PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 120, Number 2, February 1994

JORDAN $*$-DERIVATIONS OF STANDARD OPERATOR ALGEBRAS

PETER ŠEMRL

(Communicated by Palle E. T. Jorgensen)

ABSTRACT. Let H be a real or complex Hilbert space, $\dim H > 1$, and $\mathcal{B}(H)$ the algebra of all bounded linear operators on H. Assume that $\mathcal{A}$ is a standard operator algebra on H. Then every additive Jordan $*$-derivation $J: \mathcal{A} \to \mathcal{B}(H)$ is of the form $J(A) = AT - TA^*$ for some $T \in \mathcal{B}(H)$.

Let $\mathcal{A}$ be a real or complex $*$-algebra and $\mathcal{A}_1$ any subalgebra of $\mathcal{A}$. An additive (linear) mapping $D: \mathcal{A}_1 \to \mathcal{A}$ is called an additive (linear) Jordan derivation if $D(a^2) = aD(a) + D(a)a$ for all $a \in \mathcal{A}_1$. An additive (real-linear) Jordan $*$-derivation $J: \mathcal{A}_1 \to \mathcal{A}$ is defined as an additive (real-linear) mapping satisfying $J(a^2) = aJ(a) + J(a)a^*$ for all $a \in \mathcal{A}_1$. It is easy to verify that for an arbitrary element $b \in \mathcal{A}$ the mapping $D_b: \mathcal{A}_1 \to \mathcal{A}$ defined by $D_b(a) = ab - ba$ ($D_b(a) = ab - ba^*$) is a Jordan derivation (Jordan $*$-derivation).

The study of Jordan $*$-derivations has been motivated by the problem of the representability of quasi-quadratic functionals by sesquilinear ones (for the results concerning this problem we refer to [5, 7, 9–11]). It turns out that the question of whether each quasi-quadratic functional is generated by some sesquilinear functional is intimately connected with the structure of Jordan $*$-derivations [7, 9].

Let H be a real or complex Hilbert space. By $\mathcal{B}(H)$ we mean the algebra of all bounded linear operators on H. We denote by $\mathcal{F}(H)$ the subalgebra of bounded finite rank operators. We shall call a subalgebra $\mathcal{A}$ of $\mathcal{B}(H)$ standard, provided $\mathcal{A}$ contains $\mathcal{F}(H)$. It is easy to see that $\mathcal{F}(H)$ is a prime ring; that is, $A, B \in \mathcal{F}(H)$ and $A\mathcal{F}(H)B = \{0\}$ imply $A = 0$ or $B = 0$.

Assume that H is an infinite-dimensional Hilbert space. Let $\mathcal{A}$ be a standard operator algebra on H. Suppose that $D: \mathcal{A} \to \mathcal{B}(H)$ is an additive Jordan derivation. Every finite rank operator is a linear combination of idempotent operators of rank one. If P is an idempotent operator of rank one and $\lambda = \mu^2$ is a scalar, then $D(\lambda P) = \mu(PD(\mu P) + D(\mu P)P)$ has rank at most two. Thus, D maps $\mathcal{F}(H)$ into itself. Using the result of Herstein [3], which states that all additive Jordan derivations of prime rings of characteristic different from two are derivations, we infer that D satisfies $D(AB) = AD(B) + D(A)B$ for all pairs $A, B \in \mathcal{F}(H)$. It follows from [8] that there exists $T \in \mathcal{B}(H)$

Received by the editors May 26, 1992.
1991 *Mathematics Subject Classification*. Primary 47B47, 47D25.
This work was supported by the Research Council of Slovenia.

such that

$$D(A) = AT - TA \tag{1}$$

for all finite rank operators A. Linearizing the equation $D(A^2) = AD(A) + D(A)A$ we get $D(AB + BA) = AD(B) + BD(A) + D(A)B + D(B)A$. Together with (1) this yields that

$$B(D(A) - AT + TA) + (D(A) - AT + TA)B = 0$$

holds for all $A \in \mathscr{A}$, $B \in \mathscr{F}(H)$. Consequently, we have $D(A) = AT - TA$ for every $A \in \mathscr{A}$. Thus, every additive Jordan derivation $D: \mathscr{A} \to \mathscr{B}(H)$ is of form (1) for some $T \in \mathscr{B}(H)$. The assumption that H is infinite dimensional is indispensable in this statement [8].

It is the aim of this note to obtain a similar result for additive Jordan $*$-derivations. More precisely, we shall prove the following result:

Theorem. *Let H be a real or complex Hilbert space, $\dim H > 1$, and let $\mathscr{A}$ be a standard operator algebra on H. Suppose that $J: \mathscr{A} \to \mathscr{B}(H)$ is an additive Jordan $*$-derivation. Then there exists a unique linear operator $T \in \mathscr{B}(H)$ such that $J(A) = AT - TA^*$ holds for all $A \in \mathscr{A}$.*

Remark. Two special cases of this result have been already proved—the case when $\mathscr{A} = \mathscr{B}(H)$ [6] and the case when H is a complex Hilbert space and $\mathscr{A}$ is the algebra of all compact linear operators [1]. In this general setting we use a completely different approach as in [1, 6].

Proof. Let us denote by J_1 the restriction of J to the ideal $\mathscr{F}(H)$. We define a mapping $\phi: \mathscr{F}(H) \to \mathscr{B}(H \oplus H)$ by

$$\phi(A) = \begin{pmatrix} A & J_1(A) \\ 0 & A^* \end{pmatrix}. \tag{2}$$

Clearly, ϕ is an additive Jordan homomorphism; that is, ϕ is additive and $(\phi(A))^2 = \phi(A^2)$ holds for all finite rank operators A. It should be mentioned that relation (2) is a variation of a standard connection between linear derivations and algebra homomorphisms (see [2]). Since $\mathscr{F}(H)$ is a locally matrix algebra, by a result of Jacobson and Rickart [4], $\phi = \varphi + \psi$, where $\varphi: \mathscr{F}(H) \to \mathscr{B}(H \oplus H)$ is a ring homomorphism and $\psi: \mathscr{F}(H) \to \mathscr{B}(H \oplus H)$ is a ring antihomomorphism. We have

$$\operatorname{Im} \phi \subset \left\{ \begin{pmatrix} X & Y \\ 0 & W \end{pmatrix} \in \mathscr{B}(H \oplus H) : X, Y, W \in \mathscr{B}(H) \right\}.$$

It follows that φ and ψ are of the form

$$\varphi(A) = \begin{pmatrix} \varphi_1(A) & \varphi_2(A) \\ 0 & \varphi_3(A) \end{pmatrix}, \qquad \psi(A) = \begin{pmatrix} \psi_1(A) & \psi_2(A) \\ 0 & \psi_3(A) \end{pmatrix}, \tag{3}$$

where φ_1, φ_3 are additive homomorphisms on $\mathscr{F}(H)$, ψ_1, ψ_3 are additive antihomomorphisms on $\mathscr{F}(H)$, and the equations $\varphi_1(A) + \psi_1(A) = A$ and $\varphi_3(A) + \psi_3(A) = A^*$ are valid for all $A \in \mathscr{F}(H)$. Pick an idempotent P on H of rank one. Then P is the sum of the idempotents $\varphi_1(P)$ and $\psi_1(P)$; therefore, we have that either $\varphi_1(P) = 0$ or $\psi_1(P) = 0$. Thus, at least one of φ_1 and ψ_1 has a nonzero kernel. Since the kernels of homomorphisms and antihomomorphisms are ideals and since the only nonzero ideal of $\mathscr{F}(H)$ is

$\mathscr{F}(H)$ itself, we have $\varphi_1 = 0$ or $\psi_1 = 0$. As a consequence we get $\psi_1 = 0$ and $\varphi_1(A) = A$ for all $A \in \mathscr{F}(H)$. Similarly we show that $\varphi_3 = 0$. Thus, relations (3) can be rewritten as

$$\varphi(A) = \begin{pmatrix} A & \varphi_2(A) \\ 0 & 0 \end{pmatrix}, \qquad \psi(A) = \begin{pmatrix} 0 & \psi_2(A) \\ 0 & A^* \end{pmatrix}.$$

The mappings φ and ψ are an additive homomorphism and an additive antihomomorphism respectively, and consequently, φ_2 and ψ_2 are additive mappings satisfying

$$\varphi_2(AB) = A\varphi_2(B) \tag{4}$$

and

$$\psi_2(AB) = \psi_2(B)A^* \tag{5}$$

for all $A, B \in \mathscr{F}(H)$. Applying $J_1 = \varphi_2 + \psi_2$, $J_1(A^2) = AJ_1(A) + J_1(A)A^*$, (4), and (5), one can see that $\varphi_2(A)A^* + A\psi_2(A) = 0$ holds true for all $A \in \mathscr{F}(H)$. Linearizing this relation we get that $\varphi_2(A)B^* + \varphi_2(B)A^* + A\psi_2(B) + B\psi_2(A) = 0$ for all $A, B \in \mathscr{F}(H)$. Replacing B by CB we obtain

$$C(\varphi_2(B)A^* + B\psi_2(A)) + (\varphi_2(A)B^* + A\psi_2(B))C^* = 0$$

for every finite rank operator C. Consequently, we have

$$\varphi_2(A)B^* + A\psi_2(B) = 0 \tag{6}$$

for all finite rank operators A and B.

For any $x, y \in H$ we shall denote the inner product of these two vectors by y^*x, while xy^* shall denote the rank one operator given by $(xy^*)z = (y^*z)x$. Every rank one operator can be written in this form. For every nonzero $x \in H$ we denote $L_x = \{xy^* : y \in H\} \subset \mathscr{F}(H)$. It follows from (4) that φ_2 is a linear mapping on $\mathscr{F}(H)$. Moreover, for every nonzero $x \in H$ we have $\varphi_2(L_x) \subset L_x$. Thus, we can find for every nonzero x from H a linear mapping $S_x : H \to H$ such that $\varphi_2(xy^*) = x(S_x y)^*$. For linearly independent vectors $x, u \in H$ and for an arbitrary vector $y \in H$ we have

$$(x+u)(S_{x+u}y)^* = \varphi_2((x+u)y^*) = \varphi_2(xy^*) + \varphi_2(uy^*) = x(S_x y)^* + u(S_u y)^*.$$

This yields that $S_x = S_u$. In the case that nonzero vectors x and u are linearly dependent, we find a vector z from H such that x and z are linearly independent. Then we have $S_x = S_z = S_u$. Hence, we have proved that there exists a linear operator $S : H \to H$ such that

$$\varphi_2(xy^*) = x(Sy)^*. \tag{7}$$

One can verify using (5) that the mapping ψ_2' given by $\psi_2'(A) = (\psi_2(A))^*$ satisfies $\psi_2'(AB) = A\psi_2'(B)$. This yields the existence of a linear operator $T : H \to H$ such that

$$\psi_2(xy^*) = -Tyx^*. \tag{8}$$

Replacing A and B in (6) by xy^* and uv^* respectively and applying (7), (8) we get that $(Sy)^*v = y^*Tv$ for all $v, y \in H$. It follows from the closed graph theorem that the operators S and T are bounded. Moreover, we have

$S = T^*$. The equation $J_1 = \varphi_2 + \psi_2$ yields

$$J(A) = AT - TA^* \tag{9}$$

for every finite rank operator A.

Replacing A by $A + B$ in $J(A^2) = AJ(A) + J(A)A^*$, we get that

$$J(AB) + J(BA) = AJ(B) + BJ(A) + J(A)B^* + J(B)A^*$$

is valid for an arbitrary pair of operators A, B from $\mathscr{A}$. Applying this relation with (9) we see that

$$B(J(A) - AT + TA^*) + (J(A) - AT + TA^*)B^* = 0$$

holds true for all $A \in \mathscr{A}$ and all finite rank operators B. Thus, (9) is satisfied for all $A \in \mathscr{A}$. This completes the proof.

References

1. M. Brešar and B. Zalar, *On the structure of Jordan $*$-derivations*, Colloq. Math. **63** (1992), 163–171.
2. P. R. Chernoff, *Representations, automorphisms and derivations of some operator algebras*, J. Funct. Anal. **12** (1973), 275–289.
3. I. N. Herstein, *Jordan derivations of prime rings*, Proc. Amer. Math. Soc. **8** (1957), 1104–1110.
4. N. Jacobson and C. Rickart, *Jordan homomorphisms of rings*, Trans. Amer. Math. Soc. **69** (1950), 479–502.
5. P. Šemrl, *On quadratic functionals*, Bull. Austral. Math. Soc. **37** (1988), 27–29.
6. ——, *On Jordan $*$-derivations and an application*, Colloq. Math. **59** (1990), 241–251.
7. ——, *Quadratic functionals and Jordan $*$-derivations*, Studia Math. **97** (1991), 157–165.
8. ——, *Additive derivations of some operator algebras*, Illinois J. Math. **35** (1991), 234–240.
9. ——, *Quadratic and quasi-quadratic functionals*, Proc. Amer. Math. Soc. (to appear).
10. J. Vukman, *A result concerning additive functions in hermitian Banach $*$-algebras and an application*, Proc. Amer. Math. Soc. **91** (1984), 367–372.
11. ——, *Some functional equations in Banach algebras and an application*, Proc. Amer. Math. Soc. **100** (1987), 133–136.

Department of Mathematics, University of Ljubljana, Jadranska 19, 61000 Ljubljana, Slovenia

E-mail address: `peter.semrl@uni-lj.si`