LOWER BOUNDS FOR RELATIVE CLASS NUMBERS OF CM-FIELDS

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Abstract. Let $K$ be a CM-field that is a quadratic extension of a totally real number field $k$. Under a technical assumption, we show that the relative class number of $K$ is large compared with the absolute value of the discriminant of $K$, provided that the Dedekind zeta function of $k$ has a real zero $s$ such that $0 < s < 1$. This result will enable us to get sharp upper bounds on conductors of totally imaginary abelian number fields with class number one or with prescribed ideal class groups.

Let $K$ be a CM-field that is a quadratic extension of a totally real number field $k$.

If the Dedekind zeta function of $K$ is nonpositive at some $s_0$ that belongs to the interval $]0, 1[,$ then it is well known that we can get good lower bounds for the residue at $s = 1$ of this zeta function and for the relative class number of $K$. In Proposition A, we give explicit forms of such a result. They will enable us to consider in Corollary $c$ the class number one problem for cyclotomic fields in a more efficient way than those one can find in the literature (see $[7, 12]$).

Now, if the Dedekind zeta function of $k$ has a zero in $]0, 1[,$ then in Theorem 1 we give lower bounds for the relative class number of $K$. Our proof assumes the technical assumption $d(K) > 4N d(k)^2$ where $N$ is the degree of $k$ and where $d(k)$ and $d(K)$ are the absolute values of the discriminants of $k$ and $K$.

Let us stress that, under this previous technical assumption, one remarkable consequence of Theorem 1 is that the zeta function of $k$ has no real zeros in the open interval $]0, 1[,$ provided that the relative class number of $K$ is less than or equal to 2 (or provided that this relative class number is not “too large”). If we can deduce from this that the zeta function of $K$ is nonpositive on this interval, then from our previous lower bounds for the relative class numbers we may get very good upper bounds on the discriminants of $K$, provided that the relative class number of $K$ is less than or equal to 2 (or provided that this relative class number if not “too large”).

Our main application of these techniques is the proof in Corollary $b$ that the zeta function of the real quadratic subfield $k$ of a totally imaginary cyclic

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quartic number field $\mathbb{K}$ with ideal class group of exponent $\leq 2$ and discriminant $d(\mathbb{K})$ has no real zero in the interval $1 - 2/\log(d(\mathbb{K})) < s < 1$. This is the result we needed in [5] in order to prove that there are exactly 33 such quartic number fields.

Let $\mathbb{k}$ be an algebraic number field with class number $h(\mathbb{k})$ and regulator $\text{Reg}(\mathbb{k})$. Let $d(\mathbb{k})$ be the absolute value of the discriminant of $\mathbb{k}$. Set

$$[\mathbb{k}: \mathbb{Q}] = r_1 + 2r_2, \quad A = 2^{-r_2}d(\mathbb{k})^{1/2}e^{-r_1+r_2}/2,$$

$$\lambda(\mathbb{k}) = \frac{2^{r_1}h(\mathbb{k})\text{Reg}(\mathbb{k})}{w(\mathbb{k})}$$

where $w(\mathbb{k})$ is the number of roots of unity of $\mathbb{k}$,

$$F_k(s) = A\Gamma\left(\frac{s}{2}\right)^{r_1}\Gamma(s)^{r_2}\zeta(\mathbb{k}, s),$$

so that $F_k(s)$ has a simple pole at $s = 1$ with residue equal to $\lambda(\mathbb{k})$.

Whenever $y \in (\mathbb{R}^{*})^{r_1+r_2}$, we set

$$||y||_k = (y_1 \cdots y_{r_1} + y_{r_1+1} \cdots y_{r_1+r_2})^2,$$

$$\text{Tr}(y) = y_1 + \cdots + y_{r_1} + 2(y_{r_1+1} + \cdots + y_{r_1+r_2}),$$

so that $\text{Tr}(\lambda y) = \lambda y$.

It is well known (see [4]) that we have the integral representation

$$F_k(s) = \frac{\lambda(\mathbb{k})}{s(s-1)} + I_k(s)$$

with

$$(1) \quad I_k(s) = \sum \int_{||y|| \geq 1} \exp(-\pi d(\mathbb{k})^{-1/N}N_{\mathbb{k}/\mathbb{Q}}(\mathcal{B})^{2/N}\text{Tr}(y))[||y||^2 + ||y||^{1-s}/2] \frac{dy}{y},$$

where the sum is taken over all integral ideals $\mathcal{B} \neq 0$ of $\mathbb{k}$.

From now on, we assume that $s$ is a real number such that $1/2 \leq s < 1$.

For $y = (y_1, \ldots, y_N) \in (\mathbb{R}^{*})^N$ we set $\text{Tr}(y) = y_1 + \cdots + y_N$ and $||y|| = y_1 \cdots y_N$.

If $\mathbb{K}$ is a totally imaginary number field of degree $2N$ that is a quadratic extension of a totally real number field $\mathbb{k}$ of degree $N$, then $I_\mathbb{K}(s)$ and $I_\mathbb{k}(s)$ are integrals in $(\mathbb{R}^{*})^N$ and we have:

$$\text{Tr}_\mathbb{K}(y) = 2\text{Tr}(y)$$

and $||y||_\mathbb{K} = ||y||^2$, so that $||y||_\mathbb{K} \geq 1$ if and only if $||y|| \geq 1$;

$$\text{Tr}_\mathbb{k}(y) = \text{Tr}(y)$$

and $||y||_\mathbb{k} = ||y||$, so that $||y||_\mathbb{k} \geq 1$ if and only if $||y|| \geq 1$.

Moreover, we have the natural injection map $i_{\mathbb{K}/\mathbb{k}}$ from the group of fractional ideals of $\mathbb{k}$ in the group of fractional ideals of $\mathbb{K}$ that satisfies

$$N_{\mathbb{K}/\mathbb{Q}}(i_{\mathbb{K}/\mathbb{k}}(\mathcal{B}))^{2/2N} = N_{\mathbb{k}/\mathbb{Q}}(\mathcal{B})^{2/N}$$

whenever $\mathcal{B}$ is an integral ideal of $\mathbb{k}$. We thus get

$$(2) \quad I_\mathbb{K}(s) \geq \sum_i \int_{||y|| \geq 1} \exp(-2\pi d(\mathbb{K})^{-1/2N}N_{\mathbb{k}/\mathbb{Q}}(\mathcal{B})^{2/N}\text{Tr}(y))[||y||^s + ||y||^{1-s}] \frac{dy}{y},$$

and

$$(3) \quad I_\mathbb{k}(s) = \sum_i \int_{||y|| \geq 1} \exp(-\pi d(\mathbb{k})^{-1/N}N_{\mathbb{k}/\mathbb{Q}}(\mathcal{B})^{2/N}\text{Tr}(y))[||y||^{s/2} + ||y||^{(1-s)/2}] \frac{dy}{y},$$
where the sums are taken over all integral ideals $B \neq 0$ of $k$.

Hence, if we assume that we have $d(K) \geq 4^N d(k)^2$, noticing that we have $\|y\|^2 > \|y\|$ whenever $\|y\| > 1$, then (2) and (3) provide us with

(a) $I_K(s) > I_k(s)$.

Moreover, from (2) we have

$$I_K(s) \geq \sum_B \int_{\|y\| \geq 1} \exp(-2\pi d(K)^{-1/2} N_{k/Q}(B)^{2/N} Tr(y)) \|y\|^s \frac{dy}{y},$$

and from (3) we have

$$I_k(s) \leq 2 \sum_B \int_{\|y\| \geq 1} \exp(-\pi d(k)^{-1/2} N_{k/Q}(B)^{2/N} Tr(y)) \|y\|^{s/2} \frac{dy}{y},$$

where the sums are taken over all integral ideals $B \neq 0$ of $k$.

We change variables in (4), making the multiplicative translation $y = d(K)^{1/2} Y/2d(k)^{1/N}$. We note that, under the hypothesis $d(K) \geq 4^N d(k)^2$, the domain $\|y\| \geq 1$ is included in the domain $\|Y\| \geq 2^N d(k)/\sqrt{d(K)}$. Using $\|Y\|^s > \|y\|^{s/2}$ whenever $\|y\| > 1$, we get

(b) $I_K\left(\frac{1}{2}\right) > \left(\frac{d(K)}{4^N d(k)^2}\right)^{1/4} I_k\left(\frac{1}{2}\right),$

(c) $I_k(s) > \frac{1}{2} \left(\frac{d(K)}{4^N d(k)^2}\right)^{s/2} I_k(s), \quad \frac{1}{2} < s < 1.$

Now, as $K$ is a CM-field of degree $2N$ that is a quadratic extension of a totally real number field $k$ of degree $N$, it is well known that we have

$$\frac{\lambda(K)}{\lambda(k)} = \frac{h^*(K)}{\varphi(K)},$$

where $h^*(K)$ is the relative class number of $K$ and where $Q = 1$ or 2 (see [12, Theorem 4.12]). Moreover, if $\frac{1}{2} \leq s_0 < 1$ is a real zero of $\zeta_k$, then we have $\zeta_K(s_0) = 0$ since $K/k$ is normal, so that $F_k(s_0) = F_k(s_0) = 0$, so that

$$\frac{\lambda(K)}{\lambda(k)} = \frac{I_K(s_0)}{I_k(s_0)}.$$

Hence, we get the following theorem whose assertion (b) is much more precise than the one given in [6] (note that as soon as the totally real number field $k$ is fixed, then there are only finitely many totally imaginary number fields $K$ that are quadratic extensions of $k$ and such that $d(K) < 4^N d(k)^2$):

**Theorem 1.** Let $K$ be a CM-field of degree $2N$ that is a quadratic extension of a totally real number field $k$ of degree $N$. Let us suppose that we have $d(K) \geq 4^N d(k)^2$.

If the Dedekind zeta function of $k$ has a real zero $s_0$ such that $\frac{1}{2} \leq s_0 < 1$, then we have the three following lower bounds for the relative class number $h^*(K)$.
of $K$:

\begin{enumerate}
\item[(a)] $h^*(K) > Q_w(K) \geq 2$,
\item[(b)] $h^*(K) > \frac{Q_w(K)}{\sqrt{2^N d(k)}} d(K)^{1/4}$ if $s_0 = \frac{1}{2}$,
\item[(c)] $h^*(K) > \frac{1}{2} Q_w(K) \left( \frac{d(K)}{4^N d(k)^2} \right)^{s_0/2}$ if $\frac{1}{2} < s_0 < 1$.
\end{enumerate}

Hence, the zeta function of $k$ has no real zero in the interval $1 - 2/\log(d(K)) < s < 1$ provided that we have $h^*(K) \leq \sqrt{d(K)}/e_{2^N d(k)}$.

Remark. Theorem 1 does not apply to the class number one problem for cyclotomic fields, for $d(K) = d(k)^2$ whenever $K = Q(\zeta_n)$ with $n$ not a prime power, and $d(K) = pd(k)^2$ whenever $K = Q(\zeta_{p^e})$, with $p$ an odd prime. Nevertheless, in Corollary c, we will manage to consider the class number one problem for cyclotomic fields (with prime powers conductors). Theorem 1 does not apply to the class number one problem for totally imaginary biquadratic abelian number fields with group $(\mathbb{Z}/2\mathbb{Z})^2$, for $d(K) = d(k)^2$ whenever $K = Q(\sqrt{-p}, \sqrt{-q})$, $p$ and $q$ prime and congruent to 3 mod 4.

In fact, Theorem 1 applies nicely to class numbers problems for totally imaginary cyclic number fields with bounded degrees.

**Theorem 2.** Let $K$ be a CM-field of degree $2N$ that is a quadratic extension of a totally real number field $k$ of degree $N$. Let $\text{Res}_1(\zeta_K)$ be the residue at $s = 1$ of the Dedekind zeta function $\zeta_K$ of $k$. Let us suppose that the Dedekind zeta function $s \mapsto \zeta_K(s)$ satisfies

$$\Re \zeta_K(1 - 2/\log(d(K))) \leq 0.$$ 

Then, we have the following lower bounds for the relative class number of $K$:

$$h^*(K) \geq f(N, K) \frac{1 - 2Q_w(K)}{\text{Res}_1(\zeta_K) e^{(2\pi)^N \log(d(k))}},$$

with the two possible choices:

\begin{enumerate}
\item[(a)] $f(N, K) = 1 - \frac{2\pi Ne^{1/N}}{d(K)^{1/2N}}$
\item[(b)] $f(N, K) = \frac{2}{5} \exp \left(-\frac{2\pi N}{d(K)^{1/2N}}\right)$, whenever $N \geq 2$.
\end{enumerate}

**Proof.** We get the desired result from Proposition A thanks to

$$\frac{\text{Res}_1(\zeta_K)}{\text{Res}_1(\zeta_k)} = (2\pi)^N \sqrt{\frac{d(k)}{d(K)}} \frac{\lambda(K)}{\lambda(k)} = (2\pi)^N \frac{h^*(K)}{Q_w(K)} \sqrt{\frac{d(k)}{d(K)}}.$$

**Proposition A.** Let $K$ be a totally imaginary number field of degree $2N$. If its Dedekind zeta function $s \mapsto \zeta_K(s)$ is such that $\zeta_K(s_0) \leq 0$ for some $s_0$ real in $[\frac{1}{2}, 1[$, then we have the following effective lower bounds for the residue at $s = 1$ of this zeta function:

\begin{enumerate}
\item[(a)] $\text{Res}_1(\zeta_K) \geq (1 - s_0)d(K)^{(s_0-1)/2} \left\{ 1 - \frac{2\pi N}{d(K)^{s_0/2N}} \right\}$;
\item[(b)] $\text{Res}_1(\zeta_K) \geq \frac{2}{5} (1 - s_0)d(K)^{(s_0-1)/2} \exp \left(-\frac{2\pi N}{d(K)^{1/2N}}\right)$, whenever $N \geq 2$.
\end{enumerate}
Proof. From (1) where we use only the term of the sum corresponding to the ideal \( B \) equal to the ring of algebraic integers of \( K \), and where we disregard the term with \( \|y\|_{K}^{1-s/2} \), we get

\[
\frac{\sqrt{d(K)}}{(2\pi)^{N}} \frac{\text{Res}_{1}(z_{K})}{s_{0}(1-s_{0})} = \frac{\lambda(K)}{s_{0}(1-s_{0})} \geq \int_{\|y\| \geq 1} \exp(-2\pi d(K)^{-1/2}Tr(y))\|y\|_{K}^{s_{0}}dy/y.
\]

Setting \( y = d(K)^{1/2N}y \), we get

\[
\text{Res}_{1}(z_{K}) \geq s_{0}(1-s_{0})d(K)^{(s_{0}-1)/2} \int_{\|y\| \geq d(K)^{-1/2}} \exp(-2\pi Tr(Y))\|Y\|^{s_{0}}dY/Y = (1-s_{0})d(K)^{(s_{0}-1)/2}\left\{ f(s_{0}) - J_{K}(s_{0}) \right\}
\]

with

\[
f(s) = s \left[ \frac{\Gamma(s)}{(2\pi)^{s}} \right]^{N} \quad \text{and} \quad J_{K}(s) = s \int_{\|y\| \leq d(K)^{-1/2}} \exp(-2\pi Tr(Y))\|Y\|^{s}dY/Y.
\]

Since \( \{ Y ; \|Y\| \leq d(K)^{-1/2} \} \) is included in \( \{ Y ; \exists i \in \{1, \ldots, N\} / Y_{i} \leq d(K)^{-1/2N} \} \), we have (using \( e^{-2\pi y} < 1, y > 0 \))

\[
J_{K}(s) \leq Ns \left[ \frac{\Gamma(s)}{(2\pi)^{s}} \right]^{N-1} \int_{0}^{d(K)^{-1/2N}} e^{-2\pi y_{y}^{s}}dy/y \leq Nf(s) \left( \frac{2\pi}{s\Gamma(s)} \right)^{s}d(K)^{-s/2N}.
\]

Hence,

\[
\frac{\text{Res}_{1}(z_{K})}{(2\pi)^{N}} \geq (1-s_{0})d(K)^{(s_{0}-1)/2}f(s_{0}) \left\{ 1 - Ns \left( \frac{2\pi}{s\Gamma(s)} \right)^{s}d(K)^{-s_{0}/2N} \right\}.
\]

Since \( s \mapsto f(s) \) decreases on \( [0, 1[ \), we have \( f(s_{0}) \geq f(1) = (1/2\pi)^{N} \). Since \( s \mapsto (2\pi)^{s}/s\Gamma(s) \) increases on \( [0, 1[ \), we get the desired first result. In order to get the second desired result, we start from (6) and use the third point of the following lemma with \( x = d(K)^{-1/2N} \) (so that from the Minkowski's lower bound \( d(K)^{1/2N} \geq \pi N^{2}/((2N))^{1/2} \) we have \( x \leq \frac{1}{2}, 2N \geq 4 \):

**Lemma.** Set \( P_{N}(t) = \sum_{n=0}^{N-1} t^{n}/n! \). Then,

(i) \[
\int_{Tr(Y) \geq t} \exp(-Tr(Y))dY = P_{N}(t)e^{-t}, \quad N \geq 1.
\]

(ii) \[
\int_{Tr(Y) \leq t} \exp(-Tr(Y))dY = 1 - P_{N}(t)e^{-t}, \quad N \geq 1.
\]

(iii) \[
\int_{1 \geq \|y\| \geq x} \exp(-2\pi Tr(y))dy = \frac{e^{-2\pi Nx}}{(2\pi)^{N}}(1 - P_{N}(2\pi N(1-x))e^{-2\pi N(1-x)}).
\]

(iv) \( x_{N} = 1 - P_{N}(\pi N)e^{-\pi N} \) is an increasing sequence that converges towards 1, so that \( x_{N} \geq \frac{4}{3}, N \geq 2 \).

**Proof.** Part (iii) is proved from (ii) using the fact that the domain \( \{ y ; y \in (\mathbb{R}_{+}^{*})^{N}, y_{i} \geq x \text{ and } N \geq Tr(y) \} \) is included in the domain \( \{ y ; y \in (\mathbb{R}_{+}^{*})^{N} \text{ and } \|y\|_{K}^{1-s/2} \} \).
$1 \geq \|y\| \geq x^N$ and changing variables making the translation $y_i = x + Y_i/2\pi$.
Part (iv) follows from the inequality $P_{N+1}((N+1)\pi) \leq e^N P_N(N\pi)$. Indeed, we have

$$P_{N+1}((N+1)\pi) = \sum_{n=0}^{N} \frac{(\pi N)^n}{n!} \left( \frac{N + 1}{N} \right)^n \leq \left( \frac{N + 1}{N} \right)^N \sum_{n=0}^{N} \frac{(\pi N)^n}{n!}$$

$$\leq e \left( P_N(N\pi) + \frac{(N\pi)^N}{N!} \right) = e \left( P_N(N\pi) + \pi \frac{(N\pi)^{N-1}}{(N-1)!} \right)$$

$$\leq e(1 + \pi) P_N(N\pi). \quad \square$$

**Remark.** Theorems 1 and 2 apply nicely to the determination of CM-fields $K$ with "small" class numbers, provided that the fields $K$ are CM-fields that are quadratic extensions of totally real number fields $k$ such that $\zeta_K/\zeta_k$ is non-negative on $[0, 1]$. Indeed, Theorem 1 then implies that the zeta functions $\zeta_k$ have no real zero in the interval $1 - 2/\log(d(k)) \leq s < 1$. Hence, $\zeta_k(s_0) < 0$ and $\zeta_K(s_0) \leq 0$ with $s_0 = 1 - 2/\log(d(K))$, so that Theorem 2 provides us with good lower bounds for $h^*(K)$. Since we seek "small" class numbers, this will provide us with upper bounds on $d(K)$.

Let us point out that these assumptions "$\zeta_K/\zeta_k$ are nonnegative on $[0, 1]$" are satisfied as soon as the number fields $K$ are totally imaginary and cyclic over $\mathbb{Q}$ and such that 4 divides $[K: \mathbb{Q}] = 2N$, for $\zeta_K/\zeta_k$ is then a product of $L$-functions that come in conjugate pairs.

For example, we first give the following corollary, which greatly improves upon the upper bounds given in [1] or [10]:

**Corollary a.** Let $K$ be a cyclic quartic totally imaginary number field with conductor $f$ and class number $h(K)$. If $h(K) = 1$ then $f \leq 4500$. If $h(K) = 2$ then $f \leq 10000$.

Let $K$ be a cyclic octic totally imaginary number field with conductor $f$ and class number $h(K)$. If $h(K) = 1$ then $f = 32$ or $f$ is prime and $f \leq 3000$.

**Proof.** We only prove the first point. Let $k$ be the real quadratic subfield of $K$, let $f_k$ be the conductor of $k$, and let $L(s, \chi_{f_k})$ be the $L$-function of $k$. First, $f_k$ divides $f$, so that we have $f_k \leq f$. Moreover, $d(k) = f_k$ and $d(K) = f_k f^2$. Hence, $d(K)/4^2 d(k)^2 = f^2/16 f_k$ is greater than or equal to 1 as soon as we have $f \geq 16$. Hence, from Theorem 1(a) we deduce that the Dedekind zeta function of $k$ has no zero on the interval $[1/2, 1]$ as soon as $h^*(K) = 1$ or 2, provided that we have $f \geq 16$. Since the Dedekind zeta function of $K$ can be written $\zeta_K(s) = \zeta_k(s)|L(s, \chi_f)|^2$, $s \in [0, 1]$, we can apply Theorem 2. Since $\Res_1(\zeta_k) = L(1, \chi_{f_k}) \leq 1/2 \log(f_k) + 1 \leq 1/2 \log(f) + 1$ (see [8, Lemma 8.4]) and since $5 f^2 \leq d(K) \leq f^3$, Theorem 2 provides us with the following lower bound from which we get the desired results:

$$h(K) \geq h^*(K) \geq \frac{2}{3 e^2} \left( 1 - \frac{4 \pi e^{1/2}}{(5 f^2)^{1/4}} \right) \frac{f}{\log(f) + 2} \log(f). \quad \square$$

**Corollary b.** Let $K$ be a cyclic quartic totally imaginary number field with conductor $f$. Then the Dedekind zeta function of the real quadratic subfield $k$ of $K$ has no zero in the interval $[1 - 2/\log(d(K)), 1]$ provided that the ideal class group of $K$ is of exponent $\leq 2$. 

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Proof. If the ideal class group of $K$ is of exponent $\leq 2$, then $k$ is principal and $f_k = 8$ or $f_k$ is prime and such that $f_k \equiv 1 \pmod{4}$. Conversely, if $f_k \equiv 1 \pmod{4}$ is prime, if $k$ is principal, and if we define $f_2$ by means of $f = f_k f_2$, then the ideal class group of $K$ has 2-rank $t - 1$ where $t$ is the number of prime ideals that ramify in the quadratic extension $K/k$. Hence, $t \leq 1 + 2\omega(f_2)$ where $\omega(f_2)$ is the number of prime divisors of $f_2$ (the proofs of these assertions can be found in [5]). Let us suppose that the Dedekind zeta function of $k$ had a real zero $s_0$ such that $1 - 2/\log(d(K)) \leq s_0 < 1$. Then, as $d(K) = f_k f_2^2 = f_k^3 f_2^2$ and $d(k) = f_k$, Theorem 1 would imply $4^{\omega(f_2)} \geq h(K) \geq h^*(K) \geq \sqrt{f_k f_2}/4^e$.

Now, $f_k \geq 211$ implies $\sqrt{f_k}/4^e > 4/3$, so that we would have $4^{\omega(f_2)} > 4 f_2/3$. Since 4 divides $f_2$ as soon as $f_2$ is even, this inequality is never satisfied. On the other hand, if $5 \leq f_k \leq 211$, then $s \mapsto L(s, \chi_k)$ has no zero on $]0, 1[$ (see [9]). Thus, we get the desired result. □

Lower bounds for the relative class numbers of cyclotomic fields. Now, we would like to show that Theorem 2 applies to CM-fields with unbounded degrees. For example, we show that Theorem 2(b) enables us to get good upper bounds on the conductors of the cyclotomic fields (with prime-power conductors) with relative class numbers equal to 1. We first give a less tedious proof and more precise form of Lemma 11.5 of Washington [12]; i.e., we give an upper bound on $\text{Res}_1(k)$ with $k$ being the maximal totally real subfield of a cyclotomic field with prime power conductor.

We define $g(b) = b - 1 + H(b, 1)$, where

$$H(b, s) = \sum_{n \geq 0} \left( \frac{1}{(n+b)^s} - \frac{1}{(n+1)^s} \right), \quad \begin{cases} \Re(s) > 0, \\ b > 0. \end{cases}$$

Whenever $\chi : \mathbb{N} \mapsto \mathbb{C}$ is a complex-valued function which is periodic mod $m$, such that $\chi(m) = 0$ and $\sum_{a=1}^{m-1} \chi(a) = 0$, we have

$$L(s, \chi) = \sum_{n \geq 1} \frac{\chi(n)}{n^s} = \frac{1}{m^s} \sum_{a=1}^{m-1} \chi(a) H \left( \frac{a}{m}, s \right).$$

Consequently, whenever $\chi_m$ is a nontrivial (not necessarily primitive) even Dirichlet character mod $m$ we have $\sum_{a=1}^{m-1} a \chi_m(a) = 0$ and

$$L(1, \chi_m) = \frac{1}{m} \sum_{a=1}^{m-1} \chi(a) g \left( \frac{a}{m} \right).$$

Lemma (i). $g(b) \geq 0$ and $g(b)^2 + g(b)g(1-b) \leq 1/b^2$, $0 < b < 1$.

Proof. Follows from the following two inequalities:

$$g(b) = b - 1 + \frac{b}{b} - 1 + \sum_{n \geq 1} \left( \frac{1}{n+b} - \frac{1}{n+1} \right) \geq b + \frac{1}{b} - 2 = \frac{(1-b)^2}{b} \geq 0,$$

$$g(b) = b - 1 + \frac{b}{b} - \sum_{n \geq 1} \frac{b}{n(n+b)} \leq b - 1 + \frac{1}{b} - \sum_{n \geq 1} \frac{b}{n(n+1)} = \frac{1}{b} - 1. \quad \Box$$

Lemma (ii). $| \prod_{\chi_m \text{ even}, \chi_m \neq 1} L(1, \chi_m) | \leq (\pi^2/6)^{(\phi(m)-2)/4}$, where the product is over the (not necessarily primitive) even Dirichlet characters mod $m$.
Proof.

\[
\sum_{\chi_m \text{ even}} \chi_m(a)\overline{\chi}_m(b) = \begin{cases} 
\phi(m)/2 - 1 & \text{if } a \equiv \pm b \pmod{m} \\
0 & \text{and } \gcd(a, m) = 1, \\
0 & \text{otherwise}.
\end{cases}
\]

Hence, by the arithmetic-geometric mean inequality,

\[
\left( \prod_{\chi_m \text{ even} \atop \chi_m \neq 1} |L(1, \chi_m)|^2 \right)^{2/(\phi(m) - 2)} \leq \frac{2m^{-2}}{\phi(m) - 2} \sum_{a=1}^{m-1} \sum_{b=1}^{m-1} g\left(\frac{a}{m}\right) g\left(\frac{b}{m}\right) \left( \sum_{\chi_m \text{ even} \atop \chi_m \neq 1} \chi_m(a)\overline{\chi}_m(b) \right)
\]

\[
\leq \frac{1}{m^2} \sum_{a=1}^{m-1} g\left(\frac{a}{m}\right)^2 + g\left(\frac{a}{m}\right) g\left(1 - \frac{a}{m}\right) \leq \frac{1}{a^2}
\]

\[
\leq \sum_{a=1}^{\infty} \frac{1}{a^2} = \frac{\pi^2}{6} \prod_{\rho \text{ prime} \atop \rho \text{ divides } m} \left(1 - \frac{1}{\rho^2}\right) \leq \frac{\pi^2}{6}. \quad \square
\]

Now, in order to apply Theorem 2 to the cyclotomic case, and thanks to the fact that the Dedekind zeta function of a CM cyclic number field factorises on \( \mathbb{L} \) into a product of \( L \)-functions that come in conjugate pairs, apart from the two \( L \)-functions associated to the principal character and to some quadratic character, we must find an explicit zero-free region for an \( L \)-function associated to a quadratic character.

Lemma (iii). Let \( \chi \) be a quadratic nonprincipal character \( \pmod{f} \). Set \( N = \frac{1}{2}\phi(f) \). Then, for \( \sigma \geq 1/\log(3) \) we have

\[
|L'(\sigma, \chi)| \leq \sum_{n=2}^{N+2} \frac{\log(n)}{n^\sigma} \leq (N + 2)^{1-\sigma} \sum_{n=2}^{N+2} \frac{\log(n)}{n}.
\]

Proof. We have

\[
L'(\sigma, \chi) = -\chi(2) \frac{\log(2)}{2^\sigma} - \sum_{k \geq 0} \left( \frac{(k+1)f+2}{\sum_{n=kf+3}^{(k+1)f+3}} \chi(n) \frac{\log(n)}{n^\sigma} \right).
\]

Now, \( n \mapsto \log(n)/n^\sigma \) decreases for \( n \geq 3 \) provided that we have \( \sigma \geq 1/\log(3) \). Moreover, in each set of \( f \) consecutive integers there are \( N \) of them such that \( \chi(n) = -1 \) and \( N \) of them such that \( \chi(n) = +1 \). Hence, for each \( k \geq 0 \) we have

\[
\left| \sum_{n=kf+3}^{(k+1)f+2} \chi(n) \frac{\log(n)}{n^\sigma} \right| \leq u_k - v_k,
\]
with
\[ u_k = \sum_{n = k^f + 3}^{k^f + N + 2} \frac{\log(n)}{n^\sigma} \quad \text{and} \quad v_k = \sum_{n = (k + 1)^f + 3 - N}^{(k + 1)^f + 2} \frac{\log(n)}{n^\sigma}. \]

Since \((u_k)_{k \geq 0}\) and \((v_k)_{k \geq 0}\) are decreasing sequences converging towards 0, and since \(u_k + 1 \leq v_k\), \(k \geq 0\), we get
\[
|L'(\sigma, \chi)| \leq \frac{\log(2)}{2^\sigma} + u_0 - (u_0 - u_1) - (u_1 - u_2) + \cdots
\]
\[
\leq \frac{\log(2)}{2^\sigma} + u_0 = \sum_{n = 2}^{N + 2} \frac{\log(n)}{n^\sigma}.
\]

**Theorem 3** (see [12, Lemma 11.10]). Let \(\chi\) be a primitive quadratic character of conductor \(f\). Then
\[
L(\sigma, \chi) \geq 0 \quad \text{for} \quad \sigma \leq \sigma_0 = 1 - \frac{2}{\sqrt{f} \log(f)} \quad \text{if} \quad \chi(-1) = +1,
\]
\[
L(\sigma, \chi) \geq 0 \quad \text{for} \quad \sigma \leq \sigma_1 = 1 - \frac{2\pi}{\sqrt{f} \log(f)} \quad \text{if} \quad \chi(-1) = -1.
\]

Hence, \(L(\sigma, \chi) \geq 0\) for \(\sigma \geq \sigma_1 = 1 - 2/(f - 2) \log(f)\).

**Proof.** Since \(\sigma \mapsto L(\sigma, \chi)\) has no real zero in the open interval \([0, 1]\) for \(f \leq 24\) (see [9]), we may assume that we have \(f \geq 24\). Let us first assume that \(\chi\) is even, and let \(k_2\) be the real quadratic field with conductor \(f\). Then
\[
L(1, \chi) = 2h \log(e_0)/\sqrt{f} \geq \log(f - 4)/\sqrt{f} \quad \text{where} \quad e_0 = \sqrt{f - 4} + \sqrt{f}/2 \geq \sqrt{f - 4} \quad \text{is the fundamental unit of} \quad k_2 \quad \text{and where} \quad h \geq 1 \quad \text{is the class number of} \quad k_2.
\]
Let \(\sigma\) be such that \(\sigma_0 \leq \sigma \leq 1\). Then \(L(\sigma, \chi) \geq 0\). Indeed, if we had \(L(\sigma, \chi) < 0\), then from Lemma (iii) above and since we have \(N \leq (f - 1)/2\), we would get a contradiction from
\[
\frac{\log(f - 4)}{\sqrt{f}} \leq L(1, \chi) = L(1, \chi) - L(\sigma, \chi)
\]
\[
\leq (1 - \sigma) \max_{\sigma_0 \leq \sigma \leq 1} L'(\sigma, \chi)
\]
\[
\leq (1 - \sigma_0) \exp\left(\frac{2 \log((f + 3)/2)}{\sqrt{f} \log(f)}\right) \sum_{2 \leq n \leq (f + 3)/2} \frac{\log(n)}{n}
\]
\[
\leq (1 - \sigma_0) \frac{1}{2} \log(f - 4) \log(f) = \frac{\log(f - 4)}{\sqrt{f}}
\]
where the last inequality is valid for \(f \geq 24\).

In the same way, we get the desired result if \(\chi\) is odd using \(L(1, \chi) \geq \pi/\sqrt{f}\), \(f \geq 5\).

The third result follows from the first and second ones. \(\square\)

**Corollary c** (see [12, Corollary 11.17]). Let \(p\) be an odd prime. Then we have the following lower bound for the relative class number \(h^*(K)\) of the cyclotomic field \(K = \mathbb{Q}(\zeta_{p^a})\), \(a \geq 1\), of degree \(2N = [K : \mathbb{Q}] = \phi(p^a)\):
\[
h^*(K) \geq \frac{1}{128} \left(\frac{N}{39}\right)^{N/2} \frac{1}{\log(2N)}.
\]
so that $2N = \phi(p^a) \geq 100$ implies $h^*(K) > 1$. Moreover, $p \geq 89$ implies $h^*(K) > 1$.

**Proof.** Set $h(p) = (1 - 1/p)p^{1/(p-1)}$, so that we have $h(5) \geq h(p) \geq 1$. We note that
\[
d(K) = (2N/h(p))^{2N} \leq (2N)^{2N}, \quad w(K) = 2p^a \geq 2N,
\]
\[
d(K)/d(k) = \sqrt{pd(K)} = \sqrt{p}(2N/h(p))^N,
\]
\[
\text{Resi}(\zeta_k) \leq (\pi^2/6)^{(N-1)/2} \quad \text{[thanks to Lemma (ii)].}
\]
Noticing that we have $d(K) \geq p^{p-2}$, then thanks to Theorem 3 we may apply Theorem 2(b), so that we get the following lower bound from which we get the desired results:
\[
h^*(K) \geq \frac{2\pi\sqrt{6}}{15e} p^{1/4} e^{-\pi h(p)} \left( \frac{3N}{\pi^4 h(p)} \right)^{N/2} \frac{1}{\log(2N)}.
\]

**REFERENCES**