PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 120, Number 3, March 1994

A DECOMPOSITION THEOREM FOR $\mathbf{R}^n$

PÉTER KOMJÁTH

(Communicated by Andreas R. Blass)

ABSTRACT. $\mathbf{R}^n$ is the union of countably many sets, none containing two points a rational distance apart.

0. INTRODUCTION

The aim of this paper is to prove that it is possible to color the points of $\mathbf{R}^n$, the n-dimensional Euclidean space, with countably many colors such that if two points are a rational distance apart then they get different colors. This was conjectured by Erdős. If G_n denotes the graph on $\mathbf{R}^n$ where points a rational distance apart are joined, then the result can be reformulated as the statement that the chromatic number of G_n is countable. Erdős observed that the result for G_2 can be deduced from an earlier theorem by him and Hajnal [4], namely, that a graph is countably chromatic, unless it contains $K(2, \omega_1)$, the complete bipartite graph on classes of size 2 and ω_1, respectively. It is easy to see that G_2 does not contain this latter graph. One might be tempted to think that a similar argument works in general. It is, however, fairly easy to find even a $K(\omega, 2^\omega)$ in G_3: take a line, a point p outside it, and let all points on the line, at rational distance from p, constitute one class. The other class will be the points of a full circle, perpendicular to the line and containing p. And even a graph omitting $K(\omega, \omega)$ can be uncountably chromatic by an old result of Erdős and Hajnal [4].

One of the early instances of theorems producing "paradoxical" decompositions of Euclidean spaces can be found in [1], where Ceder showed that the plane can be colored by countably many colors so that the three corners of an equilateral triangle never get the same color. His clever proof "defines" the coloring from a Hamel basis. Most other proofs, however, use a different technique. In order to show that $\mathbf{R}^n$ possesses a certain coloring, one proves, by transfinite induction on $|X|$, that every $X \subseteq \mathbf{R}^n$ does. A prime example can be found in [2]. In another such example, see [6], we prove that $\mathbf{R}^3$ can be colored with countably many colors with no monochromatic regular tetrahedron. This has recently been extended to higher dimensions by J. Schmerl [7].

Received by the editors October 20, 1990 and, in revised form, July 2, 1992.
1991 *Mathematics Subject Classification*. Primary 03E05, 52C10.
Research partially supported by Hungarian National Science Fund OTKA No. 2117.

Our present proof is even more complicated. When coloring the points of a certain set of transfinite induction, it may happen for a point x that there is no color left, all possible colors having been used by the points already colored which are at a rational distance from x. To overcome this difficulty, we assign a set of colors to sets that occur as the points in rational distance form a certain point. For technical reasons we need the more complicated notion of configuration, an intersection of finitely many such sets. By transfinite induction we assign color sets to configurations, along a well-selected well-ordering.

Notation. We use standard axiomatic set theory notation, for example, cardinals are identified with initial ordinals. d denotes Euclidean distance.

1. Configurations

If α, β are real numbers, let $\alpha \equiv \beta$ mean that $\alpha - \beta \in Q$. For the rest of the proof, we fix n.

Definition. A set $F \subseteq \mathbf{R}^n$ is a *configuration*, if $F = \varnothing$, $\mathbf{R}^n$, $\{a\}$ for some $a \in \mathbf{R}^n$, or there exist *distinct* points $a_1, \dots, a_t \in \mathbf{R}^n$ and reals $\alpha_1, \dots, \alpha_t$ such that

$$F = \{x \in \mathbf{R}^n : d^2(x, a_i) \equiv \alpha_i \text{ (all } 1 \le i \le t)\}.$$

Definition. If $a \in \mathbf{R}^n$, then $G(a) = \{x \in \mathbf{R}^n : d^2(x, a) \in Q\}$. This is clearly a configuration.

Lemma 1.1. *There are continuum many configurations.*

Proof. As every configuration is determined by a finite sequence of points and reals, there are at most continuum many of them. And the number of singletons is already continuum.

Lemma 1.2. *The intersection of two configurations is again a configuration.*

Proof. Immediate from the definition.

Lemma 1.3. *If $F_1 \supseteq F_2$ are configurations, $|F_2| \ge 2$, and F_1 is determined by $a_1, \dots, a_t$ and $\alpha_1, \dots, \alpha_t$, then there exist points $b_1, \dots, b_s$ and reals $\beta_1, \dots, \beta_s$, such that F_2 is determined by $a_1, \dots, a_t$, $b_1, \dots, b_s$ and $\alpha_1, \dots, \alpha_t$, $\beta_1, \dots, \beta_s$.*

Proof. Simply select $b_1, \dots, b_s$, $\beta_1, \dots, \beta_s$ determining F_2 as in the definition of configuration. If $a_i = b_j$ and $\alpha_i \not\equiv \beta_j$, then we have conflicting demands, so $F_2 = \varnothing$.

Lemma 1.4. *There is no strictly decreasing sequence $F_0 \supset \cdots \supset F_{n+5}$ of configurations.*

Proof. Assume that $\mathbf{R}^n = F_0 \supset \cdots \supset F_{n+3}$, with $|F_{n+3}| > 1$. We can assume, by Lemma 1.3, that there are sequences $a_0, \dots, \alpha_0, \dots,$ such that F_i is determined by $a_0, \dots, a_{t(i)}, \alpha_0, \dots, \alpha_{t(i)}$ for some natural numbers $0 \le t(0) < \cdots < t(n+2)$. We can assume as well that $a_0 = 0$. Using scalar products, the congruences on distances can be written as $x^2 \equiv \alpha_0$, $(x - a_j)^2 \equiv x^2 - 2(x, a_j) + a_j^2 \equiv \alpha_j$. Using the former congruence, the latter one can be rewritten as $(x, a_j) \equiv \beta_j$ for some $\beta_j \in \mathbf{R}$ calculable from a_0, a_j, α_j $(j > 0)$. As there is no strictly increasing sequence of vector subspaces of length $n+2$ in

$\mathbf{R}^n$, there is an i such that all vectors a_j, $t(i-1) < j \leq t(i)$, are linear combinations of the vectors a_k, $k \leq t(i-1)$. If $a_j = \sum \lambda_k a_k$, then the equation we have for a_j is $\beta_j \equiv (x, a_j) = \sum \lambda_k (x, a_k) \equiv \sum \lambda_k \beta_k$, so, unless $\beta_j \not\equiv \sum \lambda_k \beta_k$ in which case $F_i = \varnothing$, all the equations defining F_i relative to those defining F_{i-1} can be omitted, i.e., $F_{i-1} = F_i$.

Lemma 1.5. *The intersection of arbitrarily many configurations is again a configuration.*

Proof. Let $\mathscr{F}$ be a nonempty family of configurations. Select, as long as possible, $F_0, \dots, F_s \in \mathscr{F}$ such that the series of intersections $F_0 \cap \cdots \cap F_i$ is properly decreasing. By Lemma 1.4, the procedure must stop after finitely many steps, and then we get an intersection $F_0 \cap \cdots \cap F_s$, which is the intersection of all members of $\mathscr{F}$ and is a configuration by Lemma 1.2.

Lemma 1.6. *For every $X \subseteq \mathbf{R}^n$ there is a unique least configuration $F \supseteq X$.*

Proof. Immediate from Lemma 1.5.

Definition. If F_1, F_2 are configurations, (F_1, F_2) is the least configuration $F \supseteq F_1, F_2$.

Definition. If F is a configuration, $F^\perp = \{y \in \mathbf{R}^n : d^2(x, y) \in Q \text{ (all } x \in F)\}$. As $F^\perp = \mathbf{R}^n \cap \bigcap \{G(x) : x \in F\}$, it is a configuration, by Lemma 1.5.

Lemma 1.7. *If F is a configuration, then $F \cap F^\perp$ is countable.*

Proof. It is a set such that the square of the distance between any two points is in Q, so it is necessarily countable.

2. Closed and nice sets

Definition. If $X \subseteq \mathbf{R}^n$ and $\mathscr{F}$ is a set of configurations, then $(X, \mathscr{F})$ is closed if:

(2.1) $|X| + \omega = |\mathscr{F}| + \omega$;
(2.2) If $F \in \mathscr{F}$ is countable, then $F \subseteq X$;
(2.3) $\varnothing, \mathbf{R}^n \in \mathscr{F}$;
(2.4) if $x \in X$, then $G(x) \in \mathscr{F}$;
(2.5) $x \in X$ iff $\{x\} \in \mathscr{F}$;
(2.6) if $F \in \mathscr{F}$, then $F^\perp, F \cap F^\perp \in \mathscr{F}$; and
(2.7) if $F \notin \mathscr{F}$, then there exist finitely many subsets $F_1, \dots, F_t$ of F and supersets $F^1, \dots, F^s$ of F, all in $\mathscr{F}$, such that, for all $F', F'' \in \mathscr{F}$, if $F' \subseteq F \subseteq F''$, then there are i, j with $F' \subseteq F_i$, $F^j \subseteq F''$.

Definition. A quadruple $(X, \mathscr{F}, X', \mathscr{F}')$ is nice if $X \subseteq X' \subseteq \mathbf{R}^n$, $\mathscr{F} \subseteq \mathscr{F}'$ are sets of configurations, $(X, \mathscr{F})$ is closed, and:

(2.8) $|X' - X| + \omega = |\mathscr{F}' - \mathscr{F}| + \omega$;
(2.9) if $F \in \mathscr{F}'$ is countable, then $F \subseteq X'$;
(2.10) if $x \in X'$, then $G(x) \in \mathscr{F}'$;
(2.11) $x \in X'$ iff $\{x\} \in \mathscr{F}'$;
(2.12) if $F \in \mathscr{F}'$, then $F^\perp, F \cap F^\perp \in \mathscr{F}'$; and
(2.13) if $F_1, F_2 \in \mathscr{F}' - \mathscr{F}$, then $F_1 \cap F_2, (F_1, F_2) \in \mathscr{F}'$.

Lemma 2.1. *If $(X, \mathscr{F}, X', \mathscr{F}')$ is nice, then $(X', \mathscr{F}')$ is closed.*

Proof. We really only have to check (2.7) for $(X', \mathscr{F}')$. If $F \notin \mathscr{F}'$, then, in particular, $F \notin \mathscr{F}$, so there exist $F_1, \dots, F_t, F^1, \dots, F^s$, as in (2.7). Let F_{t+1} be a maximal subset of F in $\mathscr{F}' - \mathscr{F}$ (exists, by Lemma 1.4). If now $F' \subseteq F$, $F' \in \mathscr{F}'$, then either $F' \in \mathscr{F}$, and then $F' \subseteq F_i$ for some $1 \le i \le t$, or $F' \in \mathscr{F}' - \mathscr{F}$, and if we let $F'' = (F', F_{t+1})$, then $F'' \in \mathscr{F}'$, so either $F'' \in \mathscr{F}$, and again $F' \subseteq F_i$ for some $1 \le i \le t$, or $F'' \in \mathscr{F}' - \mathscr{F}$, and then $F'' = F_{t+1}$ by maximality, so $F' \subseteq F_{t+1}$. A similar argument works for the upper approximations.

Lemma 2.2. *If λ is a limit ordinal, $\{X_\alpha : \alpha \le \lambda\}$, $\{\mathscr{F}_\alpha : \alpha \le \lambda\}$ are continuous, increasing sequences such that $(X, \mathscr{F}, X_\alpha, \mathscr{F}_\alpha)$ is nice for $\alpha < \lambda$, then $(X, \mathscr{F}, X_\lambda, \mathscr{F}_\lambda)$ is nice, too.*

Proof. (2.8) is an easy computation, using the assumption that the sequences are continuous and increasing. Of the other clauses, only (2.13) needs explanation. If $F_1, F_2 \in \mathscr{F}_\lambda - \mathscr{F}$, then $F_1, F_2 \in \mathscr{F}_\alpha - \mathscr{F}$ for some $\alpha < \lambda$, so $F_1 \cap F_2$, $(F_1, F_2) \in \mathscr{F}_\alpha \subseteq \mathscr{F}_\lambda$.

Lemma 2.3. *If $(X, \mathscr{F}, X', \mathscr{F}')$ is nice, $\kappa = |X' - X| = |\mathscr{F}' - \mathscr{F}| > \omega$, then there exist continuous, increasing sequences $\{X_\alpha : \alpha \le \kappa\}$, $\{\mathscr{F}_\alpha : \alpha \le \kappa\}$ such that $X_\kappa = X'$, $\mathscr{F}_\kappa = F'$, $|X_\alpha - X| < \kappa$, $|\mathscr{F}_\alpha - \mathscr{F}| < \kappa$ $(\alpha < \kappa)$, and*

(2.14) *$(X, \mathscr{F}, X_\alpha, \mathscr{F}_\alpha)$ is nice $(\alpha < \kappa)$; and*

(2.15) *$(X_\alpha, \mathscr{F}_\alpha, X_{\alpha+1}, \mathscr{F}_{\alpha+1})$ is nice $(\alpha < \kappa)$.*

Proof. Enumerate $X' - X$ as $\{x_\alpha : \alpha < \kappa\}$ and $\mathscr{F}' - \mathscr{F}$ as $\{F_\alpha : \alpha < \kappa\}$. Let $(X_\alpha, \mathscr{F}_\alpha)$ be minimal such that $X_\alpha \supseteq \{x_\beta : \beta < \alpha\}$, $\mathscr{F}_\alpha \supseteq \{F_\beta : \beta < \alpha\}$, and $(X, \mathscr{F}, X_\alpha, \mathscr{F}_\alpha)$ is nice. These sets exist, as we only have to close under the algebraic operations described in (2.9)–(2.13). By a well-known Löwenheim-Skolem type argument, in every structure with countably many finitary operations, for every subset Y of the ground set there is a set $\supset Y$ of size $|Y| + \omega$ closed under the operations. As $(X_\alpha, \mathscr{F}_\alpha)$ is closed, $(X_\alpha, \mathscr{F}_\alpha, X_{\alpha+1}, \mathscr{F}_{\alpha+1})$ is nice.

3. Independent subsets

Definition. $[\omega]^\omega = \{A \subseteq \omega : |A| = \omega\}$. $A \subseteq^* B$ iff $A - B$ is finite.

Definition. A family $\{D_i : i \in I\}$ of subsets of ω is *independent* if

$$D_1 \cap \cdots \cap D_i \cap (\omega - D_{i+1}) \cap \cdots \cap (\omega - D_s)$$

is infinite for different members $D_1, \dots, D_s$ of the family.

Lemma 3.1. *If $D_1, \dots, D_k$ are independent subsets, σ is a Boolean combination of the sets $D_1, \dots, D_{k-1}$, and $D_k \cap \sigma$ is finite, then $\sigma = \varnothing$.*

Proof. σ can be written as $\sigma_1 \cup \cdots \cup \sigma_s$, where each σ_i is the intersection of some D_j's and some $(\omega - D_j)$'s. If $D_k \cap \sigma_i$ is finite, then, by independence, for some j, both D_j and $(\omega - D_j)$ occur in σ_i, so $\sigma_i = \varnothing$.

Lemma 3.2 (folklore). *There exists an independent family of size continuum.*

Proof. It suffices to give an independent family on any other countable set; our selection is $\mathscr{I}$, the set of all finite collections of rational intervals. If $\alpha \in \mathbf{R}$, let

$$D(\alpha) = \{\{I_1, \dots, I_s\} \in \mathcal{I} : \alpha \in I_1 \cup \cdots \cup I_s\}.$$

If different real numbers $\alpha_1, \dots, \alpha_s$ are given, and $0 \le i \le s$, then clearly there are infinitely many sets of the form $K = \{I_1, \dots, I_i\}$ such that $I_1 \cup \cdots \cup I_i$ contains $\alpha_1, \dots, \alpha_i$ but excludes $\alpha_{i+1}, \dots, \alpha_s$, i.e., $K \in D(\alpha_1) \cap \cdots \cap D(\alpha_i) \cap (\mathcal{I} - D(\alpha_{i+1})) \cap \cdots \cap (\mathcal{I} - D(\alpha_s))$.

4. Colorings

Let Φ be the set of all configurations. By Lemmas 1.1 and 3.2 there exists a function $\varphi: \Phi \to [\omega]^\omega$ such that its range $\{\varphi(F): F \in \Phi\}$ is independent. Fix such a φ for the rest of the proof.

Definition. If $(X, \mathcal{F})$ is closed, then the pair (f, ψ) of functions with $f: X \to \omega$, $\psi: \mathcal{F} \to [\omega]^\omega$ is good if:

(4.1) if $F_1 \subseteq F_2$, then $\psi(F_1) \subseteq \psi(F_2)$;
(4.2) $f(x) \notin \psi(G(x))$ $(x \in X)$;
(4.3) each $\psi(F)$ is a Boolean combination of finitely many $\varphi(F_i)(F_i \in \mathcal{F})$; and
(4.4) if $\psi(F_1) \cap \cdots \cap \psi(F_t) \subseteq^* \psi(F^1) \cup \cdots \cup \psi(F^s)$, then $F_i \subseteq F^j$ for some i, j.

Lemma 4.1. *If $(X, \mathcal{F}, X', \mathcal{F}')$ is nice and (f, ψ) is good with respect to $(X, \mathcal{F})$, then there exists a good extension (f', ψ') of it to $(X', \mathcal{F}')$ such that:*

(4.5) *if $F \in \mathcal{F}$, $x \in F \cap (X' - X)$, then $f'(x) \in \psi(F)$; and*
(4.6) *if $x, y \in X' - X$, $x \neq y$, $d(x, y) \in Q$, then $f'(x) \neq f'(y)$.*

Proof. By transfinite induction on $\kappa = |X' - X| + \omega$.

If $\kappa = \omega$, enumerate first $\mathcal{F}' - \mathcal{F}$ as $\{T_0, T_1, \dots\}$. We are going to define $\psi'(T_n)$ by induction on n, so that at every step $\mathcal{F}_n = \mathcal{F} \cup \{T_0, \dots, T_n\}$ satisfies (4.1), (4.3), (4.4). Assume that $\psi'(T_0), \dots, \psi'(T_{n-1})$ have already been constructed. Let F_i $(1 \le i \le t)$, F^j $(1 \le j \le s)$ be finitely many members of $\mathcal{F}_{n-1}$ such that if $F \in \mathcal{F}_{n-1}$, $F \subseteq T_n$, the $F \subseteq F_i$ for some $1 \le i \le t$, and, likewise, if $F \in \mathcal{F}_{n-1}$, $F \supseteq T_n$, then $F \supseteq F^j$ for some $1 \le j \le s$. Such families exist, as $(X, \mathcal{F})$ is closed. By the inductive hypothesis on (4.1), $\psi'(F_i) \subseteq \psi'(F^j)$ whenever $1 \le i \le t$, $1 \le j \le s$. We need to select $\psi'(T_n)$ such that $\psi'(F_i) \subseteq \psi'(T_n) \subseteq \psi'(F^j)$ for all choices of i, j. We do this by defining

$$\psi'(T_n) = \bigcap\{\psi'(F^j): 1 \le j \le s\} \cap \left[\varphi(T_n) \cup \bigcup\{\psi'(F_i): 1 \le i \le t\}\right].$$

This choice obviously ensures (4.1) and (4.3).

We show by induction on $n < \omega$ that $\mathcal{F}_n$ satisfies (4.4) as well. Assume that the statement is true for $n - 1$, and

$$\psi'(G_1) \cap \cdots \cap \psi'(G_u) \subseteq^* \psi'(G^1) \cup \cdots \cup \psi'(G^v) \tag{4.7}$$

for some $G_1, \dots, G_u$, $G^1, \dots, G^v \in \mathcal{F}_n$. If T_n occurs on both the left-hand side and the right-hand side, then the result is trivial, so assume first that T_n occurs on just the left-hand side, $G_u = T_n$. Then (4.7) can be rewritten as

$$A \cap B \cap (\varphi(T_n) \cup C) \subseteq^* D \tag{4.8}$$

where $A = \psi'(G_1) \cap \cdots \cap \psi'(G_{u-1})$, $B = \psi'(F^1) \cap \cdots \cap \psi'(F^s)$, $C = \psi'(F_1) \cup \cdots \cup \psi'(F_t)$, $D = \psi'(G^1) \cup \cdots \cup \psi'(G^v)$. (Notice that $C \subseteq B$.) A, B, C, and D are Boolean combinations of φ values from $\mathcal{F}_{n-1}$, so, by Lemma 3.1, as $\varphi(T_n)$ is independent from them, $A \cap B \subseteq^* D$ holds, i.e.,

$$\psi'(G_1) \cap \cdots \cap \psi'(G_{u-1}) \cap \psi'(F^1) \cap \cdots \cap \psi'(F^s) \subseteq^* D. \tag{4.9}$$

By the inductive hypothesis, either $G_i \subseteq G^j$ for some i, j, or $T_n \subseteq F^i \subseteq G^j$, and we are done.

A similar argument works if T_n appears on the right-hand side of (4.7).

In order to define f', enumerate $X' - X$ as $\{x_0, x_1, \dots\}$. If the $f'(x_i)$ for $i < k$ have already been selected, we try to find a value for $x = x_k$ such that $f'(x) \neq f'(x_i)$ for $i < k$, $f'(x) \notin \psi'(G(x))$, but $f'(x) \in \cap\{\psi'(F_i): 1 \leq i \leq t\}$ where $F_1, \dots, F_t$ is a finite set of minimal elements of $\mathcal{F}$ containing x. Such a family exists by (2.5) and (2.7). We can select $f'(x)$ unless $\bigcap\{\psi'(F_i): 1 \leq i \leq t\} \subseteq^* \psi'(G(x))$. But then, by (4.4), for some $1 \leq i \leq t$, $F_i \subseteq G(x)$ holds, so $x \in F_i^\perp$; therefore, $x \in F_i \cap F_i^\perp \subseteq X$, which by (2.6), Lemma 1.7, and (2.2) is a contradiction.

If $\kappa > \omega$, let $X_\alpha, \mathcal{F}_\alpha$ be given as in Lemma 2.3 for $(X, \mathcal{F}, X', \mathcal{F}')$. By transfinite recursion on $\alpha < \kappa$, let $(f_\alpha, \psi_\alpha) = (f, \psi)$ if $\alpha = 0$, $(f_{\alpha+1}, \psi_{\alpha+1})$ extend (f_α, ψ_α) according to (4.5) and (4.6), $(f_\alpha, \psi_\alpha) = (\bigcup_{\beta<\alpha} f_\beta, \bigcup_{\beta<\alpha} \psi_\beta)$ if α is a limit ordinal. We claim that $f' = \bigcup f_\alpha$, $\psi' = \bigcup \psi_\alpha$ work. (4.5) is clear, so assume that $x, y \in X' - X$, $x \neq y$, $d(x, y) \in Q$. If $x, y \in X_{\alpha+1} - X_\alpha$ for some $\alpha < \kappa$, then $f'(x) \neq f'(y)$ by assumption. If, however $x \in X_\alpha$, $y \in X_{\alpha+1} - X_\alpha$ for some $\alpha < \kappa$, then $f'(x) \notin \psi'(G(x))$ and $f'(y) \in \psi'(G(x))$, so $f'(x) \neq f'(y)$ again.

Theorem 4.2. *There exists an $f: \mathbf{R}^n \to \omega$ such that, if $d(x, y) \in Q$ and $x \neq y$, then $f(x) \neq f(y)$.*

Proof. Apply Lemma 4.1 to $(\varnothing, \{\varnothing, \mathbf{R}^n\}, \mathbf{R}^n, \Phi)$, $f = \varnothing$, and ψ, where $\psi(\varnothing) = \varphi(\varnothing) \cap \varphi(\mathbf{R}^n)$ and $\psi(\mathbf{R}^n) = \varphi(\varnothing) \cup \varphi(\mathbf{R}^n)$.

Theorem 4.3. *If $X \subseteq \mathbf{R}^n$ is uncountable, then there exists a $Y \subseteq X$, $|Y| = |X|$, omitting rational distances.*

Proof. The statement follows trivially from Theorem 4.2 if $\kappa = |X|$ has uncountable cofinality, for then one of the color classes has full cardinality in X. Assume, therefore, that $\mathrm{cf}(\kappa) = \omega$. Let F be a configuration such that it has the property that $|F \cap X| = \kappa$ but this does not hold for any proper subconfiguration of F. Such an F exists by Lemma 1.4. If now $x \in F$ and $x \notin F \cap F^\perp$, then $F \cap G(x)$ is a proper subconfiguration, so $|G(x) \cap X \cap F| < \kappa$ for all but countably many $x \in X \cap F$, and we may assume so for all. Decompose $X \cap F$ into a disjoint union $X = \bigcup X_n$, where $|X_n| = \kappa_n$, each κ_n is an uncountable regular cardinal, and $\sum \kappa_n = \kappa$. By the trivial part of this present theorem, we can assume that rational distances are omitted inside each X_n (by omitting some of their points, if necessary). For every $x \in X_n$ there is a $k < \omega$ such that $|G(x) \cap X \cap F| < \kappa_k$. As κ_n is uncountable and regular, we may assume, by omitting more points, that the same $k = k(n) < \omega$ works for every $x \in X_n$, and we may assume further that $n < k(n)$. Let $n(0) = 0$, $Y_0 = X_0$. Select $n(1) = k(0)$, and then let $Y_1 \subseteq X_{n(1)}$ be a set of size $\kappa_{n(1)}$ such that no point in it is in rational distance from a point in X_0. This is possible, as by our conditions, every point in X_0 disqualifies $< \kappa_{n(1)}$ points, and, as that cardinal is regular, only $< \kappa_{n(1)}$ points must be left out together. Then let $n(2) = k(n(1))$, select $Y_2 \subseteq X_{n(2)}$, etc. Our set will be $Y_0 \cup Y_1 \cup \cdots$.

From Theorem 4.2 it is also easy to deduce that every set $X \subseteq \mathbf{R}^n$ of positive outer measure has a subset of positive outer measure that omits rational distances: one of the color classes must intersect X in positive outer measure. We have, however, been unable to prove the existence of such a subset with the same outer measure, even in the case $n = 1$. If X is Borel, a straightforward transfinite selection gives this. We will return to this question in another paper.

Acknowledgment

I thank the referee for detecting several errors in an earlier version of the paper and for suggesting numerous stylistic improvements.

References

1. J. Ceder, *Finite subsets and countable decompositions of Euclidean spaces*, Rev. Roumaine Math. Pures Appl. **14** (1969), 1247–1251.
2. R. O. Davies, *Partitioning the plane into denumerably many sets without repeated distances*, Proc. Cambridge Philos. Soc. **72** (1972), 179–183.
3. P. Erdős, *Set theoretic, measure theoretic, combinatorial, and number theoretic problems concerning point sets in Euclidean space*, Real Anal. Exchange **4** (1978–1979), 113–138.
4. P. Erdős and A. Hajnal, *On chromatic number of graphs and set systems*, Acta Math. Hungar. **17** (1966), 61–99.
5. P. Erdős and P. Komjáth, *Countable decompositions of* $\mathbf{R}^2$ *and of* $\mathbf{R}^3$, Discrete Comput. Geom. **5** (1990), 325–331.
6. P. Komjáth, *Tetrahedron free decomposition of* $\mathbf{R}^3$, Bull. London Math. Soc. **23** (1991), 116–120.
7. J. Schmerl, *Partitioning Euclidean space* (to appear).

Department of Mathematics and Statistics, Simon Fraser University, Burnaby, British Columbia, Canada V5A 1S6

Current address: Department of Computer Science, Eötvös University, Budapest, Múzeum krt 6-8, 1088, Hungary

E-mail address: kope@cs.elte.hu