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WEAK (1, 1) ESTIMATE FOR OSCILLATORY SINGULAR INTEGRALS WITH REAL-ANALYTIC PHASES

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ABSTRACT. In this paper, we prove the uniform weak $(1, 1)$ estimate for oscillatory singular integral operators with real-analytic phase functions in the one-dimensional case. Some partial results for the higher-dimensional case is also included.

1. INTRODUCTION

Consider the operator

$$T_\lambda f(x) = \text{p.v.} \int_{\mathbf{R}^1} e^{i\lambda\Phi(x,y)} \frac{\varphi(x,y)}{x-y} f(y)\, dy, \tag{1.1}$$

where $\varphi \in C_0^\infty(\mathbf{R}^1 \times \mathbf{R}^1)$, Φ is real-valued, and $\lambda \in \mathbf{R}^1$. The main result in this paper is the following:

Theorem 1. *Suppose that T_λ is defined as in (1.1) and $\Phi(x, y)$ is real-analytic in* $\text{supp}(\varphi)$. *Then T_λ is uniformly bounded from L^1 to $L^{1,\infty}$; i.e., there is a constant C such that, for any $\alpha > 0$,*

$$m\{x: |T_\lambda(f)(x)| > \alpha\} \le C\alpha^{-1}\|f\|_1 \tag{1.2}$$

holds. C is independent of λ and α.

The space $L^{1,\infty}$ is also known as weak L^1. A function f is in this space if $\sup_s s\lambda_f(s) < \infty$, where λ_f is the distribution function of f, i.e., $\lambda_f(s) = m\{x: f(x) < s\}$.

Oscillatory integral operators with singular kernels have been studied by many authors (see, e.g., [1, 2, 6–10]). Let $x, y \in \mathbf{R}^n$, $P(x, y)$ be a polynomial, and $K(x, y)$ be a Calderón-Zygmund kernel. Define the operator

$$Tf(x) = \text{p.v.} \int_{\mathbf{R}^n} e^{iP(x,y)} K(x,y) f(y)\, dy. \tag{1.3}$$

Ricci and Stein [9] showed that T is bounded on $L^p(\mathbf{R}^n)$, for $1 < p < \infty$, and the bound depends only on the total degree of P, not on the coefficients of

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P. Later in [1], Chanillo and Christ proved that T is also a bounded mapping from L^1 to weak L^1, and the bound depends only on the total degree of P. Operators of this type, but with more general phase functions such as those given in (1.1), were studied in [7]. Define $\widetilde{T}_\lambda$ as

$$\widetilde{T}_\lambda f(x) = \text{p.v.} \int_{\mathbf{R}^n} e^{i\lambda\Phi(x,y)} K(x,y)\varphi(x,y) f(y)\, dy, \tag{1.4}$$

where K and φ are given as above. It was proved in [7] that, among other things, the operators $\widetilde{T}_\lambda$ are uniformly bounded on $L^p(\mathbf{R}^n)$, for $\lambda \in \mathbf{R}^1$ $(1 < p < \infty)$, if the phase function $\Phi(x,y)$ is real-analytic. (It is known that smoothness of the phase function alone does not suffice; see [6].) Therefore, it seems natural to conjecture that the operators $\widetilde{T}_\lambda$ are also uniformly bounded mappings from L^1 to weak L^1, when the phase function is real-analytic. Theorem 1 says that this is indeed true in the case of dimension one; the proof will be given in §2. The analysis consists of two parts. Using the ideas in [7] and the result of Chanillo and Christ on operators with polynomial phase functions, we are able to control the part with kernel supported near the diagonal. But the main estimate is for the part with kernel supported away from the diagonal. Using the L^2 theory for T_λ which was established in [7] and some standard arguments, the matter is quickly reduced to the estimate of $T_\lambda(\sum b_j)$, after one makes a Calderon-Zygmund decomposition $f = g + \sum b_j$. As pointed out by Chanillo and Christ in [1], unlike the case of Calderon-Zygmund singular integrals, one can no longer control the L^1 norm of $T_\lambda(\sum b_j)$. For operators with polynomial phase functions, Chanillo and Christ used an appropriate $L^1 \to L^2$ estimate for $T(\sum b_j)$ and obtained the desired result by applying Chebychev's inequality. This type of estimates have also appeared in other problems involving weak type estimates (see, e.g., [3, 4]).

Our main task here is to establish a similar $L^1 \to L^2$ estimate for $T_\lambda(\sum b_j)$. To achieve this, it requires detailed information about sets such as

$$E_{x,z} = \{y\colon |\partial_z^\alpha \Phi(z,y) - \partial_z^\alpha \Phi(z,x)| \le c\}.$$

In particular, we need to understand how $E_{x,z}$ moves as z varies (for fixed x). This is where the analytic case differs from the polynomial case. For if $\Phi(x,y)$ is a polynomial, it is always possible to find a α, such that $E_{x,z}$ is independent of z.

It is not hard to see that, when the dimension is higher, these sets could get very complicated. For the higher-dimensional case, we have obtained some partial result. Namely, we are able to prove that $\widetilde{T}_\lambda$ are uniformly bounded mappings from L^1 to weak L^1, under the additional assumption that the phase function is translation invariant, i.e., $\Phi(x,y) = \phi(x-y)$, for some real-analytic function ϕ. This result is stated in Theorem 2 and proved in §3.

Remark. We would like to point out that, as in [9, Theorem 1; 7, Theorem 2], the function $1/(x-y)$ in Theorem 1 can be replaced by more general kernel functions $k(x,y)$ which satisfy the following conditions:

(1) $|k(x,y)| \le A|x-y|^{-1}$, $|\nabla k(x,y)| \le A|x-y|^{-2}$,

(2) $f \to \int k(x,y) f(y)\, dy$ extends to a bounded operator on L^2.

2. Proof of Theorem 1

Now we begin our proof of Theorem 1. Let $N > 0$ be a large number (to be chosen later), and $\phi \in C_0^\infty(\mathbf{R}^1)$ with

$$\phi(t) = \begin{cases} 0 & \text{for } |t| < \frac{1}{2}, \\ 1 & \text{for } |t| \geq 1. \end{cases}$$

Let

$$T_\lambda^1 f(x) = \text{p.v.} \int_{\mathbf{R}^1} e^{i\lambda\Phi(x,y)} \frac{\varphi(x,y)}{x-y}(1-\phi)(\lambda^{1/N}(x-y))f(y)\,dy$$

and $T_\lambda^2 f = T_\lambda f - T_\lambda^1 f$. First we shall prove that there is a $C_N > 0$ such that, for any $\alpha > 0$,

$$m\{x : |T_\lambda^1 f(x)| > \alpha\} \leq C_N \alpha^{-1} \|f\|_1. \tag{2.1}$$

Let $A > 0$ and $\operatorname{supp}(\varphi) \subset [-\frac{A}{4}, \frac{A}{4}] \times [-\frac{A}{4}, \frac{A}{4}]$. Fix $a \in \mathbf{R}^1$, $|a| \leq \frac{A}{2}$. Let $I(a) = (a - \lambda^{-1/N}, a + \lambda^{-1/N})$. Assume that $f \in L^1$ and $\operatorname{supp}(f) \subset I(a)$. Using Taylor's expansion, we have

$$\Phi(x,y) = \Phi(x,x) + \sum_{k=1}^{N-1} \frac{1}{k!} \frac{\partial^k \Phi}{\partial y^k}(x,x)(y-x)^k + r_N(x,y),$$

where

$$|r_N(x,y)| \leq C|x-y|^N \quad \text{for } |x| \leq A, \ |y| \leq A.$$

For $k = 1, \ldots, n-1$, let

$$g_k(x) = \frac{\partial^k \Phi}{\partial y^k}(x,x).$$

Using Taylor's expansion again, we obtain

$$g_k(x) = \sum_{l=0}^{N-k} \frac{g_k^{(l)}(a)}{l!}(x-a)^l + h_{k,N}(x,a),$$

where $|h_{k,N}(x,a)| \leq C|x-a|^{N-k+1}$. Define the polynomial $P_a(x,y)$ by

$$P_a(x,y) = \sum_{k=1}^{N-1} \sum_{l=0}^{N-k} \frac{g_k^{(l)}(a)}{k!\,l!}(x-a)^l (y-x)^k.$$

We have

$$\Phi(x,y) = \Phi(x,x) + P_a(x,y) + \tilde{r}_{N,a}(x,y),$$

where

$$\tilde{r}_{N,a}(x,y) = \sum_{k=1}^{N-1} \frac{h_{k,N}(x,a)}{k!}(y-x)^k + r_N(x,y).$$

Let

$$R_{\lambda,a} f(x) = \int_{\mathbf{R}^1} e^{i\lambda P_a(x,y)} \frac{\varphi(x,y)}{x-y}(1-\phi)(\lambda^{1/N}(x-y))f(y)\,dy.$$

Since $\operatorname{supp}(f) \subset I_a$, we have

(2.2)
$$\begin{aligned}
&\|R_{\lambda,a}f(x) - e^{-i\lambda\Phi(x,x)}T_\lambda^1 f(x)\|_1 \\
&\quad \leq \int_{\mathbf{R}^1}\int_{I_a} |e^{i\lambda\tilde{r}_{N,a}(x,y)} - 1| \frac{|(1-\phi)(\lambda^{1/N}(x-y))\varphi(x,y)|}{|x-y|} |f(y)|\,dy\,dx \\
&\quad \leq C\lambda \int\int_{\substack{|x-y|\leq\lambda^{-1/N} \\ |y-a|\leq\lambda^{-1/N}}} \left(|x-y|^N + \sum_{k=1}^{N-1} |x-a|^{N-k+1}|x-y|^k\right) \frac{|f(y)|}{|x-y|}\,dy\,dx \\
&\quad \leq C_N\|f\|_1.
\end{aligned}$$

We note that the constant C_N is independent of a, as long as $|a| \leq \frac{A}{2}$. Using the theorem of Chanillo and Christ on operators with polynomial phase functions [1, Theorem 1], we have

$$m\{x\colon |R_{\lambda,a}f(x)| > \alpha\} \leq C_N\alpha^{-1}\|f\|_1. \tag{2.3}$$

Again, the constant C_N is independent of a (the constant depends on the degree of the polynomial, not its coefficients). Combining (2.2) and (2.3), we have

$$m\{x\colon |T_\lambda^1 f(x)| > \alpha\} \leq C_N\|f\|_1, \tag{2.4}$$

for $f \in L^1$ with $\operatorname{supp}(f) \subset I_a$, $|a| \leq \frac{A}{2}$.

Now let $I_j = (-\lambda^{-1/N}, \lambda^{-1/N}) + (2j)\lambda^{-1/N} = I(2j\lambda^{-1/N})$, $I_j^* = (-2\lambda^{-1/N}, 2\lambda^{-1/N}) + (2j)\lambda^{-1/N}$. For any $f \in L^1$, let $f = \sum f_j$, where $f_j = f\chi_{I_j}$. Obviously, we need to worry about those j's with $I_j \subset [-\frac{A}{2}, \frac{A}{2}]$ only. Since $\operatorname{supp}(f_j) \subset I_j$, by (2.4) we have

$$m\{x\colon |T_\lambda^1 f_j(x)| > \alpha\} \leq C_N\alpha^{-1}\|f_j\|_1,$$

and $\operatorname{supp}(T_\lambda^1 f_j) \subset I_j^*$. Since $\{I_j^*\}$ has finite overlapping, we get

$$\begin{aligned}
&m\{x\colon |T_\lambda^1 f(x)| > \alpha\} \\
&\quad \leq \sum_{j=-\infty}^{\infty} m\left(\left\{x\colon \left|\sum_{k=-\infty}^{\infty} T_\lambda^1 f_k(x)\right| > \alpha\right\} \cap I_j\right) \\
&\quad \leq \sum_{j=-\infty}^{\infty} (m\{x\colon |T_\lambda^1 f_{j-1}(x)| > \alpha/3\} + m\{x\colon |T_\lambda^1 f_j(x)| > \alpha/3\} \\
&\qquad\qquad + m\{x\colon |T_\lambda^1 f_{j+1}(x)| > \alpha/3\}) \\
&\quad \leq C_N\alpha^{-1} \sum_{j=\infty}^{\infty} (\|f_{j-1}\|_1 + \|f_j\|_1 + \|f_{j+1}\|_1) \leq 3C_N\alpha^{-1}\|f\|_1,
\end{aligned}$$

which proves (2.1) for all $f \in L^1$.

We now turn to the estimate of the remaining part. We want to show that, if N is sufficiently large, the estimate

$$m\{x\colon |T_\lambda^2 f(x)| > \alpha\} \leq C\alpha^{-1}\|f\|_{L^1} \tag{2.5}$$

holds for some $C > 0$. In the next few steps, we follow the argument in [1]. Given $\alpha > 0$, $f \in L^1$, we make a Calderon-Zygmund decomposition

$f = g + \sum b_j$, where $\|g\|_\infty < \alpha$, b_j is supported on an interval Q_j of length $2^{k(j)}$, and

$$Q_i \cap Q_j = \varnothing \quad \text{if } i \neq j,$$
$$\sum |Q_j| \leq C\alpha^{-1}\|f\|_1, \qquad \frac{1}{|Q_j|}\int |b_j| \leq C\alpha. \tag{2.6}$$

Let Q_j^* be the 16-fold of Q_j and $\Omega = (\bigcup Q_j^*)^c$. The uniform L^2 estimate for T_λ^2 [7, Theorem 2] implies that

$$m\{x: |T_\lambda^2 g(x)| > \alpha/2\} \leq C\alpha^{-1}\|f\|_{L^1}. \tag{2.7}$$

To complete the proof of (2.5), it suffices to prove

$$\left\|T_\lambda^2\left(\sum b_j\right)\right\|_{L^2(\Omega)}^2 \leq C\alpha^{-1}\|f\|_{L^1}. \tag{2.8}$$

If $x \in \Omega$, $y \in Q_j$, we have $|x-y| \geq 8 \cdot 2^{k(j)}$. Therefore,

$$\phi(2^{-(k(j)+3)}(x-y)) \equiv 1 \quad \text{for } x \in \Omega,\ y \in Q_j. \tag{2.9}$$

Let $S_{\lambda,j}f(x) = \int_{\mathbf{R}^1} K_{\lambda,j}(x, y)f(y)\,dy$, where

$$K_{\lambda,j}(x, y) = e^{i\lambda\Phi(x,y)}\frac{\varphi(x, y)}{x-y}\phi(\lambda^{1/N}(x-y))\phi(2^{-(k(j)+3)}(x-y)).$$

By (2.9), we get

$$T_\lambda^2\left(\sum b_j\right)(x) = \sum S_{\lambda,j}(b_j)(x) \quad \text{if } x \in \Omega.$$

Let $L_{i,j}(x, y)$ be the kernel of the operator $(S_{\lambda,i})^* S_{\lambda,j}$, i.e.,

$$L_{i,j}(x, y) = \int_{\mathbf{R}^1} \overline{K}_{\lambda,i}(z, x)K_{\lambda,j}(z, y)\,dz.$$

If we can show that there is a constant $C > 0$ such that, for any i and $x \in Q_i$,

$$\sum_{j\,:\,k(j)\leq k(i)} \left|\int_{Q_j} L_{i,j}(x, y)b_j(y)\,dy\right| \leq C\alpha, \tag{2.10}$$

we will have

$$\left\|\sum_j S_{\lambda,j}b_j\right\|_{L^2}^2 \leq C\alpha\|f\|_1,$$

which implies (2.8).

We need the following:

Lemma 2.1. *Let $m = \inf\{k \geq 1: \exists j > 0$ such that $\partial^{j+k}\Phi(0, 0)/\partial x^j\partial y^k \neq 0\}$. Assume that $m < \infty$ (otherwise $\Phi(x, y) = \Phi_1(x) + \Phi_2(y)$). Also let j_0 be the smallest integer such that $\partial^{j_0+m}\Phi(0, 0)/\partial x^{j_0}\partial y^m \neq 0$. Then there are smooth functions $\xi_1(x, y, z), \ldots, \xi_m(x, y, z)$ such that*

$$\left|\frac{\partial^{j_0}\Phi(z, y)}{\partial z x^{j_0}} - \frac{\partial^{j_0}\Phi(z, x)}{\partial z^{j_0}}\right| \geq c_0\prod_{s=1}^m |y - x\xi_s(x, y, z)|$$

holds in a small neighborhood of the origin in $\mathbf{R}^3$.

We defer the proof of this lemma until the end of this section. By the implicit function theorem, it is not hard to see that there are smooth functions $h_1(x, z), \dots, h_m(x, z)$ and $\eta_1(z), \dots, \eta_m(z)$ (defined in a small neighborhood of the origins in $\mathbf{R}^2$ and $\mathbf{R}^1$ respectively), such that

$$|y - x\xi_s(x, y, z)| \geq \tfrac{1}{2}|y - h_s(x, z)|$$

and

$$|z - h_s(x, z)| \geq \tfrac{1}{2}|z - \eta_s(x)| \quad \textit{for } s = 1, \dots, m. \tag{2.11}$$

For fixed x and z, let

$$E_s(x, z) = \{y \colon |y - h_s(x, z)| \leq \tfrac{1}{8}\lambda^{-1/N}\}$$

and

$$\Omega_{x,y} = \left\{z \colon \left|\frac{\partial^{j_0}\Phi(z, x)}{\partial z^{j_0}} - \frac{\partial^{j_0}\Phi(z, y)}{\partial z^{j_0}}\right| \leq 16^{-m}c_0\lambda^{-m/N}\right\}.$$

To prove (2.10), we let

$$\begin{aligned} M_{i,j}(x, y) = \int_{\Omega_{x,y}} & e^{-i\lambda(\Phi(z,x)-\Phi(z,y))}\frac{\varphi(z, x)\varphi(z, y)}{(z-x)(z-y)} \\ & \times \phi(\lambda^{1/N}(z-x))\phi(2^{-k(i)-3}(z-x)) \\ & \times \phi(\lambda^{1/N}(z-y))\phi(2^{-k(j)-3}(z-y))\,dz \end{aligned} \tag{2.12}$$

and $R_{i,j} = L_{i,j} - M_{i,j}$. First we show that we can choose N to be sufficiently large such that

$$\sum_{j:k(j)\leq k(i)} \left|\int_{\mathbf{R}^1} R_{i,j}(x, y)b_j(y)\,dy\right| \leq C\alpha. \tag{2.13}$$

To achieve this, we use Van der Corput's lemma. Let

$$\eta(z, x) = \frac{\partial^{j_0}\Phi}{\partial z^{j_0}(z, x)}$$

and

$$\begin{aligned} \xi^{i,j}_{\lambda,N}(x, y, z) = & \frac{\varphi(z, x)\varphi(z, y)}{(z-x)(z-y)}\phi(\lambda^{1/N}(z-x))\phi(2^{-k(i)-3}(z-x)) \\ & \times \phi(\lambda^{1/N}(z-y))\phi(2^{-k(j)_3}(z-y)). \end{aligned}$$

We have

$$R_{i,j}(x, y) = \int_{z\in\Sigma_{x,y}} e^{-i\lambda(\Phi(z,x)-\Phi(z,y))}\xi^{i,j}_{\lambda,N}(x, y, z)\,dz,$$

where

$$\begin{aligned} \Sigma_{x,y} &= \Omega^c_{x,y} \cap \left(-\frac{A}{2}, \frac{A}{2}\right) \\ &= \{z \colon |\eta(z, x) - \eta(z, y)| > 16^{-m}c_0\lambda^{-m/N}\} \cap \left(-\frac{A}{2}, \frac{A}{2}\right). \end{aligned}$$

Clearly, for fixed x and y, $\Sigma_{x,y}$ is the union of disjoint open intervals. Since we need to apply Van der Corput's lemma on each interval separately, first we prove the following:

Lemma 2.2. *There exists a* $B > 0$ *such that for each* (x, y), *with* $|x| \leq \frac{A}{4}$, $|y| \leq \frac{A}{4}$, *the set* $\Sigma_{x,y}$ *is the union of no more than* B *disjoint open intervals.*

To prove this lemma and its higher-dimensional analogue, we need the following:

Proposition 2.3 [5, p. 76]. *Let* M *be a separable oriented real-analytic manifold and* p *a real-analytic map from* M *into* $\mathbf{R}^n$. *If* S *is a real-analytic subvariety of* M *and* K *is a compact subset of* M, *then there exists an integer* d *such that*

$$\operatorname{card}(K \cap S \cap p^{-1}\{y\}) \leq d,$$

whenever $y \in \mathbf{R}^n$ *and* $\dim(S \cap p^{-1}\{y\}) \leq 0$.

Proof of Lemma 2.2. Let

$$F(x, y, z) = \partial\eta(z, x)/\partial z - \partial\eta(z, y)/\partial z$$

and $G_{x,y}(z) = F(x, y, z)$. We write $\Sigma_{x,y}$ as the union of disjoint open intervals $\Sigma_{x,y} = \bigcup_\beta J_\beta$. Except for perhaps two intervals in the collection $\{J_\beta\}$, each J_β contains at least one zero of $G_{x,y}(\cdot)$ (by Rolle's theorem). Let $M \subset [-A, A]^3$ be a neighborhood of $K = \{(x, y, z): |x| \leq \frac{A}{4}, |y| \leq \frac{A}{4}, |z| \leq \frac{A}{2}\}$, S be the zero set of F in M, and p be the projection from $\mathbf{R}^3$ to $\mathbf{R}^2$, i.e., $p(t_1, t_2, t_3) = (t_1, t_2)$. By Proposition 2.3, there is an integer $d > 0$ such that

$$\operatorname{card}(K \cap S \cap p^{-1}\{(t_1, t_2)\}) \leq d,$$

whenever $(t_1, t_2) \in \mathbf{R}^2$ and $\dim(S \cap p^{-1}\{(t_1, t_2)\}) \leq 0$.

Clearly, for given x and y, with $|x| \leq \frac{A}{4}$ and $|y| \leq \frac{A}{4}$, we have

$$K \cap S \cap p^{-1}\{(x, y)\} = \{(x, y, z) | G_{x,y}(z) = 0\} \cap K.$$

We look at the following two cases separately.

(i) If $\dim(S \cap p^{-1}\{(x, y)\}) \leq 0$, we have

$$\operatorname{card}(\{(x, y, z) | G_{x,y}(z) = 0\} \cap K) \leq d,$$

which means the number of zeros of $G_{x,y}(\cdot)$ in $|z| \leq \frac{A}{2}$ is at most d. So there are at most $(d+2)$ intervals in $\bigcup_\beta J_\beta = \Sigma_{x,y}$.

(ii) If $\dim(S \cap p^{-1}\{(x, y)\}) > 0$, then the set $\{z | G_{x,y}(z) = 0\} \cap [-A, A]$ contains infinitely many points. By the real analyticity of $G_{x,y}(\cdot)$ we conclude that $G_{x,y}(\cdot)$ is identically zero. This implies that $(\eta(z, x) - \eta(z, y))$ is independent of z. Therefore, either $\Sigma_{x,y} = (-\frac{A}{2}, \frac{A}{2})$, or $\Sigma_{x,y} = \varnothing$.

Choose $B = d + 2$. The proof of the lemma is completed. □

Now returning to the proof of (2.13), we write

$$R_{i,j}(x, y) = \sum_\beta \int_{J_\beta} e^{-i\lambda(\Phi(z,x) - \Phi(z,y))} \xi^{i,j}_{\lambda,N}(x, y, z)\, dz,$$

where the summation contains at most B terms (B is independent of x and y).

Let $\lambda_j = \max\{2^{k(j)}, (16\lambda^{1/N})^{-1}\}$. By Van der Corput's lemma, we have

$$\begin{aligned}|R_{i,j}(x, y)| &\le \sum_\beta \left| \int_{J_\beta} e^{-i\lambda(\Phi(z,x)-\Phi(z,y))} \xi^{i,j}_{\lambda,N}(x, y, z)\, dz \right| \\ &\le C(\lambda^{1-m/N})^{-1/j_0} \left(B \cdot \|\xi^{i,j}_{\lambda,N}\|_\infty + \int_{-A/2}^{A/2} \left| \frac{\partial}{\partial z} \xi^{i,j}_{\lambda,N}(x, y, z) \right| dz \right).\end{aligned} \tag{2.14}$$

We have the following simple estimates:

$$\|\xi^{i,j}_{\lambda,N}\|_\infty \le C\lambda^{1/N}\lambda_i^{-1} \tag{2.15}$$

and

$$\left\| \frac{\partial}{\partial z} \xi^{i,j}_{\lambda,N} \right\|_1 \le C\lambda^{1/N}\lambda_i^{-1}, \tag{2.16}$$

where C is independent of i, j, and λ. While (2.15) is easy to see, the proof of (2.16) requires more careful analysis. Since it contains no essential difficulties, we omit the details.

From (2.14)–(2.16), we obtain

$$|R_{i,j}(x, y)| \le C\lambda^{-(1-m/N)/j_0+1/N}\lambda_i^{-1}$$

uniformly in x and y. Finally, we have

$$\begin{aligned}&\sum_{j\,:\,k(j)\le k(i)} \left| \int_{\mathbf{R}^1} R_{i,j}(x, y) b_j(y)\, dy \right| \\ &\le C\lambda^{-(1-m/N)/j_0+1/N}\lambda_i^{-1} \sum_{Q_j\cap[-A, A]\ne\varnothing} \int_{Q_j} |b_j(y)|\, dy \\ &\le C\lambda^{-(1-m/N)/j_0+1/N}\lambda_i^{-1}\alpha \left(\sum_{Q_j\cap[-A, A]\ne\varnothing} |Q_j| \right).\end{aligned}$$

Since $k(j) \le k(i)$ and $\{Q_j\}$ are disjoint, we have

$$\lambda_i^{-1} \left(\sum_{Q_j\cap[-A, A]\ne\varnothing} |Q_j| \right) \le \lambda_i^{-1}(2A + 2\cdot 2^{k(i)}) \le CA\lambda^{1/N}.$$

Taking N such that $N > 2j_0 + m$, we get

$$\sum_{j\,:\,k(j)\le k(i)} \left| \int_{\mathbf{R}^1} R_{i,j}(x, y) b_j(y)\, dy \right| \le C\alpha,$$

which proves (2.13).

To deal with the part with $M_{i,j}$, we observe that $\lambda_j \le \lambda_i$ if $k(j) \le k(i)$ and

$$\begin{aligned}
&\sum_{j\,:\,k(j)\le k(i)} \left| \int_{\mathbf{R}^1} M_{ij}(x, y) b_j(y)\, dy \right| \\
&\le \sum_{s=1}^{m} \int_{\mathbf{R}^1} \frac{dz}{(|x-z| + \lambda_i)} \\
&\quad \times \left(\sum_{j\,:\,k(j)\le k(i)} \int_{Q_j \cap E_s(x, z)} \frac{|\phi(\lambda_j^{-1}(z-y))|}{(|z-y| + \lambda_j)} |b_j(y)|\, dy \right).
\end{aligned} \tag{2.17}$$

For fixed x and z, if $y \in Q_j \cap E_s(x, z)$ and $\phi(\lambda_j^{-1}(z-y)) \ne 0$, we have

$$|z-y| \ge \tfrac{1}{2}(|z - h_s(x, z)| + \lambda_j). \tag{2.18}$$

We also have:

(i) $\sum |Q_j| \le C\lambda^{-1/N}$, where the summation is taken over $\{j\colon Q_j \cap E_s(x, z) \ne \varnothing,\ 2^{k(j)} \le \lambda^{-1/N}\}$.

(ii) There are at most two Q_j with $2^{k(j)} \ge \lambda^{-1/N}$ and $Q_j \cap E_s(x, z) \ne \varnothing$.

Thus we have

(2.19)

$$\begin{aligned}
&\sum_{j\,:\,k(j)\le k(i)} \left| \int_{\mathbf{R}^1} M_{ij}(x, y) b_j(y)\, dy \right| \\
&\le C \sum_{s=1}^{m} \sum_{2^{k(j)} \le \lambda^{-1/N}} \int_{|z|\le c} \int_{Q_j \cap E_s(x, z)} \frac{|b_j(y)|\, dy\, dz}{(|z - \eta_s(x)| + \lambda^{-1/N})(|z-x| + \lambda_i)} \\
&\quad + C \sum_{s=1}^{m} \sum_{k(j)\le k(i),\, 2^{k(j)} \ge \lambda^{-1/N}} \int_{|z|\le c} \int_{Q_j \cap E_s(x, z)} \frac{|b_j(y)|\, dy\, dz}{(|z - \eta_s(x)| + \lambda_j)(|z-x| + \lambda_i)} \\
&\le C\alpha \sum_{s=1}^{m} \left(\int_{|z|\le c} \frac{\lambda^{-1/N}\, dz}{(|z-x| + \lambda^{-1/N})(|z - \eta_s(x)| + \lambda^{-1/N})} \right. \\
&\quad \left. + \int_{|z|\le c} \frac{\lambda_i\, dz}{(|z-x| + \lambda_i)(|z - \eta_s(x)| + \lambda_i)} \right) \le C\alpha,
\end{aligned}$$

where we have used (2.6), (2.11), (2.18), and the fact that

$$\frac{\lambda_j}{|z - \eta_s(x)| + \lambda_j} \le \frac{\lambda_i}{|z - \eta_s(x)| + \lambda_i}, \quad \text{if } \lambda_j \le \lambda_i.$$

Combining (2.13) and (2.19), we obtain (2.10). This concludes the proof of Theorem 1.

Proof of Lemma 2.1. In a small neighborhood U of the origin in $\mathbf{R}^3$, we have

$$\frac{\partial^{j_0}\Phi(z, y)}{\partial z^{j_0}} - \frac{\partial^{j_0}\Phi(z, x)}{\partial z^{j_0}} = c_0(y-x) \sum_{j=0}^{m-1} b_j(x, y, z) y^{m-1-j} x^j,$$

where c_0 is a constant, $b_0, b_1, \dots, b_{m-1}$ are smooth functions defined in U, and

$$b_j(0) = 1 \quad \text{for } j = 0, 1, \dots, m-1.$$

Let $\zeta = y/x$ we have

$$\begin{aligned}\frac{\partial^{j_0}\Phi(z,y)}{\partial z^{j_0}} - \frac{\partial^{j_0}\Phi(z,x)}{\partial z^{j_0}} &= c_0 x^{m-1}(y-x)\sum_{j=0}^{m-1} b_j(x,y,z)\zeta^{m-1-j} \\ &= c_0 x^{m-1}(y-x)b_0(x,y,z)\prod_{j=1}^{m-1}(\zeta - \zeta_j(x,y,z)),\end{aligned}$$

where $\zeta_1(x,y,z), \dots, \zeta_{m-1}(x,y,z)$ are smooth, complex-valued, and $\zeta_1(0), \dots, \zeta_{m-1}(0)$ are the $(m-1)$ complex roots of the equation $w^m - 1 = 0$.

Let $\xi_j = \mathrm{Re}(\zeta_j)$ for $j = 1, \dots, m-1$ and $\xi_m \equiv 1$. We see that

$$\left|\frac{\partial^{j_0}\Phi(z,y)}{\partial z^{j_0}} - \frac{\partial^{j_0}\Phi(z,x)}{\partial z^{j_0}}\right| \geq c_0 \prod_{j=1}^{m} |y - x\xi_j(x,y,z)|.$$

This proves the lemma. □

3. Higher dimensions

Let $x, y \in \mathbf{R}^n$ and $K(x,y)$ be a Calderon-Zygmund kernel; i.e., K is C^∞ away from the diagonal, satisfies

$$|K(x,y)| \leq A|x-y|^{-n}, \qquad |\nabla K(x,y)| \leq |x-y|^{-n-1},$$

and $f \to \int K(x,y)f(y)\,dy$ extends to a bounded operator on L^2. Let $\varphi \in C_0^\infty(\mathbf{R}^n \times \mathbf{R}^n)$ and $\Phi(x)$ be a real-analytic function. Define T_λ as

$$T_\lambda f(x) = \text{p.v.} \int_{\mathbf{R}^n} e^{i\lambda\Phi(x-y)} K(x,y)\varphi(x,y)f(y)\,dy. \tag{3.1}$$

We have the following:

Theorem 2. *Suppose that K, Φ, and T_λ are given as above. Then the operators T_λ are uniformly bounded mappings from L^1 to $L^{1,\infty}$; i.e., there exists a constant C such that, for any given $\alpha > 0$,*

$$m\{x \colon |T_\lambda f(x)| > \alpha\} \leq C\alpha^{-1}\|f\|_1, \tag{3.2}$$

where C is independent of λ.

Proof. By the same argument used in the first part of §2, the proof of Theorem 2 can be reduced to the proof of (3.2) for the following operators:

$$f \to \int_{\mathbf{R}^n} e^{i\lambda\Phi(z-y)} K(x,y)\varphi(x,y)\phi(\lambda^{1/N}(x-y))f(y)\,dy, \tag{3.3}$$

where $\phi = 1 - \phi_1$, for some $\phi_1 \in C_0^\infty$, $\phi_1 \equiv 1$ near the origin, and N is a large number (to be chosen later). For the sake of simplicity, we still call the operators in (3.3) T_λ.

Given $\alpha > 0$, $f \in L^1$, we make a Calderon-Zygmund decomposition $f = g + \sum b_j$, as in (2.6) (with the intervals replaced by cubes). By exactly the same reasoning as in §2, we see that it suffices to prove that, for any fixed i and $x \in Q_i$, the following estimate holds:

$$\sum_{j\,:\,k(j)\le k} \left| \int_{Q_j} L_{i,j}(x, y) b_j(y)\, dy \right| \le C\alpha, \tag{3.4}$$

where

$$L_{i,j}(x, y) = \int_{\mathbf{R}^n} e^{i\lambda(\Phi(z-x)-\Phi(z-y))} K(z, x)\overline{K}(z, y) \times \varphi(z, x)\varphi(z, y)\phi(\lambda^{1/N}(z-x))\phi(\lambda^{1/N}(z-y))\, dz$$

and $k = k(i)$. Suppose that Φ is not a linear function (if Φ is linear, the operators are reduced to the usual singular integral operators), then we can find a α, $|\alpha| \ge 2$, such that

$$\partial^\alpha \Phi(0)/\partial x^\alpha \ne 0.$$

Thus, there exists a unit vector ξ so that $(\xi \cdot \nabla_x)^l \Phi(0) \ne 0$ (see also [10]). We may assume that $\xi = (1, 0, \dots, 0)$. Let

$$F(x, y, z) = \frac{\partial^{l-1}\Phi}{\partial z_1^{l-1}}(z-x) - \frac{\partial^{l-1}\Phi}{\partial z_1^{l-1}}(z-y).$$

We have $F(0) = 0$ and $\partial F(0)/\partial y_1 \ne 0$. So we can find smooth functions $c(x, y, z)$ and $a(x, y', z)$ such that $c(0) \ne 0$, $a(0) = 0$, and

$$F(x, y, z) = c(x, y, z)(y_1 + a(x, y', z)),$$

where $y' = (y_2, \dots, y_n)$. We also notice that

$$a(x, x', z) + x_1 = 0 \tag{3.5}$$

holds for any x, z. Suppose that $|c(x, y, z)| \ge c_0$, and let $\Omega_{x,y} = \{z : |F(x, y, z)| \le \lambda^{-1/N}\}$ and $E_{x,z} = \{y | z \in \Omega_{x,y}\}$.

Again we write

$$L_{i,j}(x, y) = M_{i,j}(x, y) + R_{i,j}(x, y),$$

where $M_{i,j}$ and $R_{i,j}$ are defined as in (2.12). If we choose N sufficiently large (depending on l and n), we see that

$$\sum_{j\,:\,k(j)\le k} \left| \int_{Q_j} R_{i,j}(x, y) b_j(y)\, dy \right| \le C\alpha. \tag{3.6}$$

We now turn to the estimate of the part with $M_{i,j}$. First by (3.5) and the implicit function theorem, there exists a smooth function $h(x, y', z')$ with $h(0) = 0$ such that

$$|z_1 + a(x, y', z)| \ge \tfrac{1}{2}|z_1 - h(x, y', z')|, \tag{3.7}$$

where (x, y, z) is in a small neighborhood of the origin in $\mathbf{R}^{3n}$. Fix x, z, and let Q_j be a cube such that

$$Q_j \cap E_{x,z} \ne \varnothing.$$

Let $w=(w_1,\dots,w_n)\in Q_j\cap E_{x,z}$ and y be an arbitrary point in Q_j. Then

$$|w_1+a(x,w',z)|\le\lambda^{-1/N}\quad\text{and}\quad|w-y|\le\lambda_j.$$

Therefore, we have

$$|y_1+a(x,y',z)|\le A\lambda_j \tag{3.8}$$

for some constant A. Let

$$S_{z,j}=\{(y_1,y')\in\mathbf{R}^n\colon|y_1-z_1|\ge 4A\lambda_j\}.$$

We are now ready to establish the estimate for the part with $M_{i,j}$:

(3.9)

$$\begin{aligned}
&\sum_{k(j)\le k}\left|\int_{Q_j}M_{i,j}(x,y)b_j(y)\,dy\right|\\
&\quad\le C\int_{|z|\le c}\frac{dz}{(|z-x|+\lambda_i)^n}\sum_{k(j)\le k}\int_{Q_j\cap E_{x,z}}\frac{|b_j(y)|\,dy}{(|z-y|+\lambda_j)^n}\\
&\quad\le C\alpha\int_{|z|\le c}\frac{dz}{(|z-x|+\lambda_i)^n}\sum_{k(j)\le k,\,Q_j\cap E_{x,z}\ne\varnothing}\int_{Q_j}\frac{dy}{(|z-y|+\lambda_j)^n}\\
&\quad= C\alpha\int_{|z|\le c}\frac{dz}{(|z-x|+\lambda_i)^n}\\
&\qquad\times\sum_{k(j)\le k,\,Q_j\cap E_{x,z}\ne\varnothing}\left[\int_{Q_j\cap S_{z,j}}\frac{dy}{(|z-y|+\lambda_j)^n}+\int_{Q_j\cap S_{z,j}^c}\frac{dy}{(|z-y|+\lambda_j)^n}\right]\\
&\quad= I_1+I_2,
\end{aligned}$$

where I_1 is the part with $S_{z,j}$. We observe that, when $(y_1,y')\in Q_j\cap S_{z,j}$, and $Q_j\cap E_{x,z}\ne\varnothing$,

$$|z_1-y_1|\ge\tfrac{1}{2}|z_1+a(x,y',z)|\ge\tfrac{1}{4}|z_1-h(x,y',z')|.$$

So we have

(3.10)

$$\begin{aligned}
I_1&\le C\alpha\int_{|z|\le c}\frac{dz}{(|z-x|+\lambda_i)^n}\int_{|y'|\le c}\frac{\lambda^{-1/N}\,dy'}{(|z_1-h(x,y',z')|+|z'-y'|+\lambda^{-1/N})^n}\\
&\quad+C\alpha\sum_{m\le k,\,2^m\ge\lambda^{-1/N}}\int_{|z|\le c}\frac{dz}{(|z-x|+\lambda_i)^n}\\
&\qquad\times\int_{|y'|\le c}\frac{2^m\,dy'}{(|z_1-h(x,y',z')|+|z'-y'|+2^m)^n}.
\end{aligned}$$

The first term on the right-hand side of (3.10) is bounded by

(3.11)
$$
\begin{aligned}
&C\alpha\lambda^{-1/N}\int_{|z'|\le c,\,|y'|\le c} dz'dy' \\
&\quad\times\int_{z_1\in\mathbf{R}^1}(|z-x|+\lambda^{-1/N})^{-n}(|z_1-h(x,y',z')|+|z'-y'|+\lambda^{-1/N})^{-n}\,dz_1 \\
&\quad\le C\alpha\lambda^{-1/N}\int_{z_1\in\mathbf{R}^1}\int_{|z'|\le c}\frac{dz'}{(|z_1|+|z'-x'|+\lambda^{-1/N})^n} \\
&\quad\times\int_{|y'|\le c}\frac{dy'}{(|z_1|+|y'-z'|+\lambda^{-1/N})^n} \\
&\quad\le C\lambda^{-1/N}\alpha\int_{z_1\in\mathbf{R}^1}\frac{dz_1}{(|z_1|+\lambda^{-1/N})(|z_1|+\lambda^{-1/N})}\le C\alpha.
\end{aligned}
$$

The second term in (3.10) is nonzero only if $2^k\ge\lambda^{-1/N}$, in which case it is bounded by

$$
C\alpha\sum_{m\le k}\int_{|z|\le c}\frac{dz}{(|z-x|+2^k)^n}\int_{|y'|\le c}\frac{2^m\,dy'}{(|z_1-h(x,y',z')|+|z'-y'|+2^m)^n}.
$$

Using the same argument as in (3.10), we see that this is bounded by

$$
\text{(3.12)}\quad C_\alpha\sum_{m\le k}\int_{z_1\in\mathbf{R}^1}\frac{2^m\,dz_1}{(|z_1|+2^k)(|z_1|+2^m)}\le C\alpha\sum_{m\le k}2^{m-k}(k-m+1)\le C\alpha.
$$

Now we turn to the estimate of I_2, where

$$
I_2=C\alpha\int_{|z|\le c}\frac{dz}{(|z-x|+\lambda_i)^n}\sum_{k(j)\le k,\,Q_j\cap E_{x,z}\ne\varnothing}\int_{Q_j\cap S^c_{z,j}}\frac{dy}{(|z-y|+\lambda_j)^n}.
$$

This part is actually easier to control. Suppose that $Q_j\cap E_{x,z}\ne\varnothing$, $(y_1,y')\in Q_j\cap S^c_{z,j}$. From (3.7), (3.8), and the definition of $S_{z,j}$ we get

$$
|z_1-h(x,y',z')|\le 10A\lambda_j.
$$

Thus we have

(3.13)
$$
\begin{aligned}
I_2&\le C\alpha\int_{|z_1-h(x,y',z')|\le\lambda^{-1/N}}\frac{\lambda^{-1/N}\,dz\,dy'}{(|x'-z'|+\lambda^{-1/N})^n(|y'-z'|+\lambda^{-1/N})^n} \\
&\quad+C\alpha\sum_{m\le k,\,2^k\ge\lambda^{-1/N}}\int_{|z_1-h(x,y',z')|\le 2^m}\frac{2^m\,dz\,dy'}{(|z'-x'|+2^k)^n(|y'-z'|+2^m)^n} \\
&\le C\alpha.
\end{aligned}
$$

Combining (3.6) and (3.9)–(3.13), we obtain (3.4). This concludes the proof of Theorem 2. □

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