CONNECTEDNESS OF THE SPACE OF MINIMAL 2-SHEPRES IN $S^{2m}(1)$

MOTOKO KOTANI

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Abstract. Loo's theorem asserts that the space of all branched minimal 2-spheres of degree $d$ in $S^4(1)$ is connected. The main theorem in this paper is that the assertion is still true for $S^{2m}(1)$. It is shown that any branched minimal 2-sphere in $S^{2m}(1)$ can be deformed, preserving its degree, to a meromorphic function.

0. Introduction

After the celebrated research on minimal 2-spheres in the unit sphere $S^N(1)$ by Calabi [Ca], there was much attention given not only to the study of individual minimal spheres but also to the structure of the space of all minimal 2-spheres in $S^N(1)$. Calabi proved that if a minimal 2-sphere is immersed fully in $S^N(1)$, then $N$ must be even. The simplest case is the space of all minimal 2-spheres of degree $d$ in $S^2(1)$. This space has two connected components. One component is identified with the space of all meromorphic functions of degree $d$; the other is its conjugate. These two components are connected in the space of all minimal 2-spheres of degree $d$ in $S^3(1)$.

Recently, Loo [L] determined the space of all minimal 2-spheres $S^2$ of degree $d$ in the unit 4-sphere $S^4(1)$. In particular, he proved that this space is connected.

In this paper, we prove that the space of all branched minimal 2-spheres $S^2$ of degree $d$ in the unit $N$-sphere $S^N(1)$ is connected for $N \geq 3$. We shall see that any branched minimal 2-sphere $g : S^2 \rightarrow S^{2m}(1)$ of degree $d$ can be deformed to a nonfull minimal sphere of degree $d$. By repeating this process, $g$ is deformed eventually to a $\pm$ meromorphic function $S^2 \rightarrow S^2(1)$ of degree $d$. In other words, every element in the space of all minimal spheres $S^2 \rightarrow S^{2m}(1)$ of degree $d$ is connected to a $\pm$ meromorphic function of degree $d$. Every two $\pm$ meromorphic functions are connected as we noted above. From these facts, we can prove that the space of all $g : S^2 \rightarrow S^{2m}(1)$ of degree $d$ is connected.

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803
Theorem. The space of all branched minimal 2-spheres in $S^{2m}(1)$ of degree $d$ is connected for $m > 1$. In the case of $m = 1$, the space has two connected components.

Since the image of any minimal sphere $S^2 \to S^{2m+1}$ sits in a great sphere, isometric to $S^{2m}(1)$, the theorem can easily be generalized into the following:

Corollary. The space of all branched minimal 2-spheres in $S^N(1)$ of degree $d$ is connected for $N \geq 3$.

In the forthcoming paper [FGKO], the fundamental groups of the spaces are determined.

Throughout this paper, when we refer to a minimal sphere, we mean the sphere can be either immersed or branched.

1. Preliminaries

In this section we review the investigation of minimal spheres in $S^{2m}(1)$ by Barbosa in [Ba], where he associates minimal spheres in $S^{2m}(1)$ with isotropic holomorphic curves in $\mathbb{C}P^{2m}$ as their directrix curves. Let $g : S^2 \to S^{2m}(1) \subset \mathbb{R}^{2m+1}$ be a full minimal sphere with the isothermal metric $ds^2 = \rho^2 dz d\bar{z}$ induced from $g$. Then $g$ is isotropic, that is,

Proposition 1.1 (Calabi [Ca]).

$$\left( \frac{\partial^k g}{\partial z^k}, \frac{\partial^l g}{\partial \bar{z}^l} \right) = 0 \quad \text{for all} \quad k + l > 0.$$ 

As $g$ is isotropic, the space

$$V_k(x) = \text{span} \left\{ \frac{\partial g}{\partial z}, \frac{\partial^2 g}{\partial z^2}, \ldots, \frac{\partial^k g}{\partial z^k} \right\}$$

is perpendicular to its own conjugate $V_{k-1}(x)$ for all $k \leq m$.

Let $G_m = \partial^m g / \partial z^m \cap V_{m-1}^\perp$ and $\xi = G_m / |G_m|$. $G_m$ has only isolated zeros and $\xi$ is holomorphic except for zeros $G_m = 0$, where $\xi$ has at most poles.

An immediate consequence of this lemma is that $\xi$ defines a holomorphic curve, say, $\Psi : S^2 \to \mathbb{C}P^{2m}$ extending up to the zeros of $G_m$. $\Psi$ is called the directrix curve of $g$.

Lemma 1.2. Let $g : S^2 \to S^{2m}(1)$ be a full minimal sphere in $S^{2m}(1)$ and $\xi$ be a local representation of the directrix curve $\Psi$ of $g$. Then $\xi$ satisfies

$$\left( \xi, \xi \right) = \left( \xi', \xi' \right) = \cdots = \left( \xi^{m-1}, \xi^{m-1} \right) = 0,$$

where $\xi' = \partial^r \xi / \partial z^r$ is the $r$th derivative by the isothermal coordinate $z$.

A full holomorphic curve $\Psi : S^2 \to \mathbb{C}P^{2m}$ is called isotropic if its local representation $\xi$ satisfies the equations (1.1).

Next we construct a minimal sphere $g : S^2 \to S^{2m}(1)$ from an arbitrary holomorphic curve $\Psi : S^2 \to \mathbb{C}P^{2m}$, which is isotropic. Let $\Psi : S^2 \to \mathbb{C}P^{2m}$ be a holomorphic isotropic curve and $\xi$ be its local representation by a polynomial. Define $T = \xi \wedge \xi' \wedge \cdots \wedge \xi^{m-1}$ and $g = \epsilon(T \wedge \overline{T}) / |T \wedge \overline{T}|$ for $\epsilon^2 = (-1)^m$. Then we can check that $g$ is a minimal sphere in $S^{2m}(1)$ whose directrix curve is $\Psi$. 
Theorem [Ba]. There exists a canonical 1-1 correspondence between the set $S_m$ of all minimal spheres in $S^{2m}(1)$ and the set $\mathcal{H}_m$ of all holomorphic isotropic curves $\Psi : S^2 \rightarrow \mathbb{C}P^{2m}$. Moreover, $SO(2m + 1, \mathbb{C})$ acts on $\mathcal{H}_m$.

2. Deformation

Let $\mathcal{H}_m$ be the space of all holomorphic isotropic curves $\Psi : S^2 \rightarrow \mathbb{C}P^{2m}$. As we saw in §1, $SO(2m + 1, \mathbb{C})$ acts on this space. Take $\Psi \in \mathcal{H}_m$, which is full. Consider a smooth deformation

$$\Psi(t) = A(t)\Psi$$

defined by a smooth 1-parameter action

$$A(t) \in SO(2, \mathbb{C}) \subset SO(2m + 1, \mathbb{C}) \quad \text{with} \quad A(0) = I.$$ 

More concretely, let

$$E_1 = e_1 - \sqrt{-1}e_2, \quad E_2 = e_3 - \sqrt{-1}e_4, \ldots, \quad E_m = e_{2m-1} - \sqrt{-1}e_{2m},$$

$$\overline{E}_1 = e_1 + \sqrt{-1}e_2, \ldots, \quad \overline{E}_m = e_{2m-1} + \sqrt{-1}e_{2m}, \quad e_{2m+1}$$

be eigenvectors of $A(t)$ in $SO(2m + 1, \mathbb{C})$. That is,

$$A(t)E_i = e^tE_i, \quad A(t)\overline{E}_i = e^{-t}\overline{E}_i;$$

$$A(t)E_i = E_i, \quad A(t)\overline{E}_i = \overline{E}_i \quad \text{for} \quad i \neq 1;$$

$$A(t)e_{2m+1} = e_{2m+1}.$$ 

Then the local representation $\xi(t)$ of $\Psi(t)$ is given by

$$\xi(t) = e^t c_1 E_1 + \sum_{i=2}^{m} c_i E_i + e^{-t} c_1 \overline{E}_1 + \sum_{j=2}^{m} c_j \overline{E}_j + c_{2m+1} e_{2m+1}$$

for some functions $c_i, c_j$ in $\mathbb{C}$, and this extends globally to $S^2$. We have a 1-parameter family $g(t) : S^2 \rightarrow S^{2m}(1)$ given by

$$g(t) = e^t \frac{T(t) \wedge \overline{T(t)}}{|T(t) \wedge \overline{T(t)}|},$$

where

$$T(t) = \xi(t) \wedge \xi(t)' \wedge \cdots \wedge \xi^{m-1}.$$ 

$g(t)$ is full in $S^{2m}(1)$ for every $t < \infty$, and $g(\infty) = \lim_{t \to \infty} g(t)$ is contained in a smaller sphere, say, $S^{2k}(1)$ for $k < m$. It can be seen by using Plücker coordinates $P_I$ for $m$-uple multi-indices $I = (i_1, \ldots, i_m)$ that

$$g(t) = \{P_I \mid \text{for all} \ m \text{-uple multi-indices } I\}$$

and that

$$g(\infty) = \{P_1, J\}$$

for all $(m - 1)$-uple multi-indices $J$ containing neither 1 nor $\overline{1}$. 
By the Plücker coordinates, we mean

\[ P_I = \det \begin{pmatrix} c_{i_1} & c_{i_2} & \cdots \\ c_{i_1} & c_{i_2} & \cdots \\ \vdots & \vdots & \ddots & \vdots \\ c_{i_1}^{(m-1)} & c_{i_2}^{(m-1)} & \cdots & \cdots \end{pmatrix}. \]

The degree of the deformation \( g(t) \) remains constant, when \( A(t) \) satisfies

1. \( \max \{ \deg P_{1,j} \} = \max \{ \deg P_I \} \) and
2. \( \deg \{ \text{common factor } Q \text{ of } \{ P_{1,j} \} \} = \deg \{ \text{common factor } P \text{ of } \{ P_I \} \}. \)

This is so because

\[ \deg g(t) = \max \deg P_I - \deg P, \]
\[ \deg g(\infty) = \max \deg P_{1,j} - \deg Q. \]

The condition (1) holds if and only if the degree of \( c_1(z) \) attains

\[ \max_{i=1, \ldots, 2m+1} \deg \{ c_i(z) \}. \]

**Lemma 2.1.** There exists \( A(t) \in \text{SO}(2, \mathbb{C}) \subset \text{SO}(2m + 1, \mathbb{C}) \) such that the common factor \( Q \) of \( \{ P_{1,j} \} \) divides the common factor \( P \) of \( \{ P_I \} \).

**Proof.** We shall prove all \( P_I \) have \( Q \) as a factor if all \( P_{1,j} \) have common factor \( Q \). For simplicity we put \( c_i = (c_{i_1}, c_{i_2}, \ldots, c_{i_1}^{(m-1)}) \). Without loss of generality, we may assume that \( c_1 \) never vanishes.

**Step 1.** We see that

\[ \sum_{k=0}^{m} c_{i_k} P_k (-1)^{k-1} = 0 \]

for all \((m + 1)\)-uple multi-indices \( I = (i_1, \ldots, i_{m+1}) \), where \( P_k \) is the Plücker coordinate for the \( m \)-uple multi-index \( \{ i_1, \ldots, i_{m+1} \} - i_k \).

Consider the \((m + 1) \times (m + 1)\) matrix

\[ \begin{pmatrix} c_{i_1} & c_{i_2} & \cdots & c_{i_{m+1}} \\ c_{i_1} & c_{i_2} & \cdots & c_{i_{m+1}} \\ \vdots & \vdots & \ddots & \vdots \\ c_{i_1}^{(k)} & c_{i_2}^{(k)} & \cdots & c_{i_{m+1}}^{(k)} \end{pmatrix} \]

and calculate its determinant developed along the last column. Then we obtain

\[ 0 = c_{i_1} P_1 - c_{i_2} P_2 + \cdots + (-1)^m c_{i_{m+1}} P_{m+1}. \]

In the same way, considering the determinants of the \((m + 1) \times (m + 1)\) matrices

\[ \begin{pmatrix} c_{i_1} & c_{i_2} & \cdots & c_{i_{m+1}}^{(k)} \\ c_{i_1} & c_{i_2} & \cdots & c_{i_{m+1}}^{(k)} \\ \vdots & \vdots & \ddots & \vdots \\ c_{i_1}^{(k)} & c_{i_2}^{(k)} & \cdots & c_{i_{m+1}}^{(k)} \end{pmatrix} \]

for \( k = 0, \ldots, m - 1 \), we obtain Step 1.

**Step 2.** Let \( Q \) be a common factor of \( \{ P_{1,j} \} \). Then

\[ \{ P_K \mid \text{all } m \text{-uple multi-indices } K \text{ containing neither } 1 \text{ nor } \bar{1} \} \]

has \( Q \) as a common factor.

This is an immediate consequence of Step 1, applied to the index \( I = (1, k_1, \ldots, k_m) \) and using the assumption \( c_1 \neq 0 \).
Step 3. Interchanging $A(t)$ with $A^e(t)$, which has eigenvectors $E^j_1$ given by

\[ E_1 = \cos s E^j_1 - \sin s E^j_2, \quad E_2 = \sin s E^j_1 + \cos s E^j_2, \]

\[ E_j = E^j_j \quad \text{for all } j = 3, \ldots, m, \]

\[ e^s_{2m+1} = e^s_{2m+1}, \]

we see that there exists $s$ such that all Plücker coordinates $P^s_j$ have a common factor $P$ (may differ from $Q$), except for indices $l = (\bar{1}, \bar{2}, J)$ with $(m - 2)$-uple multi-indices $J$ not containing $1, \bar{1}, 2, \bar{2}$.

With respect to this new basis $E^j_1$ and $E^j_2$, $\xi$ is given by

\[ \xi = (c_1 \cos s + c_2 \sin s)E^j_1 + (-c_1 \sin s + c_2 \cos s)E^j_2 \]

\[ + \sum_{k \neq 1} c_k E^j_k + \sum_{k \neq 1} c_k E^j_k + c_{2m+1} e^s_{2m+1}, \]

and the Plücker coordinates $P^s_j$ are given by

\[ P^s_K = P_K; \]
\[ P^s_{1,K} = \cos s P_{1,K} + \sin s P_{2,K}, \]
\[ P^s_{\bar{1},K} = \cos s P_{1,K} + \sin s P_{2,K}, \]
\[ P^s_{2,K} = -\sin s P_{1,K} + \cos s P_{2,K}, \]
\[ P^s_{\bar{2},K} = -\sin s P_{1,K} + \cos s P_{2,K}, \]
\[ P^s_{1,2,K} = P_{1,2,K}, \]
\[ P^s_{\bar{1},\bar{2},K} = P_{\bar{1},\bar{2},K}, \]
\[ P^s_{1,\bar{1},K} = \cos s P_{1,\bar{1},K} + \sin s P_{2,\bar{1},K}, \]
\[ P^s_{\bar{1},2,K} = \cos s P_{1,\bar{1},K} + \sin s P_{2,\bar{1},K}, \]
\[ P^s_{2,\bar{1},K} = -\sin s P_{1,\bar{1},K} + \cos s P_{2,\bar{1},K}, \]
\[ P^s_{\bar{2},\bar{1},K} = -\sin s P_{1,\bar{1},K} + \cos s P_{2,\bar{1},K}; \]
\[ P^s_{1,2,\bar{1},K} = \cos s P_{1,2,\bar{1},K} + \sin s P_{1,2,\bar{1},K}, \]
\[ P^s_{1,\bar{2},\bar{1},K} = -\sin s P_{1,\bar{2},\bar{1},K} + \cos s P_{1,\bar{2},\bar{1},K}; \]
\[ P^s_{\bar{1},\bar{2},1,K} = \cos s P_{\bar{1},\bar{2},1,K} + \sin s P_{\bar{1},\bar{2},1,K}, \]
\[ P^s_{\bar{1},\bar{2},2,K} = -\sin s P_{\bar{1},\bar{2},1,K} + \cos s P_{\bar{1},\bar{2},2,K}; \]
\[ P^s_{1,2,\bar{1},\bar{2},K} = P_{1,2,\bar{1},\bar{2},K}, \]

where $K$ is a multi-index not containing $1, \bar{1}, 2, \bar{2}$.

By Step 2, we know that all $P^s_j$ have common factor $Q^s$ if the multi-indices $J$ do not contain $\bar{1}$. Noting that the $P^s_K$ are the same for all $s$, we conclude that the common factor of $Q^s$ does not depend on $s$. We denote this factor by $P$. Then we can see that all coefficients of $\cos s$, $\sin s$, etc., of $P^s_j$ have $P$ as a factor. It implies that $P^s_j$ has common factor $P$ unless the multi-index contains both $\bar{1}$ and $\bar{2}$ together. This gives Step 3.
Step 4. All $P_{1, \bar{z}, j}$ have $P$ as a factor.

Proof. Let $a$ be a point such that $P(l)(a) = 0$ for $l \leq N$ and for some index $J$. Then we will see that

$$\dim\{c_k(a)|k \neq \bar{1}, \bar{2}\} \leq m - 2.$$  

If there exist $(m-1)$ linearly independent vectors 

$$\{c_j(a), \ldots, c_{jm-1}(a)\},$$  

then we obtain that

$$c_j(z) = f_1^j(z)c_{j_1}(z) + \cdots + f_{jm-1}^j(z)c_{jm-1} + (z - a)^N V_j,$$

where $f_i^j(z)$ are polynomials in $z$ and $V_j$ is a vector, from the fact that $P_j, j_1, \ldots, j_{m-1}$ has $a$ as a zero of order $N$. In particular, putting $c_1, c_2, c_j$ into $P_{1, \bar{z}, j}$, we see that it has $a$ as a zero of order $N$. This contradicts the assumption. We prove that

$$\dim\{c_k(a)|k \neq \bar{1}, \bar{2}\} = r < m - 2.$$  

Hence, that there exist $r$ constant vectors $F_1, \ldots, F_r$ such that

$$\xi(a) = c_1(a)E_1 + c_2(a)E_2 + \sum_{j=1}^r c_j(a)F_j,$$

$$\xi'(a) = c_1'(a)E_1 + c_2'(a)E_2 + \sum_{j=1}^r c_j'(a)F_j,$$

$$\vdots$$

$$\xi^{(m-1)}(a) = c_1^{(m-1)}(a)E_1 + c_2^{(m-1)}(a)E_2 + \sum_{j=1}^r c_j^{(m-1)}(a)F_j.$$

Using the isotropic property of $\xi$, i.e.,

$$(\xi^k, \xi^l) = 0$$  

for all $k, l = 1, \ldots, m - 1,$

we see

$$(E_1, F_j) = 0$$  

for all $j = 1, \ldots, r,$

which contradicts the fact that

$$E_1 \in \text{span}\{F_1, \ldots, F_r\}.$$  

So there is no such point $z = a$ with $P(l)(a) = 0$ for $l \leq N$ and $P_{1, \bar{z}, j}(a) \neq 0$. In other words, all $P_{1, \bar{z}, j}$ have $P$ as a common factor.

The case $m = 3$ and $N = 1$. We prove Step 4 more precisely when $m = 3$. If there is some point $z = a$ such that $P(a) = 0$ and $P_{1, \bar{z}, k}(a) \neq 0$ for some index $k \neq 1, 2, \bar{1}, \bar{2}$, then, since

$$P_{1, \bar{z}, k}(a) = \det(c_1(a), c_2(a), c_k(a)) \neq 0,$$
the three vectors $c_1(a)$, $c_2(a)$, $c_k(a)$ are linearly independent. On the other hand, $P_{1,1,k}(a) = 0$ implies that $c_1(a)$, $c_2(a)$, $c_k(a)$ are linearly dependent and $c_1(a)$, $c_k(a)$ are independent. Therefore, we can write

$$c_1(a) = \alpha c_1(a) + \beta c_k(a).$$

In the same way we can write

$$c_1(a) = \gamma c_2(a) + \delta c_k(a)$$

by $P_{1,2,k}(a) = 0$. But since $c_1(a)$ and $c_2(a)$ are linearly independent, we conclude that

$$c_1(a)$$

is parallel to $c_k(a)$.

By the same argument we see

$$c_2(a)$$

is parallel to $c_k(a)$, $c_l(a)$ is parallel to $c_k(a)$ for $l \neq 1, 2$, by using

$$P_{1,2,k}(a) = P_{2,2,k}(a) = 0,$$

$$P_{1,1,k}(a) = P_{2,1,k}(a) = 0.$$

By the assumption that $c_1(a) \neq 0$, we put $c_j(a) = a_j c_1(a)$ for all $j \neq 1, 2$. Let $F = E_1 + A_2 E_2 + A_3 E_3 + A_4 E_4 + A_5 E_5$. Then we obtain

$$\xi(a) = c_1(a) F + c_1(a) E_1 + c_2(a) E_2,$$

$$\xi'(a) = c_1(a) F + c_1(a) E_1 + c_2(a) E_2,$$

$$\xi''(a) = c_1(a) F + c_1(a) E_1 + c_2(a) E_2.$$

The isotropic property of $\xi$ says that

$$0 = (\xi(a), \xi(a)) = 2c_1(a)\{c_1(a)(F, E_1) + c_2(a)(F, E_2)\},$$

$$0 = (\xi'(a), \xi'(a)) = 2c_1(a)\{c_1(a)(F, E_1) + c_2(a)(F, E_2)\},$$

$$0 = (\xi''(a), \xi''(a)) = 2c_1(a)\{c_1(a)(F, E_1) + c_2(a)(F, E_2)\},$$

$$0 = (\xi(a), \xi'(a)) = c_1(a)\{c_1(a)(F, E_1) + c_2(a)(F, E_2)\}$$

$$+ c_1(a)\{c_1(a)(F, E_1) + c_2(a)(F, E_2)\},$$

$$0 = (\xi(a), \xi''(a)) = c_1(a)\{c_1(a)(F, E_1) + c_2(a)(F, E_2)\}$$

$$+ c_1(a)\{c_1(a)(F, E_1) + c_2(a)(F, E_2)\},$$

$$0 = (\xi'(a), \xi'(a)) = c_1(a)\{c_1(a)(F, E_1) + c_2(a)(F, E_2)\}$$

$$+ c_1(a)\{c_1(a)(F, E_1) + c_2(a)(F, E_2)\}.$$
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Department of Mathematics, Faculty of Sciences, Toho University, Funabashi, Chiba, 274, Japan

E-mail address: kotani@tansei.cc.u-tokyo.ac.jp