THE SET OF ALL $m \times n$ RECTANGULAR REAL MATRICES
OF RANK $r$ IS CONNECTED BY ANALYTIC REGULAR ARCS

J.-Cl. EVARD AND F. JAFARI

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Abstract. It is well known that the set of all square invertible real matrices
has two connected components. The set of all $m \times n$ rectangular real matrices
of rank $r$ has only one connected component when $m \neq n$ or $r < m = n$. We
show that all these connected components are connected by analytic regular arcs.
We apply this result to establish the existence of $p$-times differentiable bases of
the kernel and the image of a rectangular real matrix function of several real
variables.

Introduction

In [3] we showed that every open connected subset of a topological vector
space is connected by regular polynomial curves. In this paper, we deal with
the set of real $m \times n$ matrices of rank $r$. In spite of the fact that this set is not
an open subset of $\mathbb{R}^{m \times n}$, we show that it is connected by regular arcs that are
not only of class $C^\infty$ but are also analytic. A similar result was established in
[2, Theorem 7.2] for complex matrices, but some new methods are necessary to
obtain arcs that are contained in the set of real matrices. We furnish a method
to construct these arcs explicitly.

The analytic connections are likely to have many applications. For example, a
method to construct continuous arcs in the set of square invertible real matrices
is furnished in [1, Proposition 1.5]. This construction was used to establish
the uniqueness of the topological degree. In this paper, we provide another
application by showing that the main result of [2] about the existence of bases of
class $C^p$ of the kernel and the image of a rectangular matrix function of
several real variables is also valid when the field is $\mathbb{R}$ instead of $\mathbb{C}$.

We will denote by $\mathbb{R}^{m \times n}$ the set of all $m \times n$ real matrices and by $\mathbb{R}^{r \times n}
the subset of $\mathbb{R}^{m \times n}$ of all matrices of rank $r$. We will denote by $I_n$ the $n \times n$
identity matrix and by $I_r^{m \times n}$ the following $m \times n$ matrix of rank $r$:

$$
I_r^{m \times n} = \begin{bmatrix}
I_r & 0 \\
0 & 0
\end{bmatrix}.
$$

In Lemma 1 we show that the two connected components of $\mathbb{R}^{n \times n}$ are con-
nected by analytic arcs that may be chosen as closed curves travelled infinitely

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many times. In Lemma 2 we establish the existence of equivalences with positive determinants between matrices of same rank. In Lemma 3 we show that \( \mathbb{R}^{m \times n} \) is connected by analytic arcs that may be chosen as closed curves travelled infinitely many times, when \( m \neq n \) or \( r < m = n \). In Theorem 4 we show that all the connected components of \( \mathbb{R}^{n \times n} \) and \( \mathbb{R}^{m \times n} \) are connected by analytic arcs that are regular. The problem of finding analytic regular closed curves travelled infinitely many times is still open. In Theorem 5 we apply Theorem 4 to establish the existence of bases of class \( C^p \) of the kernel and the image of a rectangular real matrix function of several real variables.

**Results**

**Lemma 1.** Let \( A, B \in \mathbb{R}^{n \times n} \) be such that \( \det A \) and \( \det B \) have the same sign. Then there exists an analytic mapping \( F: \mathbb{R} \to \mathbb{R}^{n \times n} \) such that, for every \( m \in \mathbb{Z} \), \( F(m) = A \) if \( m \) is even and \( F(m) = B \) if \( m \) is odd. Moreover, \( \det F(t) \) has the same sign as \( \det A \), and \( F(t + 2) = F(t) \) for every \( t \in \mathbb{R} \).

**Proof.** Let \( C = BA^{-1} \in \mathbb{R}^{n \times n} \). Since \( \det A \) and \( \det B \) have the same sign, we have \( \det C > 0 \). It is well known that \( C \) is similar in \( \mathbb{R}^{n \times n} \) to a real Jordan matrix. More precisely, there exist \( R \in \mathbb{R}^{n \times n} \) and \( J \in \mathbb{R}^{n \times n} \) such that

\[
C = RJR^{-1},
\]

where in turn \( J_1 \) has the form

\[
J_1 = \begin{bmatrix}
C(\rho_1, \theta_1) & \alpha_1 I_2 & 0 \\
& \ddots & \ddots \\
& & \alpha_{p-1} I_2 & C(\rho_p, \theta_p)
\end{bmatrix},
\]

\( \rho_1, \ldots, \rho_p > 0, \theta_1, \ldots, \theta_p \in [0, 2\pi[; \alpha_1, \ldots, \alpha_{p-1} \in \{0, 1\}, \)

\[
C(\rho, \theta) = \rho \begin{bmatrix}
\cos \theta & \sin \theta \\
-\sin \theta & \cos \theta
\end{bmatrix} \forall \rho > 0, \theta \in [0, 2\pi[;
\]

\( J_2 \) has the form

\[
J_2 = \begin{bmatrix}
\mu_1 & \beta_1 & 0 \\
& \ddots & \beta_{q-1} \\
0 & & \mu_q
\end{bmatrix},
\]

\( \mu_1, \ldots, \mu_q > 0, \beta_1, \ldots, \beta_{q-1} \in \{0, 1\} \); and \( J_3 \) has the form

\[
J_3 = \begin{bmatrix}
v_1 & \gamma_1 & 0 \\
& \ddots & \gamma_{r-1} \\
0 & & v_r
\end{bmatrix},
\]

\( v_1, \ldots, v_r < 0, \gamma_1, \ldots, \gamma_{r-1} \in \{0, 1\}, \) and \( r \) is even because \( \det C > 0 \). Let

\[
\lambda(t) = \cos^2(\pi t/2), \quad \mu(t) = \sin^2(\pi t/2) \quad \forall t \in \mathbb{R}.
\]
Let \( t \in \mathbb{R} \). For every \( k \in \{1, 2, \ldots\} \) that makes sense, let
\[
\begin{align*}
    r_k(t) &= \lambda(t) + \mu(t) \rho_k, \\
    s_k(t) &= \mu(t) \theta_k, \\
    a_k(t) &= \mu(t) \alpha_k, \\
    C_k(t) &= C(r_k(t), s_k(t)), \\
    m_k(t) &= \lambda(t) + \mu(t) \mu_k, \\
    b_k(t) &= \mu(t) \beta_k, \\
    n_k(t) &= -\lambda(t) + \mu(t) \nu_k, \\
    c_k(t) &= \mu(t) \gamma_k.
\end{align*}
\]
Let
\[
H_1(t) = \begin{bmatrix}
    C_1(t) & a_1(t) I_2 & \cdots & 0 \\
    \vdots & \ddots & \vdots & \vdots \\
    0 & \cdots & a_{p-1}(t) I_2 & C_p(t) \\
\end{bmatrix},
\]
\[
H_2(t) = \begin{bmatrix}
    m_1(t) & b_1(t) & \cdots & 0 \\
    \vdots & \ddots & \vdots & \vdots \\
    0 & \cdots & b_{q-1}(t) & m_q(t) \\
\end{bmatrix},
\]
\[
R(t) = \begin{bmatrix}
    \cos(\pi t) & \sin(\pi t) \\
    -\sin(\pi t) & \cos(\pi t) \\
\end{bmatrix},
\]
\[
H_3(t) = \begin{bmatrix}
    n_1(t) & c_1(t) & \cdots & 0 \\
    \vdots & \ddots & \vdots & \vdots \\
    0 & \cdots & c_{r-1}(t) & n_r(t) \\
\end{bmatrix} \begin{bmatrix}
    -R(t) & 0 \\
    \vdots & \vdots \\
    0 & -R(t) \\
\end{bmatrix},
\]
where the last matrix has \( \frac{r}{2} \) blocks \( R(t) \), which is possible since \( r \) is even.

\[
H(t) = \text{diag}[H_1(t), H_2(t), H_3(t)],
\]
\[
G(t) = RH(t)R^{-1}, \quad F(t) = G(t)A.
\]
Let \( m \in \mathbb{Z} \). Plainly, if \( m \) is even, then
\[
H_1(m) = I_{2p}, \quad H_2(m) = I_q, \quad H_3(m) = (-I_r)(-I_r) = I_r,
\]
\[
H(m) = I_n, \quad G(m) = RR^{-1} = I_n, \quad F(m) = I_n A = A,
\]
and, if \( m \) is odd, then
\[
H_1(m) = J_1, \quad H_2(m) = J_2, \quad H_3(m) = J_3,
\]
\[
H(m) = J, \quad G(m) = RJR^{-1} = C, \quad F(m) = CA = B.
\]
It is obvious that all the functions above are periodic with period 2. Let \( t \in \mathbb{R} \).

It is easy to see that
\[
\begin{align*}
    \det H_1(t) &= \det C_1(t) \cdots \det C_p(t) = (r_1(t))^2 \cdots (r_p(t))^2 > 0, \\
    \det H_2(t) &= m_1(t) \cdots m_q(t) > 0, \\
    \det H_3(t) &= n_1(t) \cdots n_r(t) (\det R(t))^{r/2} = (-1)^r |n_1(t)| \cdots |n_r(t)| > 0,
\end{align*}
\]
because \( r \) is even, and, finally,
\[
\det H(t) = (\det H_1(t))(\det H_2(t))(\det H_3(t)) > 0, \\
\det(G(t)) = \det H(t) > 0, \\
\det F(t) = (\det G(t))(\det A) \neq 0.
\]
Thus \( F(t) \in \mathbb{R}_n^{n \times n} \), and \( \det F(t) \) has the same sign as \( \det A \).

**Lemma 2.** Let \( m, n \in \{1, 2, \ldots\} \) and \( r \in \{0, 1, 2, \ldots\} \) be such that \( m \neq n \) or \( r < m = n \). Let \( A \in \mathbb{R}_r^{m \times n} \). Then there exist \( L \in \mathbb{R}_r^{m \times m} \) and \( R \in \mathbb{R}_n^{n \times n} \) such that \( \det L > 0, \ \det R > 0, \) and \( A = LI^m_{m \times n}R \).

**Proof.** If \( r = 0 \), then \( A = 0 = I^m_{n \times n} \), and we can choose \( L = I_m \) and \( R = I_n \). Suppose \( r \geq 1 \). It follows from the hypothesis that \( r < m \) or \( r < n \).

(a) Suppose \( r < m \). Then \( m \geq 2 \). As \( A \) is of rank \( r \), it is well known that \( A \) is equivalent in \( \mathbb{R}_r^{m \times n} \) to \( I^m_{m \times n} \). That is, there exist \( B \in \mathbb{R}_r^{m \times m} \) and \( C \in \mathbb{R}_n^{n \times n} \) such that \( A = BI^m_{m \times n}C \). If \( \det B > 0 \) and \( \det C > 0 \), we obviously choose \( L = B \) and \( R = C \). If \( \det B > 0 \) and \( \det C < 0 \), we choose

\[
L = B \text{diag}[-1, I_{m-2}, -1] \quad \text{and} \quad R = \text{diag}[-1, I_{n-1}]C,
\]
considering that \( r = n \) is possible. If \( \det B < 0 \) and \( \det C > 0 \), we choose

\[
L = B \text{diag}[I_{m-1}, -1] \quad \text{and} \quad R = C.
\]
If \( \det B < 0 \) and \( \det C < 0 \), we choose

\[
L = B \text{diag}[-1, I_{m-1}] \quad \text{and} \quad R = \text{diag}[-1, I_{n-1}]C.
\]
It is easy to check that, in all cases, \( A = LI^m_{m \times n}R, \ \det L > 0, \) and \( \det R > 0 \).

(b) Suppose \( r < n \). Then by (a), there exist \( B \in \mathbb{R}_r^{n \times n} \) and \( C \in \mathbb{R}_r^{m \times m} \) such that \( A^T = BI^m_{m \times m}C \), \( \det B > 0 \), and \( \det C > 0 \). Let \( L = C^T \) and \( R = B^T \). Then

\[
A = C^T I^m_{m \times n}B^T = LI^m_{r \times n}R,
\]
\[
\det L = \det C^T = \det C > 0, \quad \det R = \det B^T = \det B > 0. \quad \square
\]

**Lemma 3.** Let \( m, n \in \{1, 2, \ldots\} \) and \( r \in \{0, 1, 2, \ldots\} \) be such that \( m \neq n \) or \( r < m = n \). Let \( A, B \in \mathbb{R}_r^{m \times n} \). Then there exists an analytic mapping \( F: \mathbb{R} \rightarrow \mathbb{R}_r^{m \times n} \) such that, for every \( k \in \mathbb{Z} \), \( F(k) = A \) if \( k \) is even and \( F(k) = B \) if \( k \) is odd. Moreover, \( F(t + 2) = F(t) \) for every \( t \in \mathbb{R} \).

**Proof.** By Lemma 2 there exist \( A_1, B_1 \in \mathbb{R}_r^{m \times m} \) and \( A_2, B_2 \in \mathbb{R}_n^{n \times n} \) such that

\[
A = A_1 I^m_{m \times n} A_2, \quad B = B_1 I^m_{m \times n} B_2,
\]
\[
\det A_1 > 0, \quad \det A_2 > 0, \quad \det B_1 > 0, \quad \det B_2 > 0.
\]
By Lemma 1 there exist analytic mappings \( F_1: \mathbb{R} \rightarrow \mathbb{R}_r^{m \times m} \) and \( F_2: \mathbb{R} \rightarrow \mathbb{R}_n^{n \times n} \) such that, for every \( k \in \mathbb{Z} \), \( F_1(k) = A_1, \ F_2(k) = A_2 \) if \( k \) is even and \( F_1(k) = B_1, \ F_2(k) = B_2 \) if \( k \) is odd. Let

\[
F(t) = F_1(t) I^m_{m \times n} F_2(t) \quad \forall t \in \mathbb{R}.
\]
Then \( F: \mathbb{R} \rightarrow \mathbb{R}_r^{m \times n} \) is analytic, and, for every \( k \in \mathbb{Z} \), \( F(k) = A_1 I^m_{m \times n} A_2 = A \) if \( k \) is even and \( F(k) = B_1 I^m_{m \times n} B_2 = B \) if \( k \) is odd. \( \square \)

**Theorem 4.** The subset \( \mathbb{R}_r^{n \times n} \) of \( \mathbb{R}^{n \times n} \) has two connected components, whereas the subset \( \mathbb{R}_r^{m \times n} \) of \( \mathbb{R}^{m \times n} \) has only one connected component when \( m \neq n \) or
r < m = n. When r > 0, all these connected components are connected by analytic regular arcs. More precisely: Suppose r > 0. Let A, B ∈ ℜ_{m \times n}^r be such that A ≠ B, and, if r = m = n, then det A and det B have the same sign. Then there exists an analytic mapping \( F: \mathbb{R} \to \mathbb{R}_{m \times n}^r \) such that \( F(0) = A, \ F(1) = B, \) and \( F'(t) \neq 0 \) for every \( t \in [0, 1] \).

**Proof.** By Lemma 1 (if \( r = m = n \)) and Lemma 3 (if \( m \neq n \) or \( r < m = n \)), there exists an analytic mapping \( G: \mathbb{R} \to \mathbb{R}_{m \times n}^r \) such that \( G(0) = A \) and \( G(1) = B \).

Let

\[
\chi = \{ t \in [0, 1] \mid \exists \lambda(t) \in \mathbb{R}, \ G'(t) = \lambda(t)G(t) \}.
\]

**Case 1:** Suppose \( \chi \) is infinite. Then there exist \( t_1, t_2, \ldots \in \chi \) and \( t_0 \in [0, 1] \) such that \( t_0 = \lim_{k \to \infty} t_k \). For every \( t \in \mathbb{R}, \ i \in \{1, \ldots, m\}, \ j \in \{1, \ldots, n\}, \) let \( g_{ij}(t) \) denote the entry of the \( i \)-th row, \( j \)-th column of \( G(t) \). Since \( r > 0 \), there exist \( i_0 \in \{1, \ldots, m\} \) and \( j_0 \in \{1, \ldots, n\} \) such that \( g_{i_0j_0}(t_0) \neq 0 \). As \( g_{i_0j_0} \) is continuous, there exists a neighborhood \( N_{t_0} \subseteq \mathbb{R} \) of \( t_0 \) such that \( g_{i_0j_0}(t) \neq 0 \) for every \( t \in N_{t_0} \). Because \( \lim_{k \to \infty} t_k = t_0 \), there exists \( k_0 \in \mathbb{N} \) such that \( t_k \in N_{t_0} \) for every \( k \in \{k_0, k_0+1, \ldots\} \). Let \( k \in \{k_0, k_0+1, \ldots\} \). Since \( t_k \in \chi \), we have \( G'(t_k) = \lambda(t_k)G(t_k) \), which implies \( g'_{i_0j_0}(t_k) = \lambda(t_k)g_{i_0j_0}(t_k) \) and, hence, \( g_{i_0j_0}(t_k)G'(t_k) = g'_{i_0j_0}(t_k)G(t_k) \). As \( \lim_{k \to \infty} t_k = t_0 \), it follows by the Analytic Continuation Theorem that \( g_{i_0j_0}(t)G'(t) = g'_{i_0j_0}(t)G(t) \) for every \( t \in \mathbb{R} \). Let \( g = g_{i_0j_0} \). The equality \( gG' - g'G = 0 \) implies that \( (G(t)/g(t))' = 0 \) for every \( t \in \mathbb{R} \). Therefore, there exists a constant matrix \( M_0 \in \mathbb{R}_{m \times n}^r \) such that \( G(t) = g(t)M_0 \) for every \( t \in N_{t_0} \) and hence for every \( t \in \mathbb{R} \) by analytic continuation. Because \( \text{rank} \, G = r > 0 \), the equality \( G = gM_0 \) implies that \( g(t) \neq 0 \) for every \( t \in \mathbb{R} \) and \( M_0 \neq 0 \). Furthermore, \( A = G(0) = g(0)M_0 \) and \( B = G(1) = g(1)M_0 \). Let

\[
F(t) = \left\{ t(g(1) - g(0)) + g(0) \right\}M_0 \quad \forall t \in \mathbb{R}.
\]

Then

\[
F(0) = g(0)M_0 = A, \quad F(1) = g(1)M_0 = B,
\]

and

\[
F'(t) = (g(1) - g(0))M_0 \neq 0,
\]

for every \( t \in \mathbb{R} \), because \( A \neq B \) implies that \( g(0) \neq g(1) \).

**Case 2:** Suppose that \( \chi \) is finite. Then there exist \( t_0, \ldots, t_q \in [0, 1] \) such that \( 0 = t_0 < t_1 < \cdots < t_q = 1 \) and

(1) \( \chi \subseteq \{ t_0, \ldots, t_q \} \).

By Hermite interpolation, there exists a polynomial \( p \in \mathbb{R}[X] \) such that, for every \( k \in \{0, \ldots, q\} \),

(2) \( p(t_k) = \frac{1}{2} \),

(3) \( t_k \in \chi \) and \( \lambda(t_k) \neq 0 \Rightarrow p'(t_k) = \lambda(t_k) \),

(4) \( t_k \in \chi \) and \( \lambda(t_k) = 0 \Rightarrow p'(t_k) = 1 \).

For every \( t \in \mathbb{R} \), let

\[
q(t) = p(t)^2 + \frac{3}{4}, \quad F(t) = q(t)G(t).
\]
Then, by (2), \( F(0) = G(0) = A \) and \( F(1) = G(1) = B \). Moreover, for every \( t \in \mathbb{R} \), we have \( \text{rank} \ F(t) = \text{rank} \ G(t) = r \), because \( q(t) \neq 0 \). Let \( t \in [0, 1] \). Let us show that \( F'(t) \neq 0 \). Suppose \( F'(t) = 0 \). Then

\[
0 = F'(t) = q'(t)G(t) + q(t)G'(t).
\]

Consequently,

\[
G'(t) = -\frac{q'(t)}{q(t)}G(t),
\]

which implies that \( t \in \chi \). It follows by (1), (2), (3), (4) that

\[
q(t) = p(t)^2 + \frac{3}{2} = 1,
\]

\[
q'(t) = 2p(t)p'(t) = p'(t) = \begin{cases} \lambda(t) & \text{if } \lambda(t) \neq 0, \\ 1 & \text{if } \lambda(t) = 0. \end{cases}
\]

On the other hand, since \( t \in \chi \), we have

\[
G'(t) = \lambda(t)G(t),
\]

and, hence, by (5),

\[
0 = (q'(t) + q(t)\lambda(t))G(t).
\]

Since \( r > 0 \), we have \( G(t) \neq 0 \), and it follows that

\[
q'(t) + q(t)\lambda(t) = 0.
\]

Consequently, by (6) and (7),

\[
0 = \lambda(t) + \lambda(t) = 2\lambda(t) \quad \text{if } \lambda(t) \neq 0
\]

and

\[
0 = 1 + 1 \cdot 0 = 1 \quad \text{if } \lambda(t) = 0.
\]

Both cases are impossible. Therefore, \( F'(t) \neq 0 \) for every \( t \in [0, 1] \). □

**Theorem 5** (Existence of orthonormal bases of class \( C^p \) of the kernel and the image of a rectangular matrix function of \( q \) real variables). Let \( \Omega \subseteq \mathbb{R}^q \) be \( C^p \)-diffeomorphic to \( \mathbb{R}^q \). Let \( A \in C^p(\Omega, \mathbb{R}^{m \times n}) \). Then there exist

\[
u_1, \ldots, u_m \in C^p(\Omega, \mathbb{R}^m), \quad v_1, \ldots, v_n \in C^p(\Omega, \mathbb{R}^n)
\]

such that, for every \( t \in \Omega \),

(a) if \( r > 0 \), then \( (u_1(t), \ldots, u_r(t)) \) is an orthonormal basis of \( \text{Im} \ A(t) \);

(b) if \( r < m \), then \( (u_{r+1}(t), \ldots, u_m(t)) \) is an orthonormal basis of \( (\text{Im} \ A(t))^\perp \);

(c) if \( r > 0 \), then \( (v_1(t), \ldots, v_r(t)) \) is an orthonormal basis of \( (\text{Ker} \ A(t))^\perp \);

(d) if \( r < n \), then \( (v_{r+1}(t), \ldots, v_n(t)) \) is an orthonormal basis of \( \text{Ker} \ A(t) \).

**Proof.** The proof is the same as the proof of Theorem 8.2 of [2] except for the following modifications:

(a) Replace \( \mathbb{C} \) by \( \mathbb{R} \) in the proof of Theorem 8.2 of [2].

(b) In the proof of Lemma 8.1 of [2], apply Theorem 4 of this paper instead of Theorem 7.2 of [2].

(c) In the proof of Lemma 8.1 of [2], if \( \det X(t) < 0 \), then multiply the first column of \( A(t) \) and \( X(t) \) by \(-1\). □
THE SET OF ALL $m \times n$ RECTANGULAR REAL MATRICES OF RANK $r$

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WYOMING, LARAMIE, WYOMING 82071–3036

E-mail address, J.-Cl. Evard: matdifeq@corral.uwyo.edu
E-mail address, F. Jafari: fjafari@corral.uwyo.edu