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THE FILTRATION EQUATION IN A CLASS OF FUNCTIONS DECREASING AT INFINITY

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ABSTRACT. We deal with the Cauchy and external boundary problems for the nonlinear filtration equation with variable density. For each density we define a class ϕ of initial functions φ, such that for any $\varphi \in \phi$ the problem is uniquely solvable in some set of functions decreasing at infinity with respect to space variables.

It is well known that the unique solvability of the Cauchy problem for the heat equation

$$\rho u_t = \Delta u$$

with density $\rho = \text{const}$ takes place in classes of increasing functions. This means that if an initial function φ belongs to some class ϕ of functions increasing at infinity, then there exits a unique solution u of the corresponding Cauchy problem, belonging to ϕ for any $t > 0$. An analogous result is valid for the nonlinear filtration equation in the case $\rho = \text{const}$ (see [1, 5]). However, for $\rho = \rho(x)$ it is, generally speaking, incorrect. It was shown in [4] that if ρ tends to zero fast enough as $|x| \to \infty$ and $\varphi \in L^\infty(\mathbb{R}^n)$, $n \geq 3$, then the Cauchy problem can have more than one solution belonging to the class $L^\infty(\mathbb{R}^n)$ for $t > 0$. Nevertheless, the unique solvability is valid here in a more restricted class of solutions, namely in a class of functions decreasing fast enough as $|x| \to \infty$. It is essential that the behaviour of solutions for $t > 0$ as $|x| \to \infty$ is different from that of initial functions.

Here we extend the results of [4] to the case of arbitrary $\rho(x) > 0$ and initial functions φ not necessarily belonging to $L^\infty(\mathbb{R}^n)$. We also deal with an external boundary problem. In order to include the above-mentioned case, where $\rho \to 0$ fast enough at infinity, we are looking for a solution in a class of functions decreasing as $|x| \to \infty$ in some integral sense. It turns out that initial functions can belong to a more general class. Thus, due to the nature of the problem, unlike [1], our class of unique solvability is different from the class of initial functions. Let us give a precise formulation.

We are looking for a nonnegative solution $u(x, t)$ of the following problem on the domain $Q_T := \Omega \times (0, T)$, $T > 0$, where Ω is the exterior of an

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$(n-1)$-dimensional smooth closed surface $\partial\Omega$ in the space $\mathbb{R}^n$, $n \geq 3$:

$$(1)\qquad \rho(x)u_t = \Delta G(u), \qquad (x, t) \in Q_T,$$

$$(2)\qquad u|_{t=0} = \varphi(x), \qquad x \in \Omega,$$

$$(3)\qquad u|_{\partial\Omega} = 0, \qquad t \in (0, T).$$

In this problem $\varphi \geq 0$ on Ω, $\rho, \varphi \in L^\infty(\Omega(R))$, $\rho(x) \geq C(R) > 0$ on $\Omega(R)$ where $\Omega(R) = \{x: x \in \Omega,\ |x| < R\}$, for all large enough $R > 0$,

$$(4)\qquad \int_\Omega \frac{\rho\varphi}{|x|^{n-2}}\, dx < \infty,$$

$G(u)$ is a smooth function for $u \geq 0$,

$$(5)\qquad G(0) = 0,$$

$$(6)\qquad G'(u) > 0 \quad \forall u > 0,$$

$$(7)\qquad G(u) \geq \gamma u \quad \forall u \geq 1,$$

and $\gamma = \text{const} > 0$. Moreover, $G'(u)$ is increasing in some $0 < u < \delta$ if $G'(0) = 0$.

Our main result: under the imposed restrictions the problem (1), (2), (3) has a unique nonnegative solution, satisfying the following condition:

$$(8)\qquad \lim_{R\to\infty} R^{1-n} \int_{S(R)} U(x)\, ds = 0,$$

where $S(R) = \{x: x \in \Omega,\ |x| = R\}$,

$$U(x) = \int_0^T G(u(x, t))\, dt.$$

This is valid also for the Cauchy problem (1), (2), where $\Omega = \mathbb{R}^n$. We assume below that the Cauchy problem is a special case of (1), (2), (3), where $\Omega = \mathbb{R}^n$, $\partial\Omega = \varnothing$.

Thus, if $\rho \to 0$ fast enough as $|x| \to \infty$, then $\varphi(x)$ can increase for large $|x|$, provided that (4) remains valid. Then the solution of (1), (2), (3) is decreasing as $|x| \to \infty$ in the integral sense of (8).

The most important special case of (1) is the equation

$$(9)\qquad \rho(x)u_t = \Delta(u^m), \qquad m > 1.$$

Separating variables in (9), we get the stationary equation

$$(10)\qquad -\Delta(w^m) = \frac{1}{(m-1)}\rho(x)w.$$

The recent paper [2] is devoted to the existence and uniqueness of a nontrivial nonnegative solution of (10) in $\mathbb{R}^n$, $n \geq 3$. Let us notice that the asymptotics as $t \to \infty$ of the solution of the Cauchy problem (9), (2) has the form $w(x)\zeta(t)$, where w is the solution of (10). We hope to give later a proof and possible generalizations for this statement.

Suppose $R \geq R_0 > 0$, where R_0 is chosen so that the ball $|x| < R_0$ contains the surface $\partial\Omega$. Set

$$\begin{gathered}\mathbb{R}_+ = \{\xi : \xi \in \mathbb{R},\ \xi \geq 0\}, \\ S_T(R) = \{(x, t) : x \in \mathbb{R}^n,\ |x| = R,\ 0 < t < T\}, \\ \Omega(R_1, R_2) = \{x : x \in \Omega,\ R_1 < |x| < R_2\}, \\ Q_T(R) = \Omega(R) \times (0, T), \qquad T > 0, \\ \widetilde{L}^m_{\text{loc}}(\Omega) = \{v : v \in L^m(\Omega(R))\ \forall R > R_0\}, \qquad 1 \leq m \leq \infty, \\ \widetilde{L}^m_{\text{loc}}(Q_T) = \{u : u \in L^m(Q_T(R))\ \forall R > R_0\}, \qquad 1 \leq m \leq \infty.\end{gathered}$$

Let $\widetilde{C}^\infty_0(Q_T(R))$ be a set of functions $g(x, t)$ belonging to $C^\infty(Q_T(R)) \cap C(\overline{Q_T(R)})$, such that

$$g|_{\partial\Omega} = 0, \qquad g|_{S(R)} = 0, \tag{11}$$

for any $t \in (0, T)$, and $g = 0$ if $t > T - \varepsilon$, where $\varepsilon = \varepsilon(g) \in (0, T)$.

Analogously, $\widetilde{C}^\infty_0(Q_T)$ will be a set of functions $g(x, t)$ belonging to $C^\infty(Q_T) \cap C(\overline{Q}_T)$ satisfying the first of conditions (11) and such that $g = 0$ if $t > T - \varepsilon(g)$ or $|x| > R(g)$, where $\varepsilon(g) \in (0, T)$, $R(g) > R_0$.

Later on we shall use a weak solution $u_{R,\theta}$ of equation (1) on the domain $Q_T(R)$, which satisfies the initial condition (2) for $x \in \Omega(R)$ and boundary conditions (3) and

$$u|_{S_T(R)} = \theta,$$

where $\theta \in L^\infty(S_T(R))$, $\theta \geq 0$ a.e. on $S_T(R)$. Then $u_{R,\theta} \in C(Q_T(R)) \cap L^\infty(Q_T(R))$ and for any $g \in \widetilde{C}^\infty_0(Q_T(R))$ we have

$$\begin{aligned}&\int_{Q_T(R)} (G(u_{R,\theta})\Delta g + \rho g_t u_{R,\theta})\,dx\,dt + \int_{\Omega(R)} \rho\varphi g\,dx \\ &\qquad = \int_{S_T(R)} G(\theta) g_\nu\,dS_T(R),\end{aligned} \tag{12}$$

where ν is an external normal on $S_T(R)$. The existence of such solutions can be proved by an approximating procedure (see, e.g., [6, 1, 4]) using the results of [3].

For $\theta = 0$ we set $u_{R,0} = u_R$.

Definition 1. A weak solution of the problem (1), (2), (3) is a nonnegative function $u \in \widetilde{L}^1_{\text{loc}}(Q_T)$ satisfying

(A)

$$U \in \widetilde{L}^\infty_{\text{loc}}(\Omega), \tag{13}$$

where U is as in (8),

(B)

$$\int_{Q_T} (G(u)\Delta g + \rho u g_t)\,dx\,dt + \int_\Omega \rho\varphi g(x, 0)\,dx = 0 \quad \forall g \in \widetilde{C}^\infty_0(Q_T), \tag{14}$$

and

(C) there exist such sequences R_m, θ_m, $R_m > R_0$, $\theta_m \in L^\infty(S_T(R_m))$, $m = 1, 2, \dots$, that $\theta_m \geq 0$ for $(x, t) \in S_T(R_m)$ and $u_{R_m,\theta_m} \to u$ a.e. on Q_T as $m \to \infty$.

Note. By the comparison principle we obtain, using condition (C), that

$$u_R \leq u \quad \forall R \geq R_0, \tag{15}$$

a.e. on $Q_T(R)$.

Remark 1. If $u \in \widetilde{L}^{\infty}_{\text{loc}}(Q_T)$, then condition (C) is obvious. In particular, if $\varphi \in L^{\infty}(\Omega)$, then a function $u \in L^{\infty}(Q_T)$ is a weak solution if it satisfies condition (B) (cf. [4]).

Below we use the word "solution" instead of "weak solution".

Definition 2. A solution u of the problem (1), (2), (3) is minimal if for any solution v of this problem

$$u \leq v \tag{16}$$

a.e. on Q_T.

Later on, $\eta(z)$, $\eta\colon \mathbb{R} \to \mathbb{R}$, will be a function chosen so that $\eta \in C^{\infty}(\mathbb{R})$, $0 \leq \eta \leq 1$ on $\mathbb{R}$, where $\eta = 0\ \forall z \geq 1$, $\eta = 1\ \forall z \leq 0$, and $\eta' \leq 0\ \forall z \in \mathbb{R}$. For $\varepsilon \in (0, T)$, $\tau \in (0, T - \varepsilon)$, $R > R_0$, set

$$\eta_{\varepsilon\tau}(t) = \eta\left(\frac{t-\tau}{T-\varepsilon-\tau}\right),$$

$$V_R(r) = \eta\left(\frac{r}{R} - 1\right), \qquad r = |x|, \ x \in \mathbb{R}^n.$$

The following lemma is known (see e.g., [2, Lemma A.4]).

Lemma. *Suppose* $n \geq 3$, $\mu\colon \mathbb{R}^n \to \mathbb{R}^+$, $\mu \in \widetilde{L}^{\infty}_{\text{loc}}(\mathbb{R}^n)$, $\mu|x|^{2-n} \in L^1(\mathbb{R}^n)$, *and* $U = |x|^{2-n} * \mu$. *Then the function* U *satisfies condition* (8).

Theorem 1. *Suppose* $n \geq 3$, *the function* $G\colon \mathbb{R}_+ \to \mathbb{R}$ *satisfies the conditions* (5), (6), (7), ρ, φ *are as above, and* (4) *is valid. Then there exists a minimal solution of the problem* (1), (2), (3) *satisfying condition* (8).

Proof. Let y be a fixed point in $\Omega(R)$, where $R > R_0$, and let h_m be the solution of the following problem on $\Omega(R)$:

$$\Delta h = -\delta_m(x, y), \tag{17}$$

$$h|_{\partial\Omega} = h|_{S(R)} = 0, \tag{18}$$

where $m = 1, 2, \ldots$, $\delta_m(x, y)$ as a function of x is chosen so that $\delta_m \geq 0$ on $\Omega(R)$, $\delta_m \in C^{\infty}(\Omega(R))$, $\operatorname{supp}\delta_m \subset \Omega(R)$, and $\delta_m \xrightarrow[m\to\infty]{} \delta(x-y)$ in the sense of distributions over $C^{\infty}(\Omega(R))$. Then in the same sense $h_m(x, y) \to \Gamma_R(x, y)$, where $\Gamma_R(x, y)$ is the Green function of the Dirichlet problem on $\Omega(R)$. Let $g_m = h_m \eta_{\varepsilon\tau}(t)$, where $\varepsilon \in (0, T)$, $\tau \in (0, T - \varepsilon)$. Then from (12), where $g = g_m$, $\theta = 0$, $G(\theta) = 0$ we have

$$\int_{\Omega(R)} \delta_m \, dx \int_0^{\tau} G(u_R)\, dt \leq \int_{\Omega(R)} \rho\varphi h_m \, dx. \tag{19}$$

Passing to the limit as $m \to \infty$ we obtain that for all $y \in \Omega$

$$\int_0^{\tau} G(u_R(y, t))\, dt \leq c_n \int_{\Omega} \frac{\rho(x)\varphi(x)}{|x-y|^{n-2}}\, dx, \tag{20}$$

where c_n depends on n only. Suppose $r \in (R_0, R)$. Then by (4) and (20) for any $y \in \Omega(r)$ we have

$$\int_0^T G(u_R(y, t))\,dt \leq c(r). \tag{21}$$

Using the comparison principle we obtain that u_R (and consequently $G(u_R)$) is a nondecreasing function of R for $(x, t) \in Q_T$. Now from (21), (7) we can obtain that a.e. on Q_T, $u_R \to u \in \widetilde{L}^1_{\mathrm{loc}}(Q_T)$ as $R \to \infty$, $G(u) \in \widetilde{L}^1_{\mathrm{loc}}(Q_T)$. Moreover, $G(u) \in L^1(0, T)$ and (13) is valid. It follows from the lemma and (4), (20) that function u satisfies condition (8). Suppose $g \in \widetilde{C}_0^\infty(Q_T)$ and $\operatorname{supp} g \subset Q_T(R)$. Passing to the limit as $R \to \infty$ in (12), where $G(\theta) = 0$, we obtain (14). It is obvious now that condition C is valid and u is a minimal solution. Q.E.D.

Remark 2. Putting, as above, $g = g_m$, $\theta = 0$ into (12) and passing to the limit as $\varepsilon \to T - \tau$, $m \to \infty$, we obtain that $u(\cdot, \tau)$, where $\tau \in (0, T)$, belongs, like φ, to the class of functions summable over Ω with the weight $\rho|x|^{2-n}$. Generally speaking there is no uniqueness of solutions in this class (see [4]).

The uniqueness we shall prove in the class of functions satisfying the following condition at infinity

$$\lim_{R\to\infty} R^{-n} \int_{\Omega(R, 2R)} dx \int_0^T G(u)\,dt = 0, \tag{22}$$

which is slightly weaker than (8).

Theorem 2. *Let the conditions of Theorem* 1 *be satisfied, with the possible exception of* (4). *Then the problem* (1), (2), (3) *has only one solution, satisfying condition* (22).

Proof. Suppose v is a solution of our problem satisfying (22). Due to (15), a.e. on $Q_T(R)$

$$u_R \leq v. \tag{23}$$

The function u_R is nondecreasing with respect to R and $v \in \widetilde{L}^1_{\mathrm{loc}}(Q_T)$. Now we obtain from (23) that $u_R \to u \in \widetilde{L}^1_{\mathrm{loc}}(Q_T)$ a.e. on Q_T as $R \to \infty$, where u is a minimal solution of the problem (1), (2), (3), satisfying (22). It remains to prove that $v = u$. By Definition 1

$$\int_{Q_T} ((G(v) - G(u))\Delta g + (v - u)\rho g_t)\,dx\,dt = 0 \quad \forall g \in \widetilde{C}_0^\infty(Q_T), \tag{24}$$

where $v - u \geq 0$, $G(v) - G(u) \geq 0$ a.e. on Q_T. Let us choose some smooth function $f\colon \Omega \to \mathbb{R}_+$, $f \not\equiv 0$, $\operatorname{supp} f \subset \Omega$. Let h be a solution of the problem

$$\Delta h = -f, \qquad x \in \Omega, \tag{25}$$

$$h|_{\partial\Omega} = h|_\infty = 0. \tag{26}$$

Then $h \geq 0$ on Ω and for $s = 0, 1$

$$\left|\frac{\partial^s h}{\partial x_j^s}\right| \leq C(1 + |x|)^{2-n-s}, \qquad j = 1, \dots, n, \tag{27}$$

where $x = (x_1, \ldots, x_n) \in \Omega$. Suppose for $R > R_0$, $\varepsilon \in (0, T)$, and $\tau \in (0, T-\varepsilon)$,

$$g_R(x, t) = h(x) V_R(r) \eta_{\varepsilon\tau}(t).$$

Then $g_R \in \widetilde{C}_0^\infty(Q_T)$. Putting $g = g_R$ in (24) we have

$$\begin{aligned} &\int_{\Omega(2R)} f V_R \, dx \int_0^{T-\varepsilon} (G(v) - G(u)) \eta_{\varepsilon\tau}(t) \, dt \\ &\quad = \int_{Q_T} (v - u) \rho h V_R \eta'_{\varepsilon\tau} \, dx \, dt \\ &\qquad + \int_{\Omega(R, 2R)} (2 \nabla h \nabla V_R + h \Delta V_R) \, dx \int_0^{T-\varepsilon} (G(v) - G(u)) \eta_{\varepsilon\tau}(t) \, dt. \end{aligned} \tag{28}$$

The integral over $\Omega(R, 2R)$ tends to zero as $R \to \infty$ by (22) and (27). Since $\eta'_{\varepsilon\tau} \le 0$ we get from (28) that $G(v) = G(u)$ a.e. on the set $\operatorname{supp} f \times (0, T-\varepsilon)$. Thus, $v = u$ a.e. on Q_T. Q.E.D.

From the previous considerations we immediately obtain the following.

Theorem 3. *Suppose $\varphi_j \colon \Omega \to \mathbb{R}_+$, all the conditions of Theorem 1 are satisfied for the system (G, ρ, φ_j), $j = 1, 2$, and $\varphi_1 \le \varphi_2$ a.e. on Ω. Then $u_1 \le u_2$ a.e. on Q_T, where u_j is a solution of the problem (1), (2), (3) with $\varphi = \varphi_j$.*

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