'} & Z^{\gamma'} \\ Y^\beta & Z^\gamma & X^{\alpha'} \end{pmatrix},$$

where $\alpha, \beta, \gamma, \alpha', \beta'$, and γ' are positive integers (cf. [5]). We put $a = Z^{\gamma+\gamma'} - X^{\alpha'}Y^{\beta'}$, $b = X^{\alpha+\alpha'} - Y^\beta Z^{\gamma'}$, and $c = Y^{\beta+\beta'} - X^\alpha Z^\gamma$ (hence $\mathbf{p} = (a, b, c)$). Let $\mathscr{F} = \{0 < m \in \mathbf{Z} | \exists g \in \mathbf{p}^{(m)}$ such that $\ell_A(A/(X, c, g)) = m \cdot \ell_A(A/(X) + \mathbf{p})\}$ and assume that $\mathscr{F} \neq \varnothing$. We put $m_0 = \min \mathscr{F}$ and choose an element $g_0 \in \mathbf{p}^{(m_0)}$ so that $\ell_A(A/(X, c, g_0)) = m_0 \cdot \ell_A(A/(X) + \mathbf{p})$.

Let us begin with

Lemma 2.4. *$m_0 | m$ for any $m \in \mathscr{F}$.*

Proof. Let $g \in \mathbf{p}^{(m)}$ such that $\ell_A(A/(X, c, g)) = m \cdot \ell_A(A/(X) + \mathbf{p})$. Then as $m_0 \leq m$, we get by [2, Proposition (3.4)] that $\mathbf{p}^{(m)} \subseteq (c, g_0)$, whence $\mathbf{p}^{(m)} = c\mathbf{p}^{(m-1)} + g_0\mathbf{p}^{(m-m_0)}$ by [2, Proposition (3.7)(2)]. We write $g \equiv g_0 g_1 \mod(c)$ with $g_1 \in \mathbf{p}^{(m-m_0)}$. Then because $\ell_A(A/(X, c, g)) = \ell_A(A/(X, c, g_0)) + \ell_A(A/(X, c, g_1))$, we have $\ell_A(A/(X, c, g_1)) = (m - m_0) \cdot \ell_A(A/(X) + \mathbf{p})$, which yields by induction on m that $m_0 | m$.

Proposition 2.5. $g_0 T^{m_0} \notin A[\{\mathbf{p}^{(n)}T^n\}_{1 \leq n < m_0}]$.

Proof. [1] Assume the contrary. Then we have by [2, Proposition (3.7)(4)] that $R_s(\mathbf{p}) = A[\{\mathbf{p}^{(n)}T^n\}_{1 \leq n < m_0}]$. Hence $\mathbf{p}^{(m_0)} = \sum_{i=1}^{m_0-1} \mathbf{p}^{(i)}\mathbf{p}^{(m_0-i)}$. Let $\overline{A} = A/(X, Y)$ and denote by $\overline{\cdot}$ the reduction $\mathrm{mod}(X, Y)$. Then because $\overline{A}$ is a DVR and $\overline{g}_0 \in \sum_{i=1}^{m_0-1} \mathbf{p}^{(i)}\overline{A} \cdot \mathbf{p}^{(m_0-i)}\overline{A}$, we have $\overline{g}_0 = \overline{g_1 g_2}$ for some $g_1 \in \mathbf{p}^{(i)}$ and $g_2 \in \mathbf{p}^{(m_0-i)}$ with $1 \leq i \leq m_0 - 1$.

[1] This shorter proof was suggested by the referee. The authors are grateful to the referee for his helpful suggestion.

Now recall that $c \equiv Y^{\beta+\beta'} \operatorname{mod}(X)$ and we get

$$\begin{aligned}\ell_A(A/(X, c, g_1g_2)) &= (\beta+\beta')\cdot\ell_A(A/(X, Y, g_1g_2))\\ &= (\beta+\beta')\cdot\ell_A(A/(X, Y, g_0)) = m_0\cdot\ell_A(A/(X)+\mathbf{p}).\end{aligned}$$

Hence the elements c and g_1g_2 satisfy Huneke's condition (2.1) so that c and g_1 satisfy this condition as well. In fact, because $\ell_A(A/(X, c, g_1g_2)) = \ell_A(A/(X, c, g_1)) + \ell_A(A/(X, c, g_2)) = m_0\cdot\ell_A(A/(X)+\mathbf{p})$, we get by the inequalities $\ell_A(A/(X, c, g_1)) \geq i\cdot\ell_A(A/(X)+\mathbf{p})$ and $\ell_A(A/(X, c, g_2)) \geq (m_0 - i)\cdot\ell_A(A/(X)+\mathbf{p})$ that $l_A(A/(X, c, g_1)) = i\cdot l_A(A/(X)+\mathbf{p})$. Thus we have $i\in\mathscr{F}$, which contradicts the minimality of m_0.

The next result is the key in our proof of Corollary 1.2.

Corollary 2.6. *Assume that $\beta+\beta' = 3$ and that $3\nmid n_1$. Let p be a prime number for which we assume that there exists an element $g\in\mathbf{p}^{(3p)}$ satisfying the equality $\ell_A(A/(X, c, g)) = 3p\cdot\ell_A(A/(X)+\mathbf{p})$. Then if $gT^{3p}\in A[\{\mathbf{p}^{(n)}T^n\}_{1\leq n<3p}]$, we have $m_0 = 3$ and $R_s(\mathbf{p}) = A[\mathbf{p}T, \mathbf{p}^{(2)}T^2, \mathbf{p}^{(3)}T^3]$.*

Proof. As $\ell_A(A/(X)+\mathbf{p}) = n_1$ and as $c\equiv Y^3\operatorname{mod}(X)$, we have that

$$\ell_A(A/(X, c, g_0)) = m_0n_1 = 3\cdot\ell_A(A/(X, Y, g_0)).$$

Hence $3|m_0$, because $3\nmid n_1$ by our assumption. Let $m_0 = 3m_1$. Then as $m_0|3p$ by Lemma 2.4, we see $m_1 = 1$ or p. If m_1 were equal to p, then $m_0 = 3p$ and so we must have by Proposition 2.5 that $gT^{3p}\notin A[\{\mathbf{p}^{(n)}T^n\}_{1\leq n<3p}]$, which contradicts our standard assumption. Thus $m_0 = 3$ and we get by [2, Proposition (3.7)(4)] that $R_s(\mathbf{p}) = A[\mathbf{p}T, \mathbf{p}^{(2)}T^2, g_0T^3]$.

3. Reduction to the case where $\operatorname{ch} k > 0$

Let n_1, n_2, and n_3 be positive integers with $\operatorname{GCD}(n_1, n_2, n_3) = 1$. Let $C = \mathbf{Z}[X, Y, Z]$ and $\mathbf{Z}[T]$ be polynomial rings over $\mathbf{Z}$. We denote by $\varphi: C\to\mathbf{Z}[T]$ the homomorphism of rings such that $\varphi(X) = T^{n_1}$, $\varphi(Y) = T^{n_2}$, and $\varphi(Z) = T^{n_3}$. Let $I = \operatorname{Ker}\varphi$ and consider the ring C to be a graded ring whose graduation is defined by $C_0 = \mathbf{Z}$, $C_{n_1}\ni X$, $C_{n_2}\ni Y$, and $C_{n_3}\ni Z$. Then the homomorphism φ preserves the graduation so that I is naturally a graded prime ideal in C.

For a given field k, $A_k = k[[X, Y, Z]]$ denotes a formal power series ring. We put $B_k = k[X, Y, Z]$, that we shall identify with $k\otimes_{\mathbf{Z}} C$. Let $\mathbf{p}_k$ denote the ideal $\mathbf{p}(n_1, n_2, n_3)$ considered in A_k; hence, $\mathbf{p}_k = IA_k$. We put $P_k = IB_k$. Thus P_k is the defining prime ideal of the monomial curve $x = t^{n_1}$, $y = t^{n_2}$, and $z = t^{n_3}$ in $\mathbf{A}_k^3$.

The purpose of this section is to prove the following

Theorem 3.1. *Assume that $R_s(\mathbf{p}_{\mathbf{Q}})$ is a Noetherian ring. Then there exist positive integers N, m and elements f, g of $I^{(m)}$ such that if p is a prime number and if $p\geq N$, we always have*

(1) $\mathbf{p}_k^{(m)} = I^{(m)}A_k$ *and*
(2) $\ell_{A_k}(A_k/(X, f, g)A_k) = m^2\cdot\ell_{A_k}(A_k/(X)+\mathbf{p}_k)$

for the field $k = \mathbf{Z}/p\mathbf{Z}$.

Proof. We put $B = B_{\mathbf{Q}}$, $P = P_{\mathbf{Q}}$, and $M = (X, Y, Z)B$. Then as $A = A_{\mathbf{Q}}$ is a faithfully flat extension of B_M, the symbolic Rees algebra $R_s(PB_M)$

is a Noetherian ring too; therefore, by Theorem 2.1 and Proposition 2.2 we may choose two elements $f, g \in (PB_M)^{(m)}$ with m a positive integer so that $\ell_{B_M}(B_M/(X, f, g)B_M) = m^2 \cdot \ell_{B_M}(B_M/XB_M + PB_M)$. Let $N = (X, Y, Z)C$. Then as $C_N = B_M$ and $IC_N = PB_M$, we can take f and g to be in $I^{(m)}$.

Let $s \in C\backslash I$ be an element such that $sI^{(m)} \subseteq I^m$. We expand $\varphi(s) = \sum_{i=\alpha}^{\beta} s_i T^i$ in $\mathbf{Z}[T]$ with $s_\alpha \neq 0$ and choose a prime number p so that $p \nmid s_\alpha$. Then the element $\bar{s} = 1 \otimes s$ of $B_k = k \otimes_{\mathbf{Z}} C$ is not contained in $P_k = IB_k$ and $\bar{s} \cdot I^{(m)}B_k \subseteq I^m B_k = P_k^m$. Hence $I^{(m)}B_k \subseteq P_k^{(m)}$.

We now recall that $[(PB_M)^{(m)}]^2 = (f, g)B_M \cdot (PB_M)^{(m)}$ (cf. [2, Proof of Proposition (3.1)]) and choose an element $\xi = \xi(X, Y, Z) \in C\backslash N$ so that $\xi \cdot [I^{(m)}]^2 \subseteq (f, g)C \cdot I^{(m)}$. Then if p is a prime number and if $p \nmid \xi(0, 0, 0)$, the element $\bar{\xi} = 1 \otimes \xi$ of B_k is not contained in $M_k = (X, Y, Z)B_k$ and $\bar{\xi} \cdot [I^{(m)}B_k]^2 \subseteq (f, g)B_k \cdot I^{(m)}B_k$. Thus we have $[I^{(m)}A_k]^2 = (f, g)A_k \cdot I^{(m)}A_k$ in A_k.

Let us choose a prime number p so that p is a non-zero-divisor on $D = C/XC + I^{(m)}$ (this choice is possible, because $\mathrm{Ass}_{\mathbf{Z}} D$ is a finite set). Then as X is a non-zero-divisor on $C/I^{(m)}$, we get a commutative diagram

$$\begin{array}{ccccccccc} & & & & & & 0 & & \\ & & & & & & \downarrow & & \\ 0 & \to & C/I^{(m)} & \xrightarrow{X} & C/I^{(m)} & \to & D & \to & 0 \\ & & p\downarrow & & p\downarrow & & p\downarrow & & \\ 0 & \to & C/I^{(m)} & \xrightarrow{X} & C/I^{(m)} & \to & D & \to & 0 \end{array}$$

with exact rows. Consequently X is a non-zero-divisor on $B_k/I^{(m)}B_k$, while $P_k = \sqrt{I^{(m)}B_k}$ because $P_k^m \subseteq I^{(m)}B_k \subseteq P_k$; therefore, the graded ideal $I^{(m)}B_k$ is P_k-primary (notice that $\dim B_k/I^{(m)}B_k = 1$, as $\dim B_k/P_k = 1$) and so we have that $P_k^{(m)} \subseteq I^{(m)}B_k$.

Summarizing the above observations and choosing a prime number p so that $p \nmid s_\alpha$, $p \nmid \xi(0, 0, 0)$, and $p \notin \bigcup_{\mathfrak{q} \in \mathrm{Ass}_{\mathbf{Z}} D} \mathfrak{q}$, we get that $\mathbf{p}_k^{(m)} = I^{(m)}A_k$ and that $[\mathbf{p}_k^{(m)}]^2 = (f, g)A_k \cdot \mathbf{p}_k^{(m)}$ for the field $k = \mathbf{Z}/p\mathbf{Z}$. The second assertion (2) now follows from the equality $[\mathbf{p}_k^{(m)}]^2 = (f, g)A_k \cdot \mathbf{p}_k^{(m)}$ similarly as in the proof of Proposition 2.2. This completes the proof of Theorem 3.1.

4. Proof of Theorem 1.1 and Corollary 1.2

Let $n \geq 4$ be an integer such that $n \not\equiv 0 \bmod 3$ and let k be a field. In this section we explore the prime ideal $\mathbf{p} = \mathbf{p}(7n-3, (5n-2)n, 8n-3)$ in $A = k[[X, Y, Z]]$. The purpose is to prove Theorem 1.1 and Corollary 1.2.

First of all recall that $\mathbf{p}$ is generated by the maximal minors of the matrix

$$\begin{pmatrix} X^n & Y^2 & Z^{2n-1} \\ Y & Z^n & X^{2n-1} \end{pmatrix}$$

(cf. [5]). We put $a = Z^{3n-1} - X^{2n-1}Y^2$, $b = X^{3n-1} - YZ^{2n-1}$, and $c = Y^3 - X^nZ^n$ (hence $\mathbf{p} = (a, b, c)$). Notice that any pair of a, b, and c forms a regular system of parameters in $A_{\mathbf{p}}$, since there is the obvious relation

$$X^n a + Y^2 b + Z^{2n-1} c = 0. \tag{4.1}$$

Let

$$\begin{aligned} d_2 &= X^{n-1}Y^5Z^{n-1} - 3X^{2n-1}Y^2Z^{2n-1} + X^{5n-2}Y + Z^{5n-2}, \\ d_3 &= -X^{3n-2}Y^7 + 2X^{n-1}Y^5Z^{3n-1} + X^{4n-2}Y^4Z^n \\ &\quad - 5X^{2n-1}Y^2Z^{4n-1} + 3X^{5n-2}YZ^{2n} - X^{8n-3}Z + Z^{7n-2}, \\ d_3' &= Y^8Z^{2n-2} - 4X^nY^5Z^{3n-2} + X^{4n-1}Y^4Z^{n-1} + 6X^{2n}Y^2Z^{4n-2} \\ &\quad - 4X^{5n-1}YZ^{2n-1} + X^{8n-2} - XZ^{7n-3}. \end{aligned}$$

Then a direct computation easily checks

$$\begin{aligned} X^nd_2 - Yb^2 + Z^{n-1}ac &= 0, \\ X^{n-1}b^2c + ad_2 - Z^{n-1}d_3 &= 0, \\ Xd_3 + Ybc^2 + Zd_3' &= 0. \end{aligned} \tag{4.2}$$

Hence $d_2 \in \mathbf{p}^{(2)}$ and $d_3, d_3' \in \mathbf{p}^{(3)}$.

The next lemma is a special case of much more general results (cf. [3, Corollary (2.6)] and [4, Theorem (5.4)]), however, we briefly give a direct proof for the sake of completeness.

Lemma 4.3. $\mathbf{p}^{(2)} = \mathbf{p}^2 + (d_2)$ *and* $\mathbf{p}^{(3)} = \mathbf{p}\mathbf{p}^{(2)} + (d_3, d_3')$.

Proof. As $(X) + \mathbf{p}^2 + (d_2) = (X) + (Y^6, Y^4Z^{2n-1}, Y^3Z^{3n-1}, Y^2Z^{4n-2}, Z^{5n-2})$, we get

$$\ell_A(A/(X) + \mathbf{p}^2 + (d_2)) = 3(7n-3).$$

On the other hand by the additive formula of multiplicity we get

$$\ell_A(A/(X) + \mathbf{p}^{(2)}) = \ell_{A_\mathbf{p}}(A_\mathbf{p}/\mathbf{p}^2A_\mathbf{p}) \cdot \ell_A(A/(X) + \mathbf{p}) = 3 \cdot (7n-3).$$

Hence $\ell_A(A/(X) + \mathbf{p}^{(2)}) = \ell_A(A/(X) + \mathbf{p}^2 + (d_2))$ so that we have $(X) + \mathbf{p}^{(2)} = (X) + \mathbf{p}^2 + (d_2)$, which implies $\mathbf{p}^{(2)} = \mathbf{p}^2 + (d_2) + X\mathbf{p}^{(2)}$. Thus $\mathbf{p}^{(2)} = \mathbf{p}^2 + (d_2)$ by Nakayama's lemma. Consequently

$$\begin{aligned} (X) + \mathbf{p}\mathbf{p}^{(2)} + (d_3, d_3') = (X) + (&Y^9, Y^8Z^{2n-2}, Y^7Z^{2n-1}, Y^6Z^{3n-1}, \\ &Y^5Z^{4n-2}, Y^3Z^{5n-2}, YZ^{7n-3}, Z^{7n-2}), \end{aligned}$$

whence

$$\ell_A(A/(X) + \mathbf{p}\mathbf{p}^{(2)} + (d_3, d_3')) = 6(7n-3).$$

Because $\ell_A(A/(X) + \mathbf{p}^{(3)}) = \ell_{A_\mathbf{p}}(A_\mathbf{p}/\mathbf{p}^3A_\mathbf{p}) \cdot \ell_A(A/(X) + \mathbf{p})$, we have

$$\ell_A(A/(X) + \mathbf{p}^{(3)}) = \ell_A(A/(X) + \mathbf{p}\mathbf{p}^{(2)} + (d_3, d_3'))$$

and, thus, $\mathbf{p}^{(3)} = \mathbf{p}\mathbf{p}^{(2)} + (d_3, d_3')$ as required.

Proposition 4.4. *The ring* $A/(c) + \mathbf{p}^{(3)}$ *is not Cohen-Macaulay.*

Proof. Let $e^0_{XA}(A/(c) + \mathbf{p}^{(3)})$ denote the multiplicity of XA in $A/(c) + \mathbf{p}^{(3)}$. Then as $A_\mathbf{p}/cA_\mathbf{p}$ is a DVR, we get by the additive formula of multiplicity that

$$e^0_{XA}(A/(c) + \mathbf{p}^{(3)}) = 3(7n-3).$$

On the other hand we get by Lemma 4.3 and its proof that

$$(X) + (c) + \mathbf{p}^{(3)} = (X) + (Y^3, YZ^{7n-3}, Z^{7n-2}).$$

Hence

$$\ell_A(A/(X)+(c)+\mathbf{p}^{(3)}) = 3(7n-3)+1,$$

so that we get $\ell_A(A/(X)+(c)+\mathbf{p}^{(3)}) > e^0_{XA}(A/(c)+\mathbf{p}^{(3)})$. Thus $A/(c)+\mathbf{p}^{(3)}$ cannot be a Cohen-Macaulay ring.

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. Let $\operatorname{ch} k = p > 0$ and we get by the third equation in (4.2) that

$$X^p d_3^p + Y^p b^p c^{2p} + Z^p d_3'^p = 0.$$

If $p = 2$, then because $X^2 d_3^2 + Y^2 b^2 c^4 \equiv 0$ and $X^n a + Y^2 b \equiv 0 \bmod (Z^2)$ (cf. (4.1)), we have $X^2 d_3^2 + X^n abc^4 \equiv 0 \bmod (Z^2)$. Hence $d_3^2 + X^{n-2}abc^4 = Z^2 h$ for some $h \in \mathbf{p}^{(6)}$. Notice that $d_3 \equiv Z^{7n-2} \bmod (X)$ and that $Z^2 h \equiv d_3^2 \bmod (X, c)$. Then we get $h \equiv Z^{2(7n-3)} \bmod (X, c)$, because X, c, and Z^2 form a regular sequence in A. Thus $\ell_A(A/(X, c, h)) = \ell_A(A/(X, Y^3, Z^{2(7n-3)}))$ $= 6(7n-3)$ as required.

Assume that $p \geq 3$ and write $p = 2q+1$ with q a positive integer. Then since $X^p d_3^p + Y^p b^p c^{2p} \equiv 0 \bmod (Z^p)$ and since $Y^p b^p c^{2p} = (Y^2 b)^q \cdot Y b^{p-q} c^{2p}$, we get by (4.1) that

$$\begin{aligned} X^p d_3^p + Y^p b^p c^{2p} &= X^p d_3^p + (-1)^q Y b^{p-q} c^{2p} (X^n a + Z^{2n-1} c)^q \\ &= X^p d_3^p + (-1)^q \sum_{i=0}^{q} \binom{q}{i} X^{n(q-i)} Y Z^{(2n-1)i} a^{q-i} b^{p-q} c^{2p+i} \\ &\equiv 0 \mod (Z^p). \end{aligned}$$

Here notice that $n(q-i) \geq p$ or $(2n-1)i \geq p$ for each $0 \leq i \leq q$ (use the fact $n \geq 4$), and furthermore we have

$$X^p d_3^p + (-1)^q \sum_{(2n-1)i<p} \binom{q}{i} X^{n(q-i)} Y Z^{(2n-1)i} a^{q-i} b^{p-q} c^{2p+i} \equiv 0 \mod (Z^p),$$

whence

$$Z^p h = d_3^p + (-1)^q \sum_{(2n-1)i<p} \binom{q}{i} X^{n(q-i)-p} Y Z^{(2n-1)i} a^{q-i} b^{p-q} c^{2p+i}$$

for some $h \in \mathbf{p}^{(3p)}$. As $Z^p h \equiv d_3^p \bmod (X, c)$ and as X, c, and Z^p form an A-regular sequence, we get

$$h \equiv Z^{(7n-3)p} \mod (X, c)$$

so that $\ell_A(A/(X, c, h)) = 3p(7n-3) = 3p \cdot \ell_A(A/(X)+\mathbf{p})$. This proves by Proposition 2.2 the first assertion of Theorem 1.1. The ring $R_s(\mathbf{p})$ is Noetherian by Theorem 2.1 but not a Cohen-Macaulay ring by Theorem 2.3, because $A/(c)+\mathbf{p}^{(3)}$ is not Cohen-Macaulay by Proposition 4.4. This completes the proof of Theorem 1.1.

Proof of Corollary 1.2. Let us use the notation in §3 and assume that $\operatorname{ch} k = 0$ and that $R_s(\mathbf{p}_k)$ is a Noetherian ring. Then since $R_s(\mathbf{p}_{\mathbf{Q}})$ is a Noetherian ring too, by Theorem 3.1 we can choose positive integers N and m so that if p is a prime number and if $p \geq N$, there exist two elements f and g of $\mathbf{p}_k^{(m)}$ satisfying the equality $\ell_{A_k}(A_k/(X, f, g)A_k) = m^2 \cdot (7n-3)$, where $k = \mathbf{Z}/p\mathbf{Z}$.

We choose a prime number p so that $p \geq \max\{N, 2m/3\}$ and choose an element h of $\mathbf{p}_k^{(3p)}$ so that $\ell_{A_k}(A_k/(X, c, h)) = 3p \cdot (7n-3)$, where $c = Y^3 - X^n Z^n$ in A_k (the second choice is possible by Theorem 1.1). Then both the pairs c, h and f, g satisfy Huneke's condition (2.1), whence we get by [2, Proposition (3.1)(2)] that

$$G_+ = \sqrt{(cT, hT^{3p})G} = \sqrt{(fT^m, gT^m)G}$$

where $G = G_s(\mathbf{p})$ and $G_+ = \sum_{i>0} G_i$. Consequently fT^m, gT^m is a G-regular sequence because so is the sequence cT, hT^{3p} by [2, Proposition (3.7)(3)]. Hence we have

$$(f, g) \cap \mathbf{p}_k^{(i)} = (f, g) \cdot \mathbf{p}_k^{(i-m)}$$

for all $i \in \mathbf{Z}$, while $(f, g) \supset \mathbf{p}_k^{(2m-1)}$ by [2, Proposition (3.4)]; thus, $\mathbf{p}_k^{(i)} = (f, g) \cdot \mathbf{p}_k^{(i-m)}$ for all $i \geq 2m-1$. This particularly implies $R_s(\mathbf{p}) = A[\{\mathbf{p}_k^{(i)} T^i\}_{1 \leq i < 2m-1}, fT^m, gT^m]$ so that we have $hT^{3p} \in A[\{\mathbf{p}_k^{(i)} T^i\}_{1 \leq i < 2m}]$. Thus $hT^{3p} \in A[\{\mathbf{p}_k^{(i)} T^i\}_{1 \leq i < 3p}]$, because $2m \leq 3p$ by our choice of p. Since $3 \nmid 7n-3$ by our standard assumption, we get by Corollary 2.6 that $R_s(\mathbf{p}) = A[\mathbf{p}T, \mathbf{p}^{(2)}T^2, \mathbf{p}^{(3)}T^3]$. Consequently one of the conditions stated in [4, Theorem (6.1)(2)] must be satisfied for the data $\alpha = n$, $\beta = 1$, $\gamma = n$, $\alpha' = 2n-1$, $\beta' = 2$, and $\gamma' = 2n-1$, which is obviously impossible. Hence $R_s(\mathbf{p}_k)$ cannot be a Noetherian ring, if $\operatorname{ch} k = 0$. This completes the proof of Corollary (1.2).

Remark 4.5. The same proof works for the following examples, too. Let $n \geq 5$ be an integer such that $3 \nmid 7n-10$ and $n \not\equiv -7 \mod 59$. Let $\mathbf{p} = \mathbf{p}(7n-10,\ 5n^2-7n+1,\ 8n-3)$. Then $R_s(\mathbf{p})$ is not a Noetherian ring, if $\operatorname{ch} k = 0$. The simplest example of this case is the prime ideal $\mathbf{p} = \mathbf{p}(25, 91, 37) = (X^{14} - YZ^7, Y^3 - X^5 Z^4, Z^{11} - X^9 Y^2)$ in $\mathbf{Q}[[X, Y, Z]]$. The corresponding matrix is given by

$$\begin{pmatrix} X^5 & Y^2 & Z^7 \\ Y & Z^4 & X^9 \end{pmatrix}.$$

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