JORDAN *-DERIVATIONS OF STANDARD OPERATOR ALGEBRAS

PETER ŠEMRL

(Communicated by Palle E. T. Jorgensen)

Abstract. Let $H$ be a real or complex Hilbert space, $\dim H > 1$, and $\mathcal{B}(H)$ the algebra of all bounded linear operators on $H$. Assume that $\mathcal{A}$ is a standard operator algebra on $H$. Then every additive Jordan *-derivation $J: \mathcal{A} \to \mathcal{B}(H)$ is of the form $J(A) = AT - TA^*$ for some $T \in \mathcal{B}(H)$.

Let $\mathcal{A}$ be a real or complex *-algebra and $\mathcal{A}_1$ any subalgebra of $\mathcal{A}$. An additive (linear) mapping $D: \mathcal{A}_1 \to \mathcal{A}$ is called an additive (linear) Jordan derivation if $D(a^2) = aD(a) + D(a)a$ for all $a \in \mathcal{A}_1$. An additive (real-linear) Jordan *-derivation $J: \mathcal{A}_1 \to \mathcal{A}$ is defined as an additive (real-linear) mapping satisfying $J(a^2) = aJ(a) + J(a)a^*$ for all $a \in \mathcal{A}_1$. It is easy to verify that for an arbitrary element $b \in \mathcal{A}$ the mapping $D_b: \mathcal{A}_1 \to \mathcal{A}$ defined by $D_b(a) = ab - ba$ ($D_b(a) = ab - ba^*$) is a Jordan derivation (Jordan *-derivation).

The study of Jordan *-derivations has been motivated by the problem of the representability of quasi-quadratic functionals by sesquilinear ones (for the results concerning this problem we refer to [5, 7, 9–11]). It turns out that the question of whether each quasi-quadratic functional is generated by some sesquilinear functional is intimately connected with the structure of Jordan *-derivations [7, 9].

Let $H$ be a real or complex Hilbert space. By $\mathcal{B}(H)$ we mean the algebra of all bounded linear operators on $H$. We denote by $\mathcal{F}(H)$ the subalgebra of bounded finite rank operators. We shall call a subalgebra $\mathcal{A}$ of $\mathcal{B}(H)$ standard, provided $\mathcal{A}$ contains $\mathcal{F}(H)$. It is easy to see that $\mathcal{F}(H)$ is a prime ring; that is, $A, B \in \mathcal{F}(H)$ and $A\mathcal{F}(H)B = \{0\}$ imply $A = 0$ or $B = 0$.

Assume that $H$ is an infinite-dimensional Hilbert space. Let $\mathcal{A}$ be a standard operator algebra on $H$. Suppose that $D: \mathcal{A} \to \mathcal{B}(H)$ is an additive Jordan derivation. Every finite rank operator is a linear combination of idempotent operators of rank one. If $P$ is an idempotent operator of rank one and $\lambda = \mu^2$ is a scalar, then $D(\lambda P) = \mu(PD(\mu P) + D(\mu P)P)$ has rank at most two. Thus, $D$ maps $\mathcal{F}(H)$ into itself. Using the result of Herstein [3], which states that all additive Jordan derivations of prime rings of characteristic different from two are derivations, we infer that $D$ satisfies $D(AB) = AD(B) + D(A)B$ for all pairs $A, B \in \mathcal{F}(H)$. It follows from [8] that there exists $T \in \mathcal{B}(H)$
such that

\( D(A) = AT - TA \)

for all finite rank operators \( A \). Linearizing the equation \( D(A^2) = AD(A) + D(A)A \) we get \( D(AB + BA) = AD(B) + BD(A) + D(A)B + D(B)A \). Together with (1) this yields that

\[
B(D(A) - AT + TA) + (D(A) - AT + TA)B = 0
\]

holds for all \( A \in \mathcal{A}, B \in \mathcal{B}(H) \). Consequently, we have \( D(A) = AT - TA \) for every \( A \in \mathcal{A} \). Thus, every additive Jordan derivation \( D: \mathcal{A} \rightarrow \mathcal{B}(H) \) is of form (1) for some \( T \in \mathcal{B}(H) \). The assumption that \( H \) is infinite dimensional is indispensable in this statement [8].

It is the aim of this note to obtain a similar result for additive Jordan \( \ast \)-derivations. More precisely, we shall prove the following result:

**Theorem.** Let \( H \) be a real or complex Hilbert space, \( \dim H > 1 \), and let \( \mathcal{A} \) be a standard operator algebra on \( H \). Suppose that \( J: \mathcal{A} \rightarrow \mathcal{B}(H) \) is an additive Jordan \( \ast \)-derivation. Then there exists a unique linear operator \( T \in \mathcal{B}(H) \) such that \( J(A) = AT - TA^\ast \) holds for all \( A \in \mathcal{A} \).

**Remark.** Two special cases of this result have been already proved—the case when \( \mathcal{A} = \mathcal{B}(H) \) [6] and the case when \( H \) is a complex Hilbert space and \( \mathcal{A} \) is the algebra of all compact linear operators [1]. In this general setting we use a completely different approach as in [1, 6].

**Proof.** Let us denote by \( J_1 \) the restriction of \( J \) to the ideal \( \mathcal{I}(H) \). We define a mapping \( \phi: \mathcal{I}(H) \rightarrow \mathcal{B}(H \oplus H) \) by

\[
\phi(A) = \begin{pmatrix} A & J_1(A) \\ 0 & A^\ast \end{pmatrix}.
\]

Clearly, \( \phi \) is an additive Jordan homomorphism; that is, \( \phi \) is additive and \((\phi(A))^2 = \phi(A^2)\) holds for all finite rank operators \( A \). It should be mentioned that relation (2) is a variation of a standard connection between linear derivations and algebra homomorphisms (see [2]). Since \( \mathcal{I}(H) \) is a locally matrix algebra, by a result of Jacobson and Rickart [4], \( \phi = \phi + \psi \), where \( \varphi: \mathcal{I}(H) \rightarrow \mathcal{B}(H \oplus H) \) is a ring homomorphism and \( \psi: \mathcal{I}(H) \rightarrow \mathcal{B}(H \oplus H) \) is a ring antihomomorphism. We have

\[
\text{Im} \phi \subset \left\{ \begin{pmatrix} X & Y \\ 0 & W \end{pmatrix} \in \mathcal{B}(H \oplus H): X, Y, W \in \mathcal{B}(H) \right\}.
\]

It follows that \( \varphi \) and \( \psi \) are of the form

\[
\varphi(A) = \begin{pmatrix} \varphi_1(A) & \varphi_2(A) \\ 0 & \varphi_3(A) \end{pmatrix}, \quad \psi(A) = \begin{pmatrix} \psi_1(A) & \psi_2(A) \\ 0 & \psi_3(A) \end{pmatrix},
\]

where \( \varphi_1, \varphi_3 \) are additive homomorphisms on \( \mathcal{I}(H) \), \( \psi_1, \psi_3 \) are additive antihomomorphisms on \( \mathcal{I}(H) \), and the equations \( \varphi_1(A) + \psi_1(A) = A \) and \( \varphi_3(A) + \psi_3(A) = A^\ast \) are valid for all \( A \in \mathcal{I}(H) \). Pick an idempotent \( P \) on \( H \) of rank one. Then \( P \) is the sum of the idempotents \( \varphi_1(P) \) and \( \psi_1(P) \); therefore, we have that either \( \varphi_1(P) = 0 \) or \( \psi_1(P) = 0 \). Thus, at least one of \( \varphi_1 \) and \( \psi_1 \) has a nonzero kernel. Since the kernels of homomorphisms and antihomomorphisms are ideals and since the only nonzero ideal of \( \mathcal{I}(H) \) is...
\( \mathcal{F}(H) \) itself, we have \( \varphi_1 = 0 \) or \( \psi_1 = 0 \). As a consequence we get \( \psi_1 = 0 \) and \( \varphi_1(A) = A \) for all \( A \in \mathcal{F}(H) \). Similarly we show that \( \varphi_3 = 0 \). Thus, relations (3) can be rewritten as

\[
\varphi(A) = \begin{pmatrix} A & \varphi_2(A) \\ 0 & 0 \end{pmatrix}, \quad \psi(A) = \begin{pmatrix} 0 & \psi_2(A) \\ 0 & A^* \end{pmatrix}.
\]

The mappings \( \varphi \) and \( \psi \) are an additive homomorphism and an additive antihomomorphism respectively, and consequently, \( \varphi_2 \) and \( \psi_2 \) are additive mappings satisfying

(4) \( \varphi_2(AB) = A\varphi_2(B) \)

and

(5) \( \psi_2(AB) = \psi_2(B)A^* \)

for all \( A, B \in \mathcal{F}(H) \). Applying \( J_1 = \varphi_2 + \psi_2 \), \( J_1(A^2) = AJ_1(A) + J_1(A)A^* \), (4), and (5), one can see that \( \varphi_2(A)A^* + A\psi_2(A) = 0 \) holds true for all \( A \in \mathcal{F}(H) \). Linearizing this relation we get that \( \varphi_2(A)B^* + \varphi_2(B)A^* + A\psi_2(B) + B\psi_2(A) = 0 \) for all \( A, B \in \mathcal{F}(H) \). Replacing \( B \) by \( CB \) we obtain

\[
C(\varphi_2(B)A^* + B\psi_2(A)) + (\varphi_2(A)B^* + A\psi_2(B))C^* = 0
\]

for every finite rank operator \( C \). Consequently, we have

(6) \( \varphi_2(A)B^* + A\psi_2(B) = 0 \)

for all finite rank operators \( A \) and \( B \).

For any \( x, y \in H \) we shall denote the inner product of these two vectors by \( y^*x \), while \( xy^* \) shall denote the rank one operator given by \( (xy^*)z = (y^*z)x \). Every rank one operator can be written in this form. For every nonzero \( x \in H \) we denote \( L_x = \{ xy^*: y \in H \} \subset \mathcal{F}(H) \). It follows from (4) that \( \varphi_2 \) is a linear mapping on \( \mathcal{F}(H) \). Moreover, for every nonzero \( x \in H \) we have \( \varphi_2(L_x) \subset L_x \). Thus, we can find for every nonzero \( x \) from \( H \) a linear mapping \( S_x: H \rightarrow H \) such that \( \varphi_2(xy^*) = x(S_xy)^* \). For linearly independent vectors \( x, u \in H \) and for an arbitrary vector \( y \in H \) we have

\[
(x + u)(S_x+uy^*)^* = \varphi_2((x + u)y^*) = \varphi_2(xy^*) + \varphi_2(uy^*) = x(S_xy)^* + u(S_uy)^*.
\]

This yields that \( S_x = S_u \). In the case that nonzero vectors \( x \) and \( u \) are linearly dependent, we find a vector \( z \) from \( H \) such that \( x \) and \( z \) are linearly independent. Then we have \( S_x = S_z = S_u \). Hence, we have proved that there exists a linear operator \( S: H \rightarrow H \) such that

(7) \( \varphi_2(xy^*) = x(Sy)^* \).

One can verify using (5) that the mapping \( \psi_2 \) given by \( \psi_2(A) = (\varphi_2(A))^* \) satisfies \( \psi_2(AB) = A\psi_2(B) \). This yields the existence of a linear operator \( T: H \rightarrow H \) such that

(8) \( \psi_2(xy^*) = -Tyx^* \).

Replacing \( A \) and \( B \) in (6) by \( xy^* \) and \( uv^* \) respectively and applying (7), (8) we get that \( (S_y)^*v = y^*Tv \) for all \( v, y \in H \). It follows from the closed graph theorem that the operators \( S \) and \( T \) are bounded. Moreover, we have
$S = T^*$. The equation $J_1 = \varphi_2 + \psi_2$ yields

\begin{equation}
J(A) = AT - TA^*
\end{equation}

for every finite rank operator $A$.

Replacing $A$ by $A + B$ in $J(A^2) = AJ(A) + J(A)A^*$, we get that

\[ J(AB) + J(BA) = AJ(B) + BJ(A) + J(A)B^* + J(B)A^* \]

is valid for an arbitrary pair of operators $A, B$ from $\mathcal{A}$. Applying this relation with (9) we see that

\[ B(J(A) - AT + TA^*) + (J(A) - AT + TA^*)B^* = 0 \]

holds true for all $A \in \mathcal{A}$ and all finite rank operators $B$. Thus, (9) is satisfied for all $A \in \mathcal{A}$. This completes the proof.

REFERENCES