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ISOPERIMETRIC INEQUALITIES FOR IMMERSED CLOSED SPHERICAL CURVES

JOEL L. WEINER

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ABSTRACT. Let $\alpha: S^1 \to S^2$ be a C^2 immersion with length L and total curvature K. If α is regularly homotopic to a circle traversed once then $L^2 + K^2 \geq 4\pi^2$ with equality if and only if α is a circle traversed once. If α has nonnegative geodesic curvature and multiple points then $L + K \geq 4\pi$ with equality if and only if α is a great circle traversed twice.

1. INTRODUCTION

The classical isoperimetric inequality for a C^2 embedded closed spherical curve $\alpha: S^1 \to S^2$ states that

$$L^2 + A^2 - 4\pi A \geq 0, \tag{1}$$

where L is the length of α and A is the area of either component of the complement of $\alpha(S^1)$. Let k and ds be the geodesic curvature and element of arc length induced on S^1 by α, respectively, and define the total curvature K by

$$K = \int k\, ds.$$

Using the Gauss-Bonnet Theorem, (1) may be rewritten as

$$L^2 + K^2 \geq 4\pi^2. \tag{2}$$

The left-hand side of (2) makes sense even if α is an immersion, i.e., has self-intersections. Since L^2+K^2 can be made arbitrarily close to 0 for curves whose configuration is a figure-eight, it will be necessary to impose further hypotheses on α in order to obtain a positive lower bound for $L^2 + K^2$.

Recall that any C^1 immersed curve is regularly homotopic to either a circle traversed once or to a circle traversed twice [7]. In fact, a C^1 immersed curve is regularly homotopic to a circle traversed once if and only if it has an even number of double points; of course, one may have to perturb the curve into general position before one counts the number of double points.

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Theorem 1. *Let $\alpha: S^1 \to S^2$ be a C^2 immersion which is regularly homotopic to a circle traversed once. Then $L^2 + K^2 \geq 4\pi^2$. Equality holds if and only if $\alpha(S^1)$ is a circle which α traverses once.*

This theorem was recently proved by Enomoto [1]; a new proof is presented here which we believe is more transparent than the one given by Enomoto. Both proofs use the torus T in S^3 which is the inverse image of $\alpha(S^1)$ under the Hopf map, but while Enomoto works with the image of the torus T in the Grassmannian $G_{2,4}$ under the Gauss map by viewing T as being in $\mathbb{E}^4$, we work directly with the T.

Another isoperimetric inequality like (2) is obtained for curves in S^2 whose geodesic curvature k does not change sign. By appropriately orienting the curve we may assume that $k \geq 0$. This inequality follows immediately from the next result.

Theorem 2. *Let $\alpha: S^1 \to S^2$ be a C^2 immersion with at least one multiple point for which $k \geq 0$. Then $L+K \geq 4\pi$ with equality if and only if $\alpha(S^1)$ is a great circle which α traverses twice.*

Corollary 1. *With the same hypotheses as in Theorem 2, $L^2 + K^2 > 8\pi^2$.*

The lower bound in Corollary 1 is the best possible. If z and $-z$ are a pair of antipodal points of S^2, then it is possible to construct an immersion $\alpha: S^1 \to S^2$ with nonnegative geodesic curvature whose configuration consists of a pair of half great circles running between z and $-z$ and two loops, one each at z and $-z$. By letting the angle between the half great circles approach zero and the lengths of the loops approach zero, $L^2 + K^2$ approaches $8\pi^2$.

Fenchel [3] suggested that for a closed space curve with curvature $\kappa > 0$ and torsion $\tau > 0$, there may be a lower bound greater than 2π for the sum of the total curvature and total torsion. It was shown that 4π is a lower bound, in fact, the best possible lower bound, even when the condition $\tau > 0$ is weakened to $\tau \geq 0$ and not identically zero [5, 8, 9]. This same result is an easy corollary of Theorem 2.

Corollary 2. *Let $\beta: S^1 \to \mathbb{E}^3$ be a C^3 nonplanar space curve with curvature $\kappa > 0$ and torsion $\tau \geq 0$. Then $\int \kappa\, d\sigma + \int \tau\, d\sigma > 4\pi$, where $d\sigma$ is the element of arc length induced on S^1 by β.*

Proof. Let α be the tangent indicatrix of β. Then α is a C^2 immersion into S^2 with at least one multiple point [2] and its geodesic curvature is nonnegative. Also, the sum $L+K$ for α equals the sum $\int \kappa\, d\sigma + \int \tau\, d\sigma$ for β, so the result follows directly from Theorem 2 since $\alpha(S^1)$ is not a great circle.

The proofs of Theorems 1 and 2 will be presented in the remainder of the paper.

2. The proof of Theorem 1

Let $z: [a, b] \to S^2$ be a parametrization of $\alpha(S^1)$; by this we mean that z extends to a C^2 periodic function on $\mathbb{R}$ with period $b-a$ which induces α when we view S^1 as $\mathbb{R}/(b-a)\mathbb{Z}$. Associated to z is a map $e: [a, b] \to \mathrm{SO}(3)$, where we view $\mathrm{SO}(3)$ as the set of all positively oriented orthonormal frames of $\mathbb{E}^3$. If $e = (e_0, e_1, e_2)$ then $e_0 = z$ and e_1 is the unit tangent vector field along z.

Let $\mathbb{H}$ denote the quaternions and identify $\mathbb{E}^3$ with the space of pure quaternions. Let S^3 and S^2 denote the unit spheres in $\mathbb{H}$ and $\mathbb{E}^3$, respectively. Recall the double covering $\pi: S^3 \to \mathrm{SO}(3)$ defined by

$$\pi(q)(p) = \bar{q}pq \quad \text{for all } q \text{ in } S^3 \text{ and all } p \text{ in } \mathbb{E}^3.$$

The Hopf map $h: S^3 \to S^2$ is defined as follows:

$$h(q) = \pi(q)(i) = \bar{q}iq \quad \text{for all } q \text{ in } S^3.$$

Of course, $h(e^{i\phi}q) = h(q)$ for all q in S^3 and all ϕ in $\mathbb{R}$. For the given C^2 immersion $\alpha: S^1 \to S^2$ it is known that $h^{-1}(\alpha(S^1))$ is a flat torus immersed in S^3 [6]. We will denote this torus by T.

We now proceed in the manner of Pinkall [6] to study the relation of T to $\alpha(S^1)$, but the reader should note that the Hopf map defined in [6] and the one in this paper are somewhat different. Let $\eta: [a, b] \to S^3$ be a lift of z through the Hopf map h which is orthogonal to the fibers of h. We now choose the parameter s in $[a, b]$ to represent arclength along η. Necessarily, $z = \bar{\eta}i\eta$ and if we denote differentiation with respect to s using a prime, then $\eta' = u\eta$, where $|u(s)| = 1$ and $u(s)$ is in $\mathrm{span}\{j, k\}$. Then $z' = 2\bar{\eta}iu\eta$ so that $|z'| = 2$; hence $b - a = L/2$. From now on, we let $a = 0$ and $b = L/2$. Also $z'' = 4(-z + \frac{1}{2}\bar{\eta}iu'\eta)$, where $u' = 2iku$ and $k(s) =$ the geodesic curvature of $\alpha(S^1)$ at $z(s)$. Thus

$$\tfrac{1}{2}z' = \bar{\eta}iu\eta \quad \text{and} \quad \tfrac{1}{4}z'' = -z + k\bar{\eta}(-u)\eta.$$

In particular,

$$e = (\bar{\eta}i\eta, \bar{\eta}iu\eta, \bar{\eta}(-u)\eta). \tag{3}$$

The Hopf torus T is parametrized by

$$x(s, \phi) = e^{i\phi}\eta(s).$$

Since $x_s(s, \phi) = e^{i\phi}u(s)\eta(s)$ and $x_\phi(s, \phi) = ie^{i\phi}\eta(s)$, the normal $\nu(s, \phi)$ to x at $x(s, \phi)$ is given by

$$\nu(s, \phi) = iu(s)e^{-i\phi}\eta(s).$$

Since $\nu_\phi = -x_s$, x_ϕ is an asymptotic direction field as is to be expected. Also

$$\nu_s = -2kx_s - x_\phi.$$

One may now easily check that $w = x_s - kx_\phi$ is the other asymptotic direction field. If $\lambda(s) = e^{i\phi(s)}\eta(s)$ is an asymptotic curve corresponding to w then one may show that $\phi'(s) = -k(s)$. In particular, one such λ is given by

$$\lambda(s) = \exp\left(-i\int_0^s k(t)\,dt\right)\eta(s). \tag{4}$$

Of course, λ is also a lift of z through the Hopf map h. If σ is arclength along z then

$$\frac{d\lambda}{d\sigma} = \frac{1}{2}x_s \circ \lambda - \frac{k}{2}i\lambda. \tag{5}$$

Note that x_s is a unit vector field orthogonal to the generators of T while $i\lambda$ is a unit vector field in the direction of these generators, and these generators form a field of (nonintersecting) geodesics in T.

Since $u' = 2iku$ it follows that

$$u(s) = \exp\left(2i\int_0^s k(t)\,dt\right)u(0). \tag{6}$$

From (3), (4), and (6) it is straightforward to show that

$$e = (\bar{\lambda}i\lambda,\ \bar{\lambda}iu(0)\lambda,\ \bar{\lambda}(-u(0))\lambda).$$

That is, we may view λ as a lift of e through the covering π. It is well known [7] that the lift λ of e begins and ends at antipodal points of S^3 if and only if α is regularly homotopic to a circle traversed once. Hence the distance d on T between the beginning and end of λ is at least π.

Let $\mathbb{E}^2$ be the universal Riemannian covering space of T. Lift λ to a curve Λ in $\mathbb{E}^2$. The great circular generators of T are covered by a field of parallel lines in $\mathbb{E}^2$. By (5), the velocity of Λ, parametrized with respect to the arclength of α, in the direction of this field and the direction orthogonal to this field is $-\frac{k}{2}$ and $\frac{1}{2}$, respectively. If D is the distance between the beginning and end of Λ then the Pythagorean theorem implies that $L^2 + K^2 = 4D^2$. But $D \geq d \geq \pi$ and the desired inequality follows.

Now assume equality holds, i.e., $L^2 + K^2 = 4\pi^2$. Then $D = d = \pi$. Choose a Euclidean coordinate system on $\mathbb{E}^2$ so that the origin is the beginning of Λ and the first coordinate axis maps onto a generator of T. With appropriately oriented axes the end of Λ has coordinates $(-\frac{K}{2}, \frac{L}{2})$. Hence the straight line segment Γ from the origin to $(-\frac{K}{2}, \frac{L}{2})$ has length π and projects to a geodesic segment γ of length π whose beginning and end are antipodes of S^3. Thus, γ is a half great circle of S^3. Also, it is clear that γ does not lie in a generator of T because Γ is not parallel to the first axis in $\mathbb{E}^2$. Since γ is necessarily an asymptotic line of T, $\lambda = \gamma$. Hence $\alpha(S^1)$ is a circle and it is traversed once by α since $L^2 + K^2 = 4\pi^2$.

3. The proof of Theorem 2

Give $\mathbb{E}^3$ an orientation. Let $\mathscr{G}$ and $\widetilde{\mathscr{G}}$ be the space of great circles and the space of oriented great circles on S^2, respectively. We, of course, identify $\mathscr{G}$ with $\mathbb{R}P^2$ and, using the orientation of $\mathbb{E}^3$, identify $\widetilde{\mathscr{G}}$ with S^2 in the usual fashion. Then let $\mathscr{F} = \{(G, p) \in \mathscr{G} \times \mathbb{R}P^2 : p \subset G\}$ and $\widetilde{\mathscr{F}} = \{(\widetilde{G}, z) \in \widetilde{\mathscr{G}} \times S^2 : z \in \widetilde{G}\}$, where we view p as a pair of antipodal points of S^2. We may identify $\widetilde{\mathscr{F}}$ with SO(3), viewed as the space of positively oriented orthonormal frames of $\mathbb{E}^3$, as follows: Assign $(\widetilde{G}, z) \in \widetilde{\mathscr{F}}$ to the frame $(e_0, e_1, e_2) \in \mathrm{SO}(3)$, where e_0 is the unit vector (orthogonal to the plane of $\widetilde{G}$) identified with $\widetilde{G}$ and $e_1 = z$. Using the bijection just described between $\widetilde{\mathscr{F}}$ and SO(3), we pull back the kinematic density $\omega_{10} \wedge \omega_{20} \wedge \omega_{21}$ on SO(3) to $\widetilde{\mathscr{F}}$. The forms ω_{ij} are defined by $de_j \cdot e_i = \omega_{ij}$, $0 \leq i, j \leq 2$. The map $(\widetilde{G}, z) \in \widetilde{\mathscr{F}} \to (G, \{z, -z\}) \in \mathscr{F}$ is a covering space whose deck transformations preserve the kinematic density. Thus, we push down this kinematic density to $\mathscr{F}$ and denote it by dF. Using locally defined forms ω_{ij} the kinematic density on $\mathscr{F}$ may still be written $\omega_{10} \wedge \omega_{20} \wedge \omega_{21}$. There are two natural fibrations on $\mathscr{F} \subset \mathscr{G} \times \mathbb{R}P^2$ coming from the product structure on $\mathscr{G} \times \mathbb{R}P^2$, $\phi_1 : \mathscr{F} \to \mathscr{G}$ and $\phi_2 : \mathscr{F} \to \mathbb{R}P^2$. The fibration ϕ_1 suggests that we write $dF = dG \wedge ds$,

where dG is the density of great circles and ds is the density of the set of antipodal pairs on a fixed great circle. The fibration ϕ_2 suggests that we write $dF = dp \wedge d\theta$, where dp is the density of points on $\mathbb{R}P^2$ and $d\theta$ is the density of great circles through a fixed pair of antipodes.

Define $n: \mathscr{F} \to \mathbb{Z}$ by $n(G, p) = n(G)$ is the number of points in $G \cap \alpha(S^1)$ counted with multiplicity, i.e., if $y \in G \cap \alpha(S^1)$ has k preimages under α, then y is counted k times. The function n is defined off a set of measure zero. Also define $m: \mathscr{F} \to \mathbb{Z}$ by $m(G, p) = m(p)$ is the number of great circles through p tangent to $\alpha(S^1)$ counted with multiplicity. If $\alpha(S^1)^*$ denotes the polar of $\alpha(S^1)$ [3], then $m(p)$ is the number of points in $G^* \cap \alpha(S^1)^*$ counted with multiplicity, where G^* is the great circle in the plane orthogonal to the line containing p. The function m is also defined off a set of measure zero.

Lemma 1. $2\pi(L + K) = \int_{\mathscr{F}} (n + m)\, dF$.

Proof. Just note

$$\int_{\mathscr{F}} (n + m)\, dF = \int_{\mathscr{F}} n\, dG \wedge ds + \int_{\mathscr{F}} m\, dp \wedge d\theta$$
$$= \pi \int_{\mathscr{G}} n\, dG + \pi \int_{\mathbb{R}P^2} m\, dp = 2\pi(L + K).$$

The last equality uses standard results about integral geometry on S^2 and the fact that the length of the polar $\alpha(S^1)^*$ is K.

Lemma 2. $n + m \geq 4$, *generically, in particular, on the set* $\mathscr{N}$ *of all* (G, p) *such that* G *has transversal intersections with* $\alpha(S^1)$ *and* $p \cap \alpha(S^1)$ *is empty.*

Proof. Clearly, we need only show that $n + m \geq 4$ for those (G, p) in $\mathscr{N}$ with $n(G) = 2$ or 0.

Suppose first that $n(G) = 2$. Note that this implies that $G \cap \alpha(S^1)$ contains no multiple points of α. Let H_1 and H_2 be the two closed hemispheres determined by G and say that H_1 contains multiple points of α. Pick an antipodal pair p such that $(G, p) \in \mathscr{N}$ and let $p = \{z, -z\}$. Suppose every half great circle in H_1 connecting z and $-z$ meets $\alpha(S^1)$. Then there must be at least two such half great circles tangent to $\alpha(S^1)$. For if there were no half great circles tangent to $\alpha(S^1)$ in H_1 then there would be no multiple points of α in H_1, and if there were only one half great circle tangent to $\alpha(S^1)$ in H_1 then there would be two "arcs" of α in $\alpha(S^1) \cap H_1$ meeting G in four points, counting multiplicity. If not every half great circle in H_1 connecting z and $-z$ meets $\alpha(S^1)$, then the same is the case for the half great circles in H_2 connecting z and $-z$. Necessarily there is a half great circle in each hemisphere connecting z and $-z$ tangent to $\alpha(S^1)$. Thus $n(G) + m(p) \geq 4$ for this $(G, p) \in \mathscr{N}$.

Now suppose that $n(G) = 0$ and let H be the open hemisphere determined by G containing $\alpha(S^1)$. Consider the central projection of H onto a plane P. Under this projection $\alpha(S^1)$ becomes a plane curve with nonnegative curvature and self-intersections. Since the plane curve is not simple, the total curvature of such a curve is at least 4π. Hence every height function in P has at least four critical points. This means that for each antipodal pair of points $p = \{z, -z\}$ such that $(G, p) \in \mathscr{N}$, there are, counting multiplicities, at least four half great circles in H connecting z and $-z$ that are tangent to $\alpha(S^1)$. Hence $n(G) + m(p) \geq 4$ for these $(G, p) \in \mathscr{N}$.

Proof of Theorem 2. The inequality in Theorem 2 follows immediately from Lemmas 1 and 2 once we observe that $\int_{\mathscr{F}} dF = 2\pi^2$. Now suppose $L + K = 4\pi$. First observe that there must exist some $(G, p) \in \mathscr{N}$ with $n(G) = 4$. Simply choose G to be a great circle through an antipodal pair of points that α winds around twice. Since this implies that $m(G, p) = 0$ for almost all $p \subset G$, almost all great circles are not tangent to $\alpha(S^1)$. Thus $\alpha(S^1)$ is a great circle clearly traversed twice by α.

Remark. An argument similar to the above will show that $L + K < 4\pi$ if α is an embedding with nonnegative geodesic curvature, i.e., $\alpha(S^1)$ is convex. This observation along with Theorem 2 shows the following: *Suppose α_t is a regular homotopy such that α_0 and α_1 have nonnegative geodesic curvature. If α_0 is an embedding and α_1 is not, then there must exist $t \in (0, 1)$ such that the geodesic curvature of α_t changes sign.* This somewhat strengthens a result of Little [4, Proposition 3].

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Department of Mathematics, University of Hawaii at Manoa, Honolulu, Hawaii 96822
E-mail address: joel@kahuna.math.hawaii.edu