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A FUNCTION IN THE DIRICHLET SPACE SUCH THAT ITS FOURIER SERIES DIVERGES ALMOST EVERYWHERE

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ABSTRACT. An analytic function F on the disc belongs to B if $\|F\|_B = \int_0^1 \int_0^{2\pi} |F'(re^{i\theta})|\,d\theta\,dr < \infty$. Notice that $B \subsetneqq H^1 \subsetneqq L^1$, where H^1 is the Hardy space of all analytic functions F so that

$$\|F\|_{H^1} = \sup_{0<r<1} \int_0^{2\pi} |F(re^{i\theta})|\,d\theta < \infty,$$

L^1 is the Lebesgue space of integrable functions on $[0, 2\pi]$, and the inclusion $H^1 \subsetneqq L^1$ is taken in the sense of boundary values, that is, $F \in H^1 \Rightarrow \lim_{r\to 1^-} \Re F(re^{i\theta}) \in L^1$.

Kolmogorov in 1923 showed that there exists an f in L^1 so that its Fourier series diverges almost everywhere. In 1953 Sunouchi showed that there exists an f in H^1 with an almost everywhere divergent Fourier series. The purpose of this note is to announce.

Theorem 1. *There exists an $f \in B$, whose Fourier series diverges a.e.*

This problem was mentioned to the first author by Professor Guido Weiss.

An analytic function F on the disc belongs to the Dirichlet space B if $\|F\|_B = \int_0^1 \int_0^{2\pi} |F'(re^{i\theta})|\,d\theta\,dr < \infty$. Notice that $B \subsetneqq H^1 \subsetneqq L^1$, where H^1 is the Hardy space of all analytic functions F on the disc so that $\|F\|_{H^1} = \sup_{0<r<1} \int_0^{2\pi} |F(re^{i\theta})|\,d\theta < \infty$, L^1 is the Lebesgue space of all integrable functions on $[0, 2\pi]$, and the inclusion $H^1 \subsetneqq L^1$ is taken in the sense of boundary values, that is, $F \in H^1$ implies $\lim_{r\to 1} \Re F(re^{i\theta}) \in L^1$.

Kolmogorov in 1923 [4] showed that there exists an f in L^1 such that its Fourier series diverges almost everywhere. Also it was shown by Sunouchi [5], who modified the example given in [3], that there is a function in H^1 with a divergent Fourier series. Therefore a natural question to ask is: Do the functions in B have convergent Fourier series? This question was asked by Professor Guido Weiss to the first author.

In this paper we answer this question negatively. We will prove that a subclass of functions in [3], which Sunouchi [5] proved to belong to H^1 and have divergent Fourier series, indeed belongs to B.

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The example in [3] is defined as follows: Let $a_j = 4\pi j/(2n_k+1)$, $0 \le j \le n_k$, and let $m_0, m_1, \dots, m_{n_k}$ be the integers such that $2m_j+1$ is an integer multiple of $2n_k+1$, $m_0 \ge n_k^4$, and $m_{j+1} > 2m_j$. Now define

$$\phi_{n_k}(t) = \frac{1}{n_k}\sum_{j=0}^{n_k-1} K_{m_j}(t-a_j)$$

(where K_{m_j} is the Fejer kernel), which is a polynomial of order m_{n_k-1}. Then write ϕ_{n_s} in the exponential form; that is,

$$\phi_{n_k}(t) = \sum_{\gamma=-m_{n_k}}^{m_{n_k}} c_\gamma^k e^{i\gamma t}.$$

Finally define the function

$$F(t) = \sum_{k=0}^{\infty} \alpha_k e^{i\gamma_k t}\phi_{n_k}(t), \quad \text{where } \sum_{k=0}^{\infty}\alpha_k < \infty \text{ and } \alpha_k \ge 0. \tag{1}$$

Hardy and Rogosinki in [3] showed that if

$$\gamma_k + m_{n_k} < \gamma_{k+1} - m_{n_{k+1}} \quad \text{for all } k \tag{2}$$

then $F \in L^1$ and has a divergent Fourier series.

Sunouchi in [5] showed that if

$$\gamma_k - m_{n_k} \ge 0 \quad \text{for all } k \tag{3}$$

in addition to the hypothesis of Hardy-Rogosinki, then F belongs to H^1.

In this note we show that if for all k

$$\gamma_k \ge m_{n_k}^2 \tag{4}$$

along with the hypothesis of Sunouchi, then F belongs to B. We may relax the hypothesis by

$$\gamma_k > 2m_{n_k} \quad \text{for all } k. \tag{4'}$$

We will prove this as a consequence of some lemmas, but first we need some definitions.

Recall that the Fejer kernel is $K_{n-1}(t) = \frac{2}{n}[\frac{\sin nt/2}{2\sin t/2}]^2$ where n is an integer and $n \ge 1$. Let γ and n be integers. Then we define the complex-valued function $G_{\gamma,n}$ on the unit disc $D = \{z \in C;\ |z| \le 1\}$:

$$G_{\gamma,n}(z) = \frac{1}{2n}\left[\sum_{j=0}^{n-1}(n-j)z^{\gamma-j} + \sum_{j=1}^{n-1}(n-j)z^{\gamma+j}\right].$$

Notice that $G_{\gamma,n}(z)$ is analytic everywhere if $\gamma \ge n-1$, $\lim_{r\to 1} G_{\gamma,n}(re^{it}) = e^{i\gamma t}K_{n-1}(t)$, and this limit is uniform for $t \in [0, 2\pi]$.

Lemma 1. *The above-defined analytic function $G_{\gamma,n}$ belongs to H^1. Moreover, $\|G_{\gamma,n}\|_{H^1} = \pi$ for $\gamma \ge n-1$.*

Proof.

$$\|G_{\gamma,n}\|_{H^1} = \lim_{r\to 1}\int_0^{2\pi} |G_{\gamma,n}(re^{i\theta})|d\theta = \int_0^{2\pi} |e^{i\gamma\theta}K_{n-1}(\theta)|d\theta = \pi.$$

Lemma 2. *If* $\gamma \geq n^2$ *then* $f(t) = e^{i\gamma t}K_{n-1}(t)$ *is in* B. *Moreover,* $\|f\|_B \leq C$, *where* C *is independent of* γ *and* n.

Proof. Notice that the derivative of $G_{\gamma,n}$ is given by

$$G'_{\gamma,n}(z) = \gamma z^{\gamma-n}G_{n-1,n}(z) + H(z),$$

where

$$H(z) = \frac{z^{\gamma-n}}{2n}\left[\sum_{j=1}^{n-1}(n-j)jz^{n+j-1} - \sum_{j=1}^{n-1}(n-j)jz^{n-j-1}\right].$$

Therefore,

$$\begin{aligned}\|G_{\gamma,n}\|_B &= \int_0^1\int_0^{2\pi}|G'_{\gamma,n}(re^{i\theta})|d\theta dr\\ &\leq \int_0^1\int_0^{2\pi}\gamma r^{\gamma-n}|G_{n-1,n}(re^{i\theta})|d\theta\,dr + \int_0^1\int_0^{2\pi}|H(re^{i\theta})|\,d\theta\,dr\\ &= \mathrm{I} + \mathrm{II}.\end{aligned}$$

Estimate for I. Here we use Lemma 1 with $\gamma = n-1$:

$$\begin{aligned}\mathrm{I} &= \gamma\int_0^1\left(\int_0^{2\pi} r^{\gamma-n}|G_{n-1,n}(re^{i\theta})|\,d\theta\right)dr\\ &\leq \gamma\|G_{n-1,n}\|_{H^1}\cdot\int_0^1 r^{\gamma-n}\,dr = \frac{\pi\gamma}{\gamma-n+1} \leq 2\pi\end{aligned} \tag{5}$$

for $\gamma \geq n^2 \geq 2n$.

Estimate for II.

$$\begin{aligned}\mathrm{II} &= \int_0^1\int_0^{2\pi}|H(re^{i\theta})|\,d\theta\,dr \leq \frac{1}{n}\int_0^1\int_0^{2\pi} r^{\gamma-n}\sum_{j=1}^{n-1}nj\,d\theta\,dr\\ &= 2\pi\sum_{j=1}^{n-1}j\int_0^1 r^{\gamma-n}\,dr = \frac{\pi n(n-1)}{\gamma-n+1}.\end{aligned}$$

By (5) we have $\mathrm{II} \leq \pi$. Therefore, $\|G_{\gamma,n}\|_B < 3\pi$.

Now we have to show that the function F defined in (1) belongs to B; in fact, we have the following

Theorem. *The function* F *defined in* (1) *and satisfying* (2), (3), *and* (4) *belongs to* B.

Proof. We want to show that $F(t) = \sum_{k=0}^{\infty}\alpha_k e^{i\gamma_k t}\phi_{n_k}(t)$ is in B. This follows by showing that $e^{i\gamma_k t}\phi_{n_k}(t)$ is in B and $\|e^{i\gamma_k(\cdot)}\phi_{n_k}(\cdot)\|_B \leq C$, with C independent of n_k and γ_k. In fact, using Lemma 2 and the fact that $\|g(t-a)\|_B =$

$\|g(t)\|_B$, $\|cg(t)\|_B = |c|\|g(t)\|_B$, we have

$$\|e^{i\gamma_k t}\phi_{n_k}(t)\|_B \le \frac{1}{n_k}\sum_{j=0}^{n_k-1}\|e^{i\gamma_k t}K_{m_j}(t-a_j)\|_B$$

$$= \frac{1}{n_k}\sum_{j=0}^{n_k-1}\|e^{i\gamma_k(t-a_j)}K_{m_j}(t-a_j)\cdot e^{i\gamma_k a_j}\|_B = \frac{1}{n_k}\sum_{j=0}^{n_k-1}\|e^{i\gamma_k(t-a_j)}K_{m_j}(t-a_j)\|_B$$

$$= \frac{1}{n_k}\sum_{j=0}^{n_k-1}\|e^{i\gamma_k t}K_{m_j}(t)\|_B \le \frac{1}{n_k}\sum_{j=0}^{n_k-1}3\pi = \frac{3\pi(n_k)}{n_k} = 3\pi.$$

Now,

$$\|F\|_B \le \sum_{k=0}^{\infty}\alpha_k\|e^{i\gamma_k t}\phi_{n_k}(t)\|_B \le 3\pi\sum_{k=0}^{\infty}\alpha_k < \infty.$$

Therefore, $F \in B$.

Remark. With more technical arguments, we can replace condition (4) by (4′) in the theorem.

It is well known that the spaces H_ϕ over $[0, 2\pi]$ defined in [6] contain B for any nontrivial H_ϕ. Then $B \subset H_\phi$; we conjecture that $B \subsetneqq H_\phi$. It was shown by Y. Meyer [8] that the space B is in some sense a minimal space.

Corollary. *There is an $f \in H_\phi$, $H_\phi \ne \{0\}$, whose Fourier series diverges almost everywhere.*

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