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AURÉOLE OF A QUASI-ORDINARY SINGULARITY

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ABSTRACT. The auréole of an analytic germ $(X, x) \subset (\mathbb{C}^n, 0)$ is a finite family of subcones of the reduced tangent cone $|C_{X,x}|$ such that the set $D_{X,x}$ of the limits of tangent hyperplanes to X at x is equal to $\bigcup(\operatorname{Proj} C_\alpha)^\vee$. The auréole for a case of quasi-ordinary singularity is computed.

1. INTRODUCTION

When they studied the limits of tangent spaces to an analytic space, Lé and Teissier introduced the notion of auréole. Let $(X, x) \subset \mathbb{C}^n$ be a germ of analytic space. There exists a finite family $\{C_\alpha\}$ of subcones of the reduced tangent cone $|C_{X,x}|$ such that the set $D_{X,x}$ of the limits of tangent hyperplanes to X at x is equal to $\bigcup(\operatorname{Proj} C_\alpha)^\vee$. This family is called the auréole of (X, x). The auréole is an important geometric object. In this paper we will compute the auréole for a case of quasi-ordinary singularity.

A quasi-ordinary singularity is an analytic germ $(V, 0)$ of dimension d which admits a finite map (i.e., proper with finite fibers) of analytic germs $\pi : (V, 0) \to (\mathbb{C}^d, 0)$ whose discriminant locus D (the hypersurface in $\mathbb{C}^d$ over which π ramifies) has only normal crossings as singularities. In the hypersurface case, every quasi-ordinary singularity $(V, 0)$ can be parametrized by a fractional power series

$$\zeta = H(X_1^{1/n}, \dots, X_d^{1/n}) = \sum c_\alpha X_1^{\alpha_1/n} \cdots X_d^{\alpha_d/n}$$

(H a power series) in the sense that $(V, 0)$ is the image of the map $\Phi : U \to \mathbb{C}^{d+1}$ (U some neighborhood of 0 in $\mathbb{C}^d$) given by

$$\Phi(x_1, \dots, x_d) = (x_1^n, \dots, x_d^n, H(x_1, \dots, x_d)), \tag{1}$$

and $(V, 0)$ is equipped with a set of fractional monomials $\{X_1^{l_1/n} \cdots X_d^{l_d/n}\}$, called characteristic monomials, which is totally ordered by divisibility. These monomials determine quite a lot of the geometry and topology of $(V, 0)$. (For more details about quasi-ordinary singularity, see [2] or [3].)

The main result of this paper is (cf. Theorems 3.0.7, 3.0.10, and 3.0.14).

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Theorem. *Suppose the reduced discriminant locus $|D|$ is given by $X_1 \cdots X_e = 0$ and $X_1^{a_1/n} \cdots X_e^{a_e/n}$ is the smallest characteristic monomial. Then the auréole of $(V, 0) \subset (\mathbb{C}^{d+1}, 0)$ is determined by the following subcones of the reduced tangent cone $|C_{V,0}|$:*

(1) *if $n > a_1 + \cdots + a_e$, $C_I = \{(x_1, \ldots, x_d, z) \in \mathbb{C}^{d+1} \mid x_i = 0, i \in I\}$ for $I \subset \{1, 2, \ldots, e\}$ and $I \neq \varnothing$;*
(2) *if $n < a_1 + \cdots + a_e$, $C_I = \{(x_1, \ldots, x_d, z) \in \mathbb{C}^{d+1} \mid z = 0, x_i = 0, i \in I\}$ for $I \subset \{1, 2, \ldots, e\}$ such that $n > \sum_{i \in I} a_i$ or $I = \varnothing$;*
(3) *if $n = a_1 + \cdots + a_e$, the irreducible components of $C_{V,0}$.*

This result shows that the characteristic monomials determine the auréole of $(V, 0)$.

2. Auréole

Let $X \subset S \times U$ be a closed subspace with U an open set in $\mathbb{C}^n$ and $f: X \to S$ be the restriction of the first projection $S \times U \to S$ to X. Let $\mathscr{C}_f(X)$ be the closure in $S \times U \times \check{\mathbb{P}}^{n-1}$ of the set of couples (x, H) where $x \in X^\circ$ and H is the direction of a hyperplane in $\mathbb{C}^n$ containing the tangent space at x to the fiber of f. A point of $\mathscr{C}_f(X)$ is a couple (x, H) where $x \in X$ and H is a limit of hyperplanes in $\mathbb{C}^n$ tangent to the fibers of f at smooth points of the fibers. Let κ_f be the morphism induced by the projection $S \times U \times \check{\mathbb{P}}^{n-1} \to S \times U$. Then $\mathscr{C}_f(X)$ is called the relative conormal space of $f: X \to S$ and κ_f is called the relative conormal morphism. If S is a point, then we get the (absolute) conormal space $\mathscr{C}(X)$ and (absolute) conormal morphism κ. Note that $D_{X,x} = \kappa^{-1}(x)$ is the set of the limits of the tangent spaces to X at x.

Let $(X, x) \subset (\mathbb{C}^n, 0)$ be a analytic germ. Then we have the following normal/conormal diagram of (X, x):

$$\begin{array}{ccc} E_Y\mathscr{C}(X) & \xrightarrow{\tilde{e}} & \mathscr{C}(X) \\ {\scriptstyle \kappa'}\downarrow & & \downarrow{\scriptstyle \kappa} \\ E_Y X & \xrightarrow[e]{} & X \end{array}$$

where e is the blowing-up of x in X, $\tilde{e}$ is the blowing-up of $\kappa^{-1}(x)$ in $\mathscr{C}(X)$, and κ' is the morphism by the universal property of blowing-up. Let $\xi = \kappa \circ \tilde{e} = e \circ \kappa$, $|\xi^{-1}(x)| = \bigcup D_\alpha$ be the decomposition into irreducible components, and $V_\alpha = |\kappa'(D_\alpha)| \subset |e^{-1}(x)| = |\operatorname{Proj} C_{X,x}|$.

Definition 2.0.1. The collection $\{V_\alpha\}$ is called the *auréole* of X at x or the auréole of (X, x).

Let C_α be the corresponding cone of V_α in $C_{X,x}$. By abuse of language we also call C_α the auréole.

Let $\mathfrak{f}: \mathfrak{X} \to \mathbb{C}$ be the deformation to the normal cone $C_{X,x}$, $\kappa_{\mathfrak{f}}: \mathscr{C}_{\mathfrak{f}}(\mathfrak{X}) \to \mathfrak{X}$ the relative conormal morphism, and $q = \mathfrak{f} \circ \kappa_{\mathfrak{f}}: \mathscr{C}_{\mathfrak{f}}(\mathfrak{X}) \to \mathbb{C}$. We have the following result (cf. [4, 2.1.4.1]).

Proposition 2.0.1. *The cones C_α are the image in $\mathfrak{f}^{-1}(0) = C_{X,x}$ by $\kappa_{\mathfrak{f}}$ of the irreducible components of the fiber $q^{-1}(0) = \kappa_{\mathfrak{f}}^{-1}(C_{X,x})$.*

By definition, $\kappa_{\mathfrak{f}}^{-1}(C_{X,x})$ consists of the limits (q, ϕ) of $(p, H) \in \mathfrak{X}^\circ \times \check{\mathbb{P}}^{n-1}$ as p approaches $q \in C_{X,x} \times \{0\}$. p can approach q from inside the fiber $\mathfrak{f}^{-1}(0) = C_{X,x} \times \{0\}$ or from other fibers $\mathfrak{f}^{-1}(t)$ with $t \neq 0$. However, if (X, x) is a reduced hypersurface germ in $(\mathbb{C}^{d+1}, 0)$, we need only consider the second kind of limits by the following lemma.

Lemma 2.0.2. *Let $(X, 0)$ be a reduced hypersurface germ in $(\mathbb{C}^{d+1}, 0)$. Let $\mathfrak{f} : \mathfrak{X} \to \mathbb{C}$ be the deformation to the tangent cone and $\mathfrak{X}^\circ \subset \mathfrak{X} - \mathfrak{f}^{-1}(0)$ be an open dense set such that $\mathfrak{f}|\mathfrak{X}^\circ : \mathfrak{X}^\circ \to \mathbb{C}$ has smooth fibers. Then $\mathscr{C}(C_{X,0})$, the conormal space of $C_{X,0}$ identified with a subspace of $\mathbb{C}^{d+1} \times \mathbb{C} \times \check{\mathbb{P}}^d$ by the inclusion $\mathbb{C}^{d+1} \times \{0\} \times \check{\mathbb{P}}^d \hookrightarrow \mathbb{C}^{d+1} \times \mathbb{C} \times \check{\mathbb{P}}^d$, is contained in the closure of $\kappa_{\mathfrak{f}}^{-1}(\mathfrak{X}^\circ)$ in $\mathbb{C}^{d+1} \times \mathbb{C} \times \check{\mathbb{P}}^d$, where $\kappa_{\mathfrak{f}} : \mathscr{C}_{\mathfrak{f}}(\mathfrak{X}) \to \mathfrak{X}$ is the relative conormal morphism.*

Proof. Let

$$f(Z_1, \ldots, Z_{d+1}) = f_\nu(Z) + f_{\nu+1}(Z) + \cdots = 0$$

be the defining equation of $(X, 0)$, where the f_i are homogenous polynomials of degree i and f_ν is the initial form of f. The tangent cone $C_{X,0}$ is a hypersurface and is defined by $f_\nu(Z) = 0$. Then (cf. [4]) $\mathfrak{X} \subset \mathbb{C}^{d+1} \times \mathbb{C}$ and is defined by

$$T^{-\nu} f(Z) = f_\nu(Z) + T f_{\nu+1}(Z) \cdots = 0$$

and $C_{X,0}$ is defined by $f_\nu(Z) = 0$. Let $p = (z_1, \ldots, z_{d+1}) \in C_{X,0}$ be a smooth point. Since $C_{X,0}$ is a hypersurface, the tangent direction φ_p to $C_{X,0}$ at p is unique and $\varphi_p = (D_1 f_\nu, \ldots, D_{d+1} f_\nu)$ where $D_i = \partial/\partial z_i$.

We now show that (p, φ_p) is a limit of the points of $\kappa_{\mathfrak{f}}^{-1}(\mathfrak{X}^\circ)$. Let $\{t_n\} \subset \mathbb{C}^*$ be a sequence of nonzero numbers approaching 0 and $\mathfrak{X}_{t_n} = \mathfrak{f}^{-1}(t_n)$. Let $p_n \in \mathfrak{X}_{t_n} \cap \mathfrak{X}^\circ$ such that $p_n \to p$. The tangent direction to $\mathfrak{X}_{t_n}$ at p_n is

$$H_{p_n} = (h_{n,1} : \cdots : h_{n,d+1})$$

where $h_{n,i} = D_i f_\nu(z) + t_n D_i f_{\nu+1}(z) + \cdots$. Then $\lim_{n\to\infty} h_{n,i} = D_i f_\nu$ and so $\lim(p_n, H_{p_n}) = (p, \varphi_p)$. Therefore,

$$\Gamma = \{(p, \varphi_p) \mid p \in C^\circ_{X,0}\} \subset \overline{\kappa_{\mathfrak{f}}^{-1}(\mathfrak{X}^\circ)}.$$

Since $\mathscr{C}(C_{X,0})$ is the closure of Γ in $\mathbb{C}^{d+1} \times \{0\} \times \check{\mathbb{P}}^d$, it follows that $\mathscr{C}(C_{X,0}) \subset \overline{\kappa_{\mathfrak{f}}^{-1}(\mathfrak{X}^\circ)}$. □

The family $\{C_\alpha\}$ contains the irreducible components of $|C_{X,x}|$. In general it also contains much more. The cones in the family $\{C_\alpha\}$ which are not irreducible components of $|C_{X,x}|$ are called *exceptional cones.* But if (X, x) itself is a cone, there is no exceptional cones (cf. [1]).

Proposition 2.0.3. *If (X, x) itself is a cone, then*

$$D_{X,x} = \operatorname{Proj} |C_{X,x}|^\vee,$$

where $\operatorname{Proj} |C_{X,x}|^\vee$ *is the dual of* $\operatorname{Proj} |C_{X,x}|$. *So if (X, x) is a cone, then X has no exceptional cone at x.*

3. THE CASE OF A QUASI-ORDINARY SINGULARITY

Let $(V, 0) \subset (\mathbb{C}^{d+1}, 0)$ be a quasi-ordinary hypersurface singularity defined by a pseudopolynomial

$$f(Z) = Z^m + g_1(X)Z^{m-1} + \cdots + g_m(X)$$

where $g_i(X) = g_i(X_1, \ldots, X_d)$ are power series. We may assume that the quasi-ordinary projection $\pi : (V, 0) \to (\mathbb{C}^d, 0)$ is induced by the projection

$$p : (x_1, \ldots, x_d, z) \to (x_1, \ldots, x_d,).$$

Then $(V, 0)$ being quasi-ordinary means that the discriminant of f has the form

$$\Delta = X_1^{k_1} \cdots X_e^{k_e} u(X_1, \ldots, X_d), \qquad u(0, \ldots, 0) \neq 0,$$

for some $e \leq d$. Let $\zeta = H(X_1^{1/n}, \ldots, X_d^{1/n})$ be a parametrization of $(V, 0)$ with respect to π. We assume in this paper that the smallest characteristic monomial of ζ is $M = X_1^{a_1/n} \cdots X_e^{a_e/n}$, i.e., M contains the same variables X_i with those of Δ/u. Then we may assume that

$$\zeta = X_1^{a_1/n} \cdots X_e^{a_e/n} \varepsilon(X_1^{1/n}, \ldots, X_e^{1/n}, X_{e+1}, \ldots, X_d)$$

where ε is a unit (cf. [1, p. 17]). Let $K = \mathbb{C}((X_1, \ldots, X_d))$, the quotient field of $\mathbb{C}[[X_1, \ldots, X_d]]$. It can be shown that the initial form f_I of f is (cf. [2, Lemma 2.5])

$$f_I = \begin{cases} Z^m & \text{if } a_1 + \cdots + a_e > n, \\ (Z^t - \varepsilon_0^t X_1^{ta_1/n} \cdots X_e^{ta_e/n})^r & \text{if } a_1 + \cdots + a_e = n, \\ cX_1^{ma_1/n} \cdots X_e^{ma_e/n} & \text{if } a_1 + \cdots + a_e < n, \end{cases} \tag{2}$$

where $\varepsilon_0 = \varepsilon(0, \ldots 0)$, $m = [K(\zeta) : K]$, $t = [K(X_1^{a_1/n} \cdots X_e^{a_e/n}) : K]$, $r = [K(\zeta) : K(X_1^{a_1/n} \cdots X_e^{a_e/n})]$, and $c \in \mathbb{C}^*$.

Let $\mathfrak{f} : \mathfrak{X} \to \mathbb{C}$ be the deformation to the tangent cone $C_{V,0}$. Since $C_{V,0}$ is defined by $T^{-\nu} f(TX_1, \ldots, TX_d, TZ) = 0$ $(\nu = \operatorname{ord}(f_I))$, similar to (1), $\mathfrak{X} - \mathfrak{f}^{-1}(0)$ is the image of the map $\Phi : W - \{t = 0\} \to \mathbb{C}^{d+1} \times \mathbb{C}$ (W some neighborhood of 0 in $\mathbb{C}^{d+1}$) given by

$$\Phi(w_1, \ldots, w_d, t) = (w_1^n, \ldots, w_e^n, w_{e+1}, \ldots, w_d, \eta, t^n) \tag{3}$$

where $\eta = t^{a-n} w_1^{a_1} \cdots w_e^{a_e} \varepsilon(tw_1, \ldots, tw_e, t^n w_{e+1}, \ldots, t^n w_d)$ and $a = a_1 + \cdots + a_e$.

Let $\mathfrak{X}^\circ \subset \mathfrak{X}$ be the open dense subset of points where $w_1 \cdots w_e \neq 0$. Then the tangent to the fiber $\mathfrak{X}_t = \mathfrak{f}^{-1}(t)$ at $p = \Phi(w_1, \ldots, w_d, t) \in \mathfrak{X}_t^\circ = \mathfrak{X}^\circ \cap \mathfrak{X}_t$ for $t \neq 0$ is given by the direction $H_p = (h_1 : \cdots : h_{d+1})$ where

$$h_i = \begin{cases} \dfrac{t^{a-n} w_1^{a_1} \cdots w_e^{a_e}}{n w_i^n} (a_i \varepsilon + t w_i D_i \varepsilon), & 1 \leq i \leq e, \\ t^a w_1^{a_1} \cdots w_e^{a_e} D_i \varepsilon, & e < i \leq d, \\ -1, & i = d+1, \end{cases} \tag{4}$$

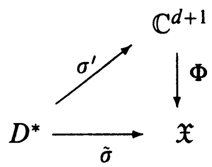
with $D_i = \partial/\partial z_i$ as before.

We are going to use Proposition 2.0.1 to compute the auréole for $(V, 0)$. For this purpose, we need a description of $\kappa_{\mathfrak{f}}^{-1}(C_{V,0})$. By definition and Lemma 2.0.2, $\kappa_{\mathfrak{f}}^{-1}(C_{V,0})$ consists of the limits of the pairs (p, H_p) as p approaches the points in $C_{V,0} \times \{0\} \subset \mathfrak{X}$ where $p = \Phi(w_1, \dots, w_d, t) \in \mathfrak{X}_t^\circ$ and $H_p = (h_1 : \cdots : h_{d+1})$ is a tangent direction to $\mathfrak{X}_t$ at p.

Lemma 3.0.4. *Let $(V, 0) \subset (\mathbb{C}^{d+1}, 0)$ be a quasi-ordinary singularity and $\mathfrak{f} : \mathfrak{X} \to \mathbb{C}$ be the deformation to the tangent cone $C_{V,0}$. Let $C \subset \mathfrak{X}$ be a curve parametrized by $\sigma : (D, 0) \to (\mathfrak{X}, p)$, D a disc in $\mathbb{C}$ centered at 0, such that*

$$\sigma(D - \{0\}) \subset \mathfrak{X} - \mathfrak{f}^{-1}(0), \quad \textit{and} \quad \sigma(0) = p \in \mathfrak{f}^{-1}(0) = C_{V,0}.$$

Then there exists a parametrization $\tilde{\sigma} : (D, 0) \to (\mathfrak{X}, p)$ of C and an analytic map $\sigma' : D^* \to \mathbb{C}^{d+1}$ such that the diagram

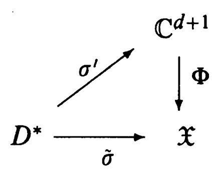


is commutative, where $D^* = D - \{0\}$ and Φ is as in (3).

Proof. Suppose $\sigma = (\sigma_1, \dots, \sigma_{d+2})$ where $\sigma_i(\tau) = a_i\tau^{\nu_i} +$ higher-order terms, $1 \le i \le d+2$. Define $\tilde{\sigma}(\tau) = \sigma(\tau^n)$. Then $\tilde{\sigma}_i = \tau^{n\nu_i}\varepsilon_i(\tau)$, $\varepsilon_i(0) \neq 0$, and the $\sqrt[n]{\varepsilon_i(\tau)}$ are analytic near $\tau = 0$. Define $\tau' : D^* \to \mathbb{C}^{d+1}$ by

$$\sigma_i'(\tau) = \begin{cases} \tau^{\nu_i}\sqrt[n]{\varepsilon_i(\tau)}, & 1 \le i \le e, \\ \tilde{\sigma}_i(\tau), & e < i \le d, \\ \tau^{\nu_{d+2}}\sqrt[n]{\varepsilon_{d+2}(\tau)}, & i = d+1, \end{cases}$$

where the branches of the $\sqrt[n]{\varepsilon_i(\tau)}$ are chosen in such a way that $\tilde{\sigma}_{d+1}(\tau) = \Phi_{d+1} \circ \sigma'(\tau)$ (Φ_{d+1} is the $(d+1)$ component of Φ). It follows that



is commutative. □

Let $(q, \varphi) \in \kappa_{\mathfrak{f}}^{-1}(C_{V,0})$. Then (q, φ) is a limit of (p, H_p) as $p \to q$ along a curve C in $\mathfrak{X}$. By Lemma 3.0.4, C is given by

$$(5) \quad \begin{cases} w_i = b_i\tau^{\nu_i} + \text{higher-order terms} & (b_i \neq 0,\ \nu_i \ge 0),\ 1 \le i \le d, \\ t = t_c\tau^{\nu_t} + \text{higher-order terms} & (t_c \neq 0,\ \nu_t > 0). \end{cases}$$

If $p = \Phi(w_1, \dots, w_d, t) \in C$, then the components of $H_p = (h_1 : \cdots : h_{d+1})$ have orders (cf. (4))

$$\text{(6)} \qquad \operatorname{ord}_\tau(h_j) \begin{cases} = (a-n)\nu_t + \left(\sum_{i=1}^e a_i\nu_i\right) - n\nu_j, & 1 \le j \le e, \\ \ge a\nu_t + \left(\sum_{i=1}^e a_i\nu_i\right), & e < j \le d, \\ = 0, & j = d+1. \end{cases}$$

Since $\lim H_p = \varphi$, $\varphi_j \ne 0$ if and only if $\operatorname{ord}_\tau(h_j) = \min_i\{\operatorname{ord}_\tau(h_i)\}$.

There are three cases.

Case I. $n > a = a_1 + \cdots + a_e$

Lemma 3.0.5. *If $n > a$, then $\varphi_{d+1} = 0$, $q_j\varphi_j = 0$ for $1 \le j \le e$, and $\varphi_j = 0$ for $e < j \le d$.*

Proof. Suppose $\varphi_{d+1} \ne 0$. Then in (6) $\operatorname{ord}_\tau(h_j) \ge \operatorname{ord}_\tau(h_{d+1}) = 0$ for each j. Let $\nu_k = \max_{1\le i\le e}\{\nu_i\}$. Then $\operatorname{ord}_\tau(h_k) \ge 0$ implies

$$(a-n)\nu_t + \sum_{i=1}^e a_i\nu_i \ge n\nu_k > \sum_{i=1}^e a_i\nu_k \ge \sum_{i=1}^e a_i\nu_i$$
$$> (a-n)\nu_t + \sum_{i=1}^e a_i\nu_i.$$

This contradiction shows that $\varphi_{d+1} = 0$.

Since $p \to q$ along C, $\lim_{\tau\to 0} p_{d+1} = \lim_{\tau\to 0} \Phi_{d+1} \circ \sigma'(\tau)$ exists; so, the order of $p_{d+1} = \eta$ (see (3)) along C satisfies

$$\operatorname{ord}_\tau(t^{a-n} w_1^{a_1} \cdots w_e^{a_e}) = (a-n)\nu_t + \sum_{i=1}^e a_i\nu_i \ge 0.$$

Now suppose $q_j \ne 0$ for some j, $1 \le j \le e$. Then $\nu_j = 0$ in (5) and

$$\operatorname{ord}_\tau(h_j) = (a-n)\nu_t + \sum_{i=1}^e a_i\nu_i \ge \operatorname{ord}_\tau(h_{d+1}) = 0.$$

Since $\varphi_{d+1} = 0$, $\varphi_j = 0$. Thus $q_j\varphi_j = 0$ for $1 \le j \le e$.

If $j > e$, then $\operatorname{ord}_\tau(h_j) \ge a\nu_t + \sum_{i=1}^e a_i\nu_i > 0 = \operatorname{ord}_\tau(h_{d+1})$ and so $\varphi_j = 0$. □

Proposition 3.0.6. *If $n > a$, the ideal J in $\mathscr{O}_{d+2}[Y_1, \dots, Y_d, Y_{d+1}]$ which defines $|\kappa_{\mathfrak{f}}^{-1}(C_{V,0})|$ in $\mathbb{C}^{d+1} \times \mathbb{C} \times \check{\mathbb{P}}^d$ is generated by $\{X_iY_i\}_{1\le i\le e}$, $\{Y_j\}_{e<j\le d+1}$, $X_1 \cdots X_e$, and T, where $\mathscr{O}_{d+2} = \mathbb{C}[[X_1, \dots, X_d, Z, T]]$.*

Proof. From (2) we know that if $n > a$, the reduced tangent cone $|C_{V,0}|$ is defined by $X_1 \cdots X_e = 0$. By Lemma 3.0.5,

$$J \subset \left(\{X_iY_i\}_{1\le i\le e}, \{Y_j\}_{e<j\le d+1}, X_1 \cdots X_e, T\right).$$

Conversely, let $(q, \varphi) \in \mathbb{C}^{d+1} \times \mathbb{C} \times \check{\mathbb{P}}^d$ such that $q_j\varphi_j = 0$, $1 \le j \le e$, $\varphi_j = 0$, $e < j \le d$, and $q_1 \cdots q_e = 0$. It suffices to show that (q, φ) is a limit of (p, H_p) as $p \to q$ along some curve C. Without loss of generality, we may assume that $q_1 = \cdots = q_c = 0$, $q_{c+1} \cdots q_e \ne 0$, $\varphi_1 \cdots \varphi_s \ne 0$, and $\varphi_{s+1} = \cdots = \varphi_e = 0$ where $1 \le s \le c \le e$. Choose positive integers $\nu_1, \nu_2, \dots, \nu_c, \nu_t$, complex numbers $b_1, b_2, \ldots, b_e, t_0$ such that $\nu_1 = \nu_2 = \cdots = \nu_s = \nu > \nu_i$ for $s < i \leq e$, and

$$\varphi_i = \frac{t_0 b_1^{a_1} \cdots b_e^{a_e}}{n b_i^n} a_i \varepsilon(0, \ldots, 0), \qquad 1 \leq i \leq c,$$

such that

$$(a-n)\nu_t + \sum_{i=1}^{e} a_i \nu_i > 0$$

if $q_{d+1} = 0$ and such that

$$(a-n)\nu_t + \sum_{i=1}^{e} a_i \nu_i = 0,$$

$$q_{d+1} = t_0^{a-n} b_1^{a_1} \cdots b_e^{a_e} \varepsilon(0, \ldots, 0)$$

if $q_{d+1} \neq 0$. Let C be the curve in $\mathfrak{X}$ given by

$$w_i = \begin{cases} b_i \tau^{\nu_i}, & 1 \leq i \leq c, \\ b_i, & c < i \leq e, \\ q_i, & e < i \leq d; \end{cases} \qquad t = t_0 \tau^{\nu_t}.$$

Then (p, H_p) approaches (q, φ) as $p \to q$ along C. □

Theorem 3.0.7. *If $n > a$, the auréole of $(V, 0)$ consists of $V_I = \operatorname{Proj} C_I$ where*

$$C_I = \{(x_1, \ldots, x_d, z) \in \mathbb{C}^{d+1} \mid x_i = 0,\ i \in I\}$$

for $I \subset \{1, 2, \ldots, e\}$ and $I \neq \varnothing$.

Proof. Let $P \subset \mathscr{O}_{d+2}[Y_1, \ldots, Y_d, Y_{d+1}]$ be a minimal prime over J ($\mathscr{O}_{d+2}$ and J are as in Proposition 3.0.6) homogeneous in Y. Since P is prime, $X_1 \cdots X_e \in J \subset P$ implies $X_j \in P$ for some j, $1 \leq j \leq e$. Also $X_i Y_i \in J \subset P$ implies X_i or $Y_i \in P$ for some i, $1 \leq i \leq e$. Therefore,

$$P_I = \left(\{X_i\}_{i \in I}, \{Y_j\}_{j \notin I}, T\right) \subset P$$

where $I \subset \{1, 2, \ldots, e\}$. It is clear that $J \subset P_I$. Since P is minimal over J and P_I is prime, $P_I = P$. These P_I's determine the irreducible components of $\kappa_{\mathfrak{f}}^{-1}(C_{V,0})$. The image of these irreducible components in $C_{V,0}$ are

$$C_I = \{(x_1, \ldots, x_d, z) \in \mathbb{C}^{d+1} \mid x_i = 0,\ i \in I\}, \qquad I \subset \{1, 2, \ldots, e\}.$$

By Proposition 2.0.1, the C_I determine the auréole of $(V, 0)$. □

Case II. $n < a = a_1 + \cdots + a_e$

Lemma 3.0.8. *Let $(q, \varphi) \in \kappa_{\mathfrak{f}}^{-1}(C_{V,0})$. If $n < a$, then $q_j \varphi_j = 0$ for $1 \leq j \leq e$, $\varphi_j = 0$ for $e < j \leq d$, and $\varphi_{j_1} \cdots \varphi_{j_k} = 0$ for $1 \leq j_1, \ldots, j_k \leq e$ such that $n \leq a_{j_1} + \cdots + a_{j_k}$.*

Proof. Since $\operatorname{ord}_\tau(h_j) > \operatorname{ord}_\tau(h_1)$ in (6) for $e+1 \leq j \leq d$, $\varphi_j = 0$.

Now assume $q_j \neq 0$, $1 \leq j \leq e$. Then $\nu_j = 0$ (cf. (5)) and

$$\mathrm{ord}_\tau(h_j) = (a-n) + \sum_{i=1}^{e} a_i \nu_i > \mathrm{ord}_\tau(h_{d+1}) = 0.$$

Thus $\varphi_j = 0$ and so $q_j \varphi_j = 0$.

Suppose $\varphi_{j_1} \cdots \varphi_{j_k} \neq 0$ and $n \leq a_{j_1} + \cdots + a_{j_k}$ where $1 \leq j_1, \ldots, j_k \leq e$. Then

$$\mathrm{ord}_\tau(h_{j_1}) = \cdots = \mathrm{ord}_\tau(h_{j_k}) = \min_i \{\mathrm{ord}_\tau(h_i)\}.$$

Let λ be this integer. Then it follows that $\lambda \leq \mathrm{ord}_\tau(h_{d+1}) = 0$. Since $\mathrm{ord}_\tau(h_i) = \mathrm{ord}_\tau(h_j)$ implies $\nu_i = \nu_j$ if $1 \leq i, j \leq e$, we have (cf. (6))

$$\nu_{j_1} = \cdots = \nu_{j_k} = \nu = \max_{1 \leq i \leq e} \{\nu_i\}.$$

Then

$$\begin{aligned}\lambda &= (a-n)\nu_t + \left(\sum_{i=1}^{e} a_i \nu_i\right) - n\nu \\ &= (a-n)\nu_t + \left(\sum_{i=1,\, i \neq j_l}^{e} a_i \nu_i\right) + (a_{j_1} + \cdots + a_{j_k} - n)\nu \geq (a-n) > 0.\end{aligned}$$

But we have shown that $\lambda \leq 0$. This contradiction shows that $\varphi_{j_1} \cdots \varphi_{j_k} = 0$. □

Proposition 3.0.9. *If $n < a$, then the ideal J in $\mathscr{O}_{d+2}[Y_1, \ldots, Y_d, Y_{d+1}]$ which defines $|\kappa_{\mathfrak{f}}^{-1}(C_{V,0})|$ in $\mathbb{C}^{d+1} \times \mathbb{C} \times \check{\mathbb{P}}^d$ is generated by $\{X_i Y_i\}_{1 \leq i \leq e}$, $\{Y_j\}_{e < j \leq d}$, Z, T, and $\{Y_{j_1} \cdots Y_{j_k}\}_{1 \leq j_1, \ldots, j_k \leq e,\, n \leq a_{j_1} + \cdots + a_{j_k}}$.*

Proof. Let N be the ideal generated by the elements as stated. We want to show that $J = N$. It is clear by Lemma 3.0.8 that $J \subset N$.

Conversely, let (q, φ) be in the zero locus of N. It suffices to show that $(q, \varphi) \in \kappa_{\mathfrak{f}}^{-1}(C_{V,0})$. Renumbering the variables if necessary, we may assume

$$\varphi = (\varphi_1 : \cdots : \varphi_c : 0 : \cdots : 0 : \varphi_{d+1})$$

and

$$q = (0, \ldots, 0, q_{r+1}, \ldots, q_d, 0, \ldots, 0)$$

where $\varphi_1 \cdots \varphi_c \neq 0$, $q_{r+1} \cdots q_d \neq 0$, and $c \leq r \leq e$. Then $n > a_1 + \cdots + a_c$. Similar to the proof of Proposition 3.0.6, we choose positive integers $\nu_1, \ldots, \nu_r, \nu_t$, nonnegative integers $\nu_{e+1}, \ldots, \nu_d$, and nonzero complex numbers $b_1, \ldots, b_d, t_0$ such that

$$\nu_1 = \cdots = \nu_c = \max_{1 \leq i \leq r} \{\nu_i\} = \nu > \nu_j \quad \text{for } j = c+1, \ldots, r;$$

$$(a-n)\nu_t + \left(\sum_{i=1}^{e} a_i \nu_i\right) - n\nu \begin{cases} < 0 & \text{if } \varphi_{d+1} = 0, \\ = 0 & \text{if } \varphi_{d+1} \neq 0; \end{cases}$$

$$b_i^n = q_i, \qquad r < i \leq e;$$

$$\lim b_i \tau^{\nu_i} = q_i, \qquad e < i \leq d;$$

and

$$\frac{t_0^{a-n} b_1^{a_1} \cdots b_e^{a_e}}{n b_i^n} a_i \varepsilon(0, \dots, 0) = \begin{cases} -\varphi_i/\varphi_{d+1} & \text{if } \varphi_{d+1} \neq 0, \\ \varphi_i & \text{if } \varphi_{d+1} = 0. \end{cases}$$

Let C be the curve in $\mathfrak{X}$ given by

$$w_i = \begin{cases} b_i \tau^{\nu_i}, & 1 \le i \le r \text{ or } e < i \le d, \\ b_i, & r < i \le e; \end{cases} \qquad t = t_0 \tau^{\nu_t}.$$

Then (p, H_p) approaches (q, φ) along C. Therefore, $(q, \varphi) \in \kappa_{\mathfrak{f}}^{-1}(C_{V,0})$. This completes the proof. □

Theorem 3.0.10. *If $n < a$, the auréole of $(V, 0)$ consists of $V_I = \operatorname{Proj} C_I$ where*

$$C_I = \{(x_1, \dots, x_d, z) \in \mathbb{C}^{d+1} \mid z = 0, x_i = 0, i \in I\}$$

for $I \subset \{1, 2, \dots, e\}$ such that $n > \sum_{i \in I} a_i$ or $I = \varnothing$.

Proof. Let P_I be the ideal in $\mathscr{O}_{d+2}[Y_1, \dots, Y_d, Y_{d+1}]$ generated by $\{X_i\}_{i\in I}$, $\{Y_j\}_{j \notin I}$, Z, and T for some $I \subset \{1, 2, \dots, e\}$ such that $n > \sum_{i\in I} a_i$. It is obvious that $X_i Y_i \in P_I$ for $1 \le i \le e$. If $n \le a_{j_1} + \cdots + a_{j_k}$, then some $j_l \notin I$ since $n > \sum_{i \in I} a_i$; thus, $Y_{j_1} \cdots Y_{j_k} \in (Y_{j_l}) \subset P_I$. Hence, $J \subset P_I$, where J is as in Proposition 3.0.9. It is also clear that P_I is prime and homogeneous in the Y_j. We will show that these P_I are the minimal primes over J and homogeneous in Y_j.

Now, let $P \supset J$ be any prime ideal homogeneous in Y_j. If $n \le a_{j_1} + \cdots + a_{j_k}$ for $1 \le j_1, \dots, j_k \le e$, then $Y_{j_1} \cdots Y_{j_k} \in J \subset P$; so, $Y_{j_l} \in P$ for some j_l. Considering all such monomials $Y_{j_1} \cdots Y_{j_k}$, we get

$$\left(\{X_{i_l} Y_{i_l}\}_{l=1,\dots,k}, \{Y_j\}_{j \neq i_l}, Z, T\right) \subset P.$$

We may assume that $n > a_{i_1} + \cdots + a_{i_k}$ or $k = 0$. If $n \le a_{i_1} + \cdots + a_{i_k}$, then $Y_{i_1} \cdots Y_{i_k} \in J \subset P$. Then $Y_{i_l} \in P$ for some i_l, say i_k, and then

$$\left(\{X_{i_l} Y_{i_l}\}_{l=1,\dots,k-1}, \{Y_j\}_{j \neq i_l}, Z, T\right) \subset P.$$

Repeating this procedure, we get a set $I' \subset \{1, 2, \dots, e\}$ with $n > \sum_{i \in I'} a_i$ or $I' = \varnothing$ such that

$$\left(\{X_i Y_i\}_{i \in I'}, \{Y_j\}_{j \notin I'}, Z, T\right) \subset P.$$

Since $X_i Y_i \in P$ implies $X_i \in P$ or $Y_i \in P$, there is a subset I of I' with $n > \sum_{i \in I} a_i$ or $I = \varnothing$ such that

$$\left(\{X_i\}_{i \in I}, \{Y_j\}_{j \notin I}, Z, T\right) = P_I \subset P.$$

We have shown that:

(1) $J \subset P_I$ for any $I \subset \{1, 2, \dots, e\}$ such that $n > \sum_{i \in I} a_i$ or $I = \varnothing$;
(2) If $P \supset J$ is prime, then $P_I \subset P$ for some such I.

It follows that the P_I are the minimal prime ideals over J homogeneous in Y. These P_I determine the irreducible components of $\kappa_{\mathfrak{f}}^{-1}(C_{V,0})$ in $\mathbb{C}^{d+1} \times \mathbb{C} \times \check{\mathbb{P}}^d$. The images of these components in $C_{V,0}$ (identified as a subspace of

$\mathbb{C}^{d+1} \times \mathbb{C} \times \check{\mathbb{P}}^d$) under $\kappa_{\mathfrak{f}}$ are

$$\{(x_1, \ldots, x_d, z, 0) \in \mathbb{C}^{d+2} \mid z = 0, x_i = 0, i \in I\}$$

for $I \subset \{1, 2, \ldots, e\}$ such that $n > \sum_{i \in I} a_i$ or $I = \varnothing$. Then

$$C_I = \{(x_1, \ldots, x_d, z) \in \mathbb{C}^{d+1} \mid z = 0, x_i = 0, i \in I\}$$

determine the auréole of $(V, 0)$ by Proposition 2.0.1. □

Case III. $n = a = a_1 + \cdots + a_e$

Lemma 3.0.11. *Let* $(q, \varphi) \in \kappa_{\mathfrak{f}}^{-1}(C_{V,0})$. *If* $n = a$, *then* $nq_i\varphi_i + a_i q_{d+1}\varphi_{d+1} = 0$ *for* $1 \le i \le e$, $\varphi_j = 0$ *for* $e < j \le d$, *and* $\varphi_1^{ta_1/n} \cdots \varphi_e^{ta_e/n} = \lambda \varphi_{d+1}^t$ *where*

$$\lambda = \frac{a_1^{ta_1/n} \cdots a_e^{ta_e/n}}{(-1)^t n^t} \varepsilon(0, \ldots, 0)^t$$

and t *are as in* (2).

Proof. Let (p, H_p) approach (q, φ) as p approaches q along a curve C in $\mathfrak{X}$. It is obvious that $\varphi_j = 0$ for $e < j \le d$ since $\operatorname{ord}_\tau(h_j) > 0 = \operatorname{ord}_\tau(h_{d+1})$ for $e < j \le d$ in (6).

Let $k = \min_{1 \le i \le d+1}\{\operatorname{ord}_\tau(h_i)\}$. Then $k \le 0$. Since (see (4) for notation)

$$H_p = (h_1 : \cdots : h_{d+1}) = (\tau^{-k} h_1 : \cdots : \tau^{-k} h_{d+1})$$

at $p \in C - \{q\}$, $\lim H_p = \varphi$ implies $\lim_{\tau \to 0} \tau^{-k} h_i = \varphi_i$ for $1 \le i \le d+1$. We have

$$a_i q_{d+1} \varphi_{d+1} = \lim_{\tau \to 0} a_i w_1^{a_1} \cdots w_e^{a_e} \varepsilon(tw_1, \ldots, t^n w_d)(-\tau^{-k})$$

and

$$\begin{aligned} nq_i\varphi_i &= \lim_{\tau \to 0} n w_i^n \tau^{-k} h_i = \lim_{\tau \to 0} \tau^{-k} w_1^{a_1} \cdots w_e^{a_e}(a_i\varepsilon + tw_i D_i \varepsilon) \\ &= \lim_{\tau \to 0} \tau^{-k} w_1^{a_1} \cdots w_e^{a_e} a_i \varepsilon = -a_i q_{d+1} \varphi_{d+1}. \end{aligned}$$

Hence, $nq_i\varphi_i + a_i q_{d+1}\varphi_{d+1} = 0$ for $1 \le i \le e$.

If $\varphi_{d+1} \ne 0$, then

$$\begin{aligned} \frac{\varphi_1^{ta_1/n} \cdots \varphi_e^{ta_e/n}}{\varphi_{d+1}^t} &= \lim_{\tau \to 0} \frac{\prod_{i=1}^e (\tau^{-k} h_i)^{ta_i/n}}{(-\tau^{-k})^t} \\ &= (-1)^t \lim_{\tau \to 0} \frac{(w_1^{a_1} \cdots w_e^{a_e})^t \prod_{i=1}^e (a_i \varepsilon + tw_i D_i \varepsilon)^{ta_i/n}}{n^t w_1^{ta_1} \cdots w_e^{ta_e}} \\ &= (-1)^t \frac{a_1^{ta_1/n} \cdots a_e^{ta_e/n} \varepsilon(0)^t}{n^t} = \lambda \end{aligned}$$

and so $\varphi_1^{ta_1/n} \cdots \varphi_e^{ta_e/n} = \lambda \varphi_{d+1}^t$.

If $\varphi_{d+1} = 0$, then $k < 0$ and $\operatorname{ord}_\tau(h_i) = k$ for some $j = 1, 2, \ldots, e$, say, $\operatorname{ord}_\tau(h_1) = k$. If we can prove that there exists at least one h_i, $1 \le i \le e$, such that $\operatorname{ord}_\tau(h_i) > k$, then $\varphi_i = 0$ and we have $\varphi_1^{ta_1/n} \cdots \varphi_e^{ta_e/n} = \lambda \varphi_{d+1}^t (= 0)$. This is done by the following lemma. □

Lemma 3.0.12. *If*

$$k = \min_{1\le i\le d+1}\{\operatorname{ord}_\tau(h_i)\} < 0,$$

then there exists an h_i, $1 \le i \le e$, *such that* $\operatorname{ord}_\tau(h_i) > k$.

Proof. Suppose the lemma is not true. Then $\operatorname{ord}_\tau(h_i) = k$ for all $i = 1, 2, \dots, e$. This implies $\nu_1 = \cdots = \nu_e$ (cf. (6)). But then

$$k = \operatorname{ord}_\tau(h_1) = \left(\sum_{i=1}^{e} a_i\right)\nu_1 - n\nu_1 = 0.$$

This contradicts $k < 0$. □

Proposition 3.0.13. *If* $n = a$, *the ideal* J *in* $\mathscr{O}_{d+2}[Y_1, \dots, Y_{d+1}]$ *which defines* $\kappa_{\mathfrak{f}}^{-1}(C_{V,0})$ *in* $\mathbb{C}^{d+1}\times\mathbb{C}\times\check{\mathbb{P}}^d$ *is generated by* $\{nX_iY_i + a_iZY_{d+1}\}_{1\le i\le e}$, $\{Y_j\}_{e<j\le d}$, T, $Z^t - \varepsilon(0)X_1^{ta_1/n}\cdots X_e^{ta_e/n}$, *and* $\lambda Y_{d+1}^t - Y_1^{ta_1/n}\cdots Y_e^{ta_e/n}$, *where* λ *is as in Lemma* 3.0.11.

Proof. Let N be the ideal generated by the elements as stated. By Lemma 3.0.11, $J \subset N$. To show that $N \subset J$, it is enough to show that, if $(q, \varphi) \in \mathbb{C}^{d+1}\times\mathbb{C}\times\check{\mathbb{P}}^d$ is in the zero locus of N, then $(q, \varphi) \in \kappa_{\mathfrak{f}}^{-1}(C_{V,0})$. We consider two cases.

1. $\varphi_{d+1} \ne 0$. In this case $\varphi_1\cdots\varphi_e \ne 0$.

If $q_{d+1} \ne 0$, then $q_1\cdots q_e \ne 0$. Choose integers $\nu_{e+1}, \dots, \nu_d$ $(= 0$ or $1)$ and complex numbers $b_1, \dots, b_d$ such that

$$b_i^n = q_i, \qquad 1 \le e \le e;$$

$$\lim_{\tau\to 0} b_i\tau^{\nu_i} = q_i, \qquad e < i \le d;$$

$$b_1^{a_1}\cdots b_e^{a_e}\varepsilon(0, \dots, 0) = q_{d+1};$$

and

$$\frac{b_1^{a_1}\cdots b_e^{a_e}}{nb_i^n}a_i\varepsilon(0, \dots, 0) = -\frac{\varphi_i}{\varphi_{d+1}}, \qquad 1 \le i \le e.$$

Then (q, φ) is the limit of (p, H_p) along the curve given by

$$w_i = \begin{cases} b_i, & 1 \le i \le e, \\ b_i\tau^{\nu_i}, & e < i \le d; \end{cases} \qquad t = \tau.$$

If $q_{d+1} = 0$, then $nq_i\varphi_i + a_iq_{d+1}\varphi_{d+1} = 0$ and $\varphi_1\cdots\varphi_e \ne 0$ imply $q_i = 0$, $1 \le i \le e$. Choose nonnegative integers $\nu_1 = \cdots = \nu_e = \nu > 0$, $\nu_j = 0$ or 1, $e < j \le d$, and nonzero complex numbers $b_1, \dots, b_d$ such that

$$\nu > \nu_j, \qquad e < j \le d;$$

$$\lim_{\tau\to 0} b_i\tau^{\nu_i} = q_i, \qquad e < i \le d;$$

and

$$\frac{b_1^{a_1}\cdots b_e^{a_e}}{nb_i^n}a_i\varepsilon(0, \dots, 0) = -\frac{\varphi_i}{\varphi_{d+1}}, \qquad 1 \le i \le e.$$

Then (q, φ) is the limit of (p, H_p) along the curve given by

$$w_i = b_i \tau^{\nu_i}, \quad 1 \leq i \leq d; \qquad t = \tau.$$

2. $\varphi_{d+1} = 0$. In this case $\varphi_1 \cdots \varphi_e = 0$.

We may assume $\varphi_1 \cdots \varphi_c \neq 0$, $\varphi_{c+1} = \cdots = \varphi_e = 0$ for some c, $1 \leq c < e$. Choose integers $\nu_1 = \cdots = \nu_c = \nu > 0$, $\nu_j = 0$ or 1, $c < j \leq d$, and complex numbers $b_1, \ldots, b_d$ such that

$$\nu > \nu_j, \qquad e < j \leq d;$$

$$\lim_{\tau \to 0} b_i \tau^{\nu_i} = q_i, \qquad e < i \leq d;$$

$$\lim_{\tau \to 0} b_i^n \tau^{n\nu_i} = q_i, \qquad c < i \leq e;$$

and

$$\frac{b_1^{a_1} \cdots b_e^{a_e}}{n b_i^n} a_i \varepsilon(0, \ldots, 0) = \varphi_i, \qquad 1 \leq i \leq c.$$

Then (q, φ) is the limit of (p, H_p) along the curve given by

$$w_i = b_i \tau^{\nu_i}, \quad 1 \leq i \leq e; \qquad t = \tau.$$

In either case, $(q, \varphi) \in \kappa_{\mathfrak{f}}^{-1}(C_{V,0})$. This completes the proof. □

Theorem 3.0.14. *If $n = a$, then $(V, 0)$ has no exceptional cones and so the family $\{C_I\}$ consists of the irreducible components of $C_{V,0}$ only.*

Proof. Let $(V', 0) = (C_{V,0}, 0)$ and $\mathfrak{f}' : \mathfrak{X}' \to \mathbb{C}$ be the deformation of V' to the tangent cone $C_{V',0}$ and $\kappa_{\mathfrak{f}'} : \mathscr{C}_{\mathfrak{f}'}(\mathfrak{X}') \to \mathfrak{X}'$ the relative conormal space. Then repeating the proof of Proposition 3.0.13 for $(V', 0)$, $\mathfrak{f}'$, and $\kappa_{\mathfrak{f}'}$, we will get the same ideal J as in Proposition 3.0.13. Then by Proposition 2.0.1 $(V, 0)$ has the same auréole as that of $(C_{V,0}, 0)$. Then the theorem follows from Proposition 2.0.3. □

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