PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 120, Number 2, February 1994

A COMPACTUM THAT CANNOT BE AN ATTRACTOR OF A SELF-MAP ON A MANIFOLD

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(Communicated by James E. West)

ABSTRACT. A one-dimensional compactum (in fact, a certain solinoid) is constructed, such that there does not exist a self-map on a manifold having this compactum as attractor.

1. THE COUNTEREXAMPLE

In [2] it is shown that a finite-dimensional compactum can be an attractor of a flow on a manifold if and only if it has the shape of a finite polyhedron. Here a flow is a continuous mapping $X \times \mathbb{R} \to X$ satisfying the usual functional equation [5, Chapter 4, §7, Theorem 12], and X is a topological manifold. If $\mathbb{R}$ is replaced by the integers $\mathbb{Z}$ we simply get a cyclic group of homeomorphisms, generated by some element $f : X \approx X$. We may generalize still further and consider arbitrary self-maps $f : X \to X$ instead of homeomorphisms. A compact subset $A \subseteq X$ with $f(A) = A$ is an attractor of f if there is a neighborhood U of A in X, such that $f(U) \subseteq U$ and for each neighborhood V of A in X there is $n \in \mathbb{N}$ with $f^n(U) \subseteq V$; this implies $f^m(U) \subseteq V$ for all $m \geq n$. In this situation the classification of attractors seems to be much more complicated than in [2], but at least we are able to produce an example of a one-dimensional compactum that can never be an attractor of a self-map.

It is well known that the *dyadic* solenoid is an attractor of a homeomorphism of a three-dimensional manifold (see [4; 2, Example 3]). The dyadic solenoid is the limit of an inverse sequence of circles $S^1_{(n)} = \{z \in \mathbb{C} \mid |z| = 1\}$, and the bonding map $g_n : S^1_{(n+1)} \to S^1_{(n)}$ is given by $g_n(z) = z^2$. Of course, we might as well choose different bonding maps such as $g_n(z) = z^{\nu_n}$, where $(\nu_n)_{n \in \mathbb{N}}$ is an arbitrary sequence of integers. The compacta obtained in this way are *generalized* solenoids.

Theorem 1. *The generalized solenoid obtained from a sequence of pairwise relatively prime integers ν_n cannot be an attractor of a self-map on a manifold.*

Received by the editors July 7, 1992.
1991 *Mathematics Subject Classification*. Primary 55P55, 57N25, 54H20, 58F12.
Key words and phrases. Dynamical systems, attractors, shape, solenoid.

For the proof the following lemma is needed.

Lemma 1. *We suppose A is an attractor of $f : X \to X$ and denote by $\varphi := \check{H}^* (f_{|A}) : \check{H}^*(A) \to \check{H}^*(A)$ the induced homomorphism of Čech cohomology groups with coefficients in $\mathbb{Z}$. Then $\check{H}^*(A)$ contains a finitely generated, φ-invariant subgroup G with $\bigcup_{n=1}^{\infty} \varphi^{-n}(G) = \check{H}^*(A)$.*

Proof. Let U be a domain of attraction as above. A is the inverse limit of a sequence of finite polyhedra P_n with projections $g_n : A \to P_n$ [3, Chapter I, §5.2, Corollary 4]. The inclusion map $i : A \hookrightarrow U$ can be factored over one of these g_n up to homotopy [3, Chapter I, §5.3, Theorem 9], because the interior of U is a manifold and hence an ANR. We may assume $n = 1$: $i \simeq hg_1$ for a suitable map $h : P_1 \to U$ and set

$$G' := \operatorname{im} \check{H}^* (g_1) \quad \text{and} \quad G := \bigcap_{m=0}^{\infty} \varphi^{-m} (G').$$

Any given element $\xi \in \check{H}^*(A)$ can be written $\xi = \check{H}^* (g_k) (\eta)$ for some k [3, p. 128]. There is a neighborhood V of A in X and a map $g'_k : V \to P_k$ extending g_k. Let n be such that $f^m(U) \subseteq V$ for all $m \geq n$. Then $g'_k f^m h g_1 : X \to P_k$ is homotopic to $g_k f^m$ and consequently

$$\varphi^m(\xi) = \check{H}^* (g_k f^m) (\eta) = \check{H}^* (g_1) \check{H}^* (g'_k f^m h) (\eta) \in G'.$$

This means $\varphi^m \varphi^n(\xi) \in G'$ for all $m \geq 0$ and therefore $\varphi^n(\xi) \in G$. □

Proof of Theorem 1. We suppose the contrary and denote by H the first Čech cohomology group of our solenoid; it is isomorphic to the subgroup of $\mathbb{Q}$ consisting of all numbers $m/\nu_1 \cdots \nu_n$ with $m \in \mathbb{Z}$, $n \in \mathbb{N}$. Let G be the subgroup of H from Lemma 1; it must be cyclic with a generator $M/\nu_1 \cdots \nu_N$. The homomorphism $\varphi : H \to H$ must be of the form $\varphi(r) = rp/q$. We choose a number $k > N$ such that ν_k is relatively prime to p. Then for all $m \geq 0$ we have $\varphi^m (1/\nu_k) \notin G$, contradicting Lemma 1. □

Remark. The subgroup G in Lemma 1 behaves like an "attractor" of the endomorphism φ: each element of $\check{H}^*(A)$ is eventually moved into G. This puts restrictions on the map $f_{|A} : A \to A$; in the case of the dyadic solenoid φ must be multiplication by a number 2^m with $m > 0$. Observing $\check{H}^1(A) = \mathbf{HTop}(A, S^1)$ one can deduce that in the shape category $f_{|A}$ coincides with ϑ^m, where ϑ is the shift map of the dyadic solenoid [4, Lemma 29].

Note. For expanding attractors of diffeomorphisms Theorem 1 can also be obtained by using Williams's characterization [6] and the classification of solenoids [1, p. 154].

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