PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 120, Number 3, March 1994

# AN EXTENSION OF THE HEINZ-KATO THEOREM

TAKAYUKI FURUTA

(Communicated by Palle E. T. Jorgensen)

*Dedicated to Professor Huzihiro Araki on his sixtieth birthday with respect and affection*

ABSTRACT. An extension of the Heinz-Kato theorem is given.

In this paper, we shall extend the famous and well-known Heinz-Kato theorem. Please note that a capital letter means a bounded linear operator on a complex Hilbert space $H$.

**Theorem A** (Heinz-Kato [1, 2]). *Let $T$ be an operator on a Hilbert space $H$. If $A$ and $B$ are positive operators such that $\|Tx\| \le \|Ax\|$ and $\|T^*y\| \le \|By\|$ for all $x, y \in H$, then the following inequality holds for all $x, y \in H$:*

$$(1) \qquad |(Tx, y)| \le \|A^\alpha x\| \, \|B^{1-\alpha} y\| \quad \text{for any } \alpha \in [0, 1].$$

We shall show an extension of Theorem A as follows.

**Theorem 1.** *Let $T$ be an operator on a Hilbert space $H$. If $A$ and $B$ are positive operators such that $\|Tx\| \le \|Ax\|$ and $\|T^*y\| \le \|By\|$ for all $x, y \in H$, then the following inequality holds for all $x, y \in H$:*

$$(2) \qquad |(T|T|^{\alpha+\beta-1}x, y)| \le \|A^\alpha x\| \, \|B^\beta y\|$$

*for any $\alpha$ and $\beta$ such that $\alpha, \beta \in [0, 1]$ and $\alpha + \beta \ge 1$.*

We remark that Theorem A follows by Theorem 1 putting $\alpha + \beta = 1$ in Theorem 1, so that Theorem 1 can be considered as an extension of the Heinz-Kato theorem.

In order to give a proof of Theorem 1, we need the following Theorem B [1, 3] for which there are a lot of proofs—among them, a nice one given in [4].

**Theorem B** (Löwner-Heinz). *If $A \ge B \ge 0$, then $A^\alpha \ge B^\alpha$ holds for each $\alpha \in [0, 1]$.*

Also we cite the following obvious lemma.

**Lemma.** *Let $S$ be positive operator. Then:*

(i) *$(Sx, x) = 0$ holds for some vector $x$ iff $Sx = 0$.*
(ii) *$N(S^q) = N(S)$ holds for any positive real number $q$, where $N(X)$ denotes the kernel of an operator $X$.*

Received by the editors May 4, 1992 and, in revised form, June 20, 1992.
1991 *Mathematics Subject Classification.* Primary 47A30; Secondary 47A63.
*Key words and phrases.* Positive operator, operator inequality.

*Proof of Theorem* 1. First of all, the hypothesis $\|Tx\| \le \|Ax\|$ for all $x \in H$ is equivalent to

$$|T|^2 \le A^2. \tag{3}$$

Also the hypothesis $\|T^*y\| \le \|By\|$ for all $y \in H$ is equivalent to

$$|T^*|^2 \le B^2. \tag{4}$$

Applying Theorem B to (3) and (4), for any $x, y \in H$ we have

$$(|T|^{2\alpha}x, x) \le (A^{2\alpha}x, x) \quad \text{for each } \alpha \in [0, 1], \tag{5}$$

$$(|T^*|^{2\beta}y, y) \le (B^{2\beta}y, y) \quad \text{for each } \beta \in [0, 1]. \tag{6}$$

Let $N(X)$ denote the kernel of an operator $X$. Let $T = U|T|$ be the polar decomposition of an operator $T$, where $U$ is partial isometry and $|T| = (T^*T)^{1/2}$ and $N(U) = N(|T|)$.

In the case $\alpha, \beta \in [0, 1]$ such that $\beta > 0$ and $\alpha + \beta \ge 1$, we recall the following well-known relation on the polar decomposition of $T$:

$$|T^*|^{2\beta} = U|T|^{2\beta}U^* \quad \text{holds for any } \beta > 0. \tag{7}$$

Then for all $x, y \in H$ we have

$$\begin{aligned} |(T|T|^{\alpha+\beta-1}x, y)|^2 &= |(U|T|^{\alpha+\beta}x, y)|^2 = |(|T|^{\alpha}x, |T|^{\beta}U^*y)|^2 \\ &\le \| |T|^{\alpha}x\|^2 \| |T|^{\beta}U^*y\|^2 = (|T|^{2\alpha}x, x)(U|T|^{2\beta}U^*y, y) \\ &= (|T|^{2\alpha}x, x)(|T^*|^{2\beta}y, y) \quad \text{(by (7))} \\ &\le (A^{2\alpha}x, x)(B^{2\beta}y, y) \quad \text{(by (5) and (6))} \end{aligned} \tag{8}$$

for any $\alpha$ and $\beta$ such that $\alpha, \beta \in [0, 1]$ and $\alpha + \beta \ge 1$; that is, (2) holds because the result is trivial in the case $\beta = 0$.

Hence the proof of Theorem 1 is complete.

*Remark* 1. In the case $\alpha > 0$ and $\beta > 0$, the equality in (2) holds for some $x$ and $y$ iff $|T|^{2\alpha}x$ and $|T|^{\alpha+\beta-1}T^*y$ are linearly dependent and $|T|^{2\alpha}x = A^{2\alpha}x$ and $|T^*|^{2\beta}y = B^{2\beta}y$ hold for some $x$ and $y$ together.

In fact, in the case $\alpha > 0$ and $\beta > 0$, the equality in the first inequality of (8) holds iff $|T|^{\alpha}x$ and $|T|^{\beta}U^*y$ are linearly dependent, that is, $|T|^{2\alpha}x$ and $|T|^{\alpha+\beta-1}|T|U^*y$ are linearly dependent by (ii) of Lemma, namely, $|T|^{2\alpha}x$ and $|T|^{\alpha+\beta-1}T^*y$ are linearly dependent.

The equality in the last inequality of (8) holds iff the equality of (5) and the equality of (6) hold together, that is, $|T|^{2\alpha}x = A^{2\alpha}x$ and $|T^*|^{2\beta}y = B^{2\beta}y$ hold together for some vector $x$ and $y$ by (i) of Lemma; so the proof of the equality is complete.

*Remark* 2. The condition $\alpha + \beta \ge 1$ in Theorem 1 is unnecessary if $T$ is a positive operator or invertible operator. This is easily seen in the proof of Theorem 1.

*Remark* 3. We remark that a condition for which $|T|^{2\alpha}x$ and $|T|^{\alpha+\beta-1}T^*y$ are linearly dependent is equivalent to that $T|T|^{\alpha+\beta-1}x$ and $|T^*|^{2\beta}y$ are linearly dependent. In fact, the former condition is equivalent to that $|T|^{\alpha}x$ and $|T|^{\beta}U^*y$ are linearly dependent as stated in the proof of the equality in the first inequality of (8), and this condition is equivalent to that $U|T|^{\alpha+\beta}x$ and $U|T|^{2\beta}U^*y$ are linearly dependent by (ii) of Lemma and $N(U) = N(|T|)$, that is, $T|T|^{\alpha+\beta-1}x$ and $|T^*|^{2\beta}y$ are linearly dependent by (7).

## REFERENCES

1. E. Heinz, *Beiträge zur Strüngstheorie der Spektralzerlegung*, Math. Ann. **123** (1951), 415–438.

2. T. Kato, *Notes on some inequalities for linear operators*, Math. Ann. **125** (1952), 208–212.

3. K. Löwner, *Über monotone Matrixfunktion*, Math. Z. **38** (1934), 177–216.

4. G. K. Pedersen, *Some operator monotone function*, Proc. Amer. Math. Soc. **36** (1972), 309–310.

DEPARTMENT OF APPLIED MATHEMATICS, FACULTY OF SCIENCE, SCIENCE UNIVERSITY OF TOKYO, 1-3 KAGURAZAKA, SHINJUKU, TOKYO 162, JAPAN