PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 120, Number 3, March 1994

VON NEUMANN'S INEQUALITY FOR COMMUTING, DIAGONALIZABLE CONTRACTIONS. I

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(Communicated by Theodore W. Gamelin)

ABSTRACT. We obtain a sufficient condition for an n-tuple T of commuting, diagonalizable contractions on a finite-dimensional space to satisfy von Neumann's inequality $\|p(T)\| \leq \|p\|_\infty$ for any polynomial p in n variables, where $\|p\|_\infty$ denotes the supremum of $|p|$ over the unit polydisk in $\mathbf{C}^n$. We apply this condition to the case where T acts on a two- or three-dimensional space. In addition, we prove that von Neumann's inequality for commuting, diagonalizable contractions on a three-dimensional space implies von Neumann's inequality for arbitrary commuting contractions on a three-dimensional space.

1. INTRODUCTION

Von Neumann's inequality [11], a fundamental tool in operator theory, states that $\|p(T)\| \leq \|p\|_\infty$ for any (linear) contraction T on a Hilbert space and any polynomial p with complex coefficients. Here $\|p\|_\infty$ denotes the supremum norm of p over the unit disk $\mathbf{D}$. Attempts to generalize von Neumann's inequality have met with varying success. Andô [1] showed that von Neumann's inequality holds for any pair $T = (T_1, T_2)$ of commuting contractions and any polynomial p in two variables, where $\|p\|_\infty$ now refers to the supremum norm over the unit bidisk $\mathbf{D}^2$ in $\mathbf{C}^2$. Varopoulos [10] proved that the generalization to a triple of commuting contractions fails. Specific examples are given in the addendum to [10] and in [2]. Drury [3] proved von Neumann's inequality for an arbitrary n-tuple of commuting contractions on a two-dimensional Hilbert space. Recently, Lewis and Wermer [6] rediscovered this fact in the special case where the contractions are diagonalizable and asked whether the extra diagonalizability assumption is sufficient to ensure that von Neumann's inequality holds for contractions on spaces of arbitrary dimension. Although this is not the case (an example is presented in the companion to this paper [7]), we will show in §2 that, with an additional hypothesis, the inequality does hold. We then show in §3 that this hypothesis can always be satisfied in the two-dimensional case, so that our theorem generalizes Drury's result. We also give in §3 an explicit, easily computable method of checking the hypothesis in the three-dimensional case. Our methods involve the use of the model theory of Sz.-Nagy and Foaiş.

Received by the editors October 28, 1991 and, in revised form, July 1, 1992; presented at AMS meeting #876, Dayton, Ohio, October 1992.

1991 *Mathematics Subject Classification*. Primary 47A63; Secondary 15A60.

2. The main theorem

We say that a subspace M of $\mathbf{C}^N$ is a reducing subspace of the n-tuple $T = (T_1, \ldots, T_n)$ of operators on $\mathbf{C}^N$ if both M and $M^\perp$ are invariant under every T_j.

Theorem 1. *Suppose that $T = (T_1, \ldots, T_n)$ is an n-tuple of commuting, diagonalizable contractions on $\mathbf{C}^N$ that has no nontrivial reducing subspace. If there exists a diagonalizable contraction X on $\mathbf{C}^N$ that commutes with every T_j such that $I - X^*X$ has rank 1, then von Neumann's inequality holds for T.*

The hypothesis that T have no nontrivial reducing subspace is not really a restriction, as shown by the following proposition.

Proposition 2. *Suppose that M is a reducing subspace for T. Then von Neumann's inequality holds for T if and only if it holds for both $T|M$ and $T|M^\perp$.*

Proof. Since M reduces T, it also reduces $p(T)$ for any polynomial p. We have $p(T)|M = p(T|M)$ and $\|p(T)\| = \max(\|p(T)|M\|, \|p(T)|M^\perp\|)$ (and similarly for $M^\perp$). The proposition follows immediately from these facts. □

Fix an n-tuple $T = (T_1, \ldots, T_n)$ of commuting, diagonalizable contractions on $\mathbf{C}^N$. Since any set of commuting, diagonalizable operators is simultaneously diagonalizable, there is a basis $V = \{v_1, \ldots, v_N\}$ of $\mathbf{C}^N$ consisting entirely of eigenvectors for every T_j. For w in $\mathbf{C}^N$, let D_w denote the operator defined by $D_w v_j = w_j v_j$ and on the rest on $\mathbf{C}^N$ by linearity. Note that each T_j is of the form D_w for an appropriate choice of w and that $p(T)$ is also of this form for any polynomial p in n variables.

To prove von Neumann's inequality for T, it is enough (by scaling) to show that if p is a polynomial with $\|p\|_\infty \leq 1$, then $p(T)$ is a contraction. Our first lemma identifies contractions of the form D_w.

Lemma 3. *Let w be in $\mathbf{C}^N$. Then D_w is a contraction if and only if the matrix*

$$((1 - w_j\bar{w}_k)\langle v_j, v_k\rangle)_{j,k=1}^N \tag{1}$$

is positive semidefinite.

Proof. Let c be in $\mathbf{C}^N$, and set $v = c_1v_1 + \cdots + c_Nv_N$. Then

$$\|v\|^2 = \left\langle \sum_{j=1}^N c_jv_j, \sum_{k=1}^N c_kv_k \right\rangle = \sum_{j,k=1}^N c_j\bar{c}_k\langle v_j, v_k\rangle$$

and

$$\|D_wv\|^2 = \left\langle \sum_{j=1}^N c_jw_jv_j, \sum_{k=1}^N c_kw_kv_k \right\rangle = \sum_{j,k=1}^N c_j\bar{c}_kw_j\bar{w}_j\langle v_j, v_k\rangle.$$

If D_w is a contraction, then $\|v\|^2 - \|D_wv\|^2 \geq 0$. Substituting, we have

$$\sum_{j,k=1}^N c_j\bar{c}_k(1 - w_j\bar{w}_k)\langle v_j, v_k\rangle \geq 0.$$

Since c was arbitrary, the matrix (1) is positive semidefinite.

The argument is reversible, so the lemma is proved. □

On the basis of Proposition 2, we will assume for the rest of this section that T has no nontrivial reducing subspaces. A consequence of this assumption is that there is no partition of the basis of eigenvectors V into two nonempty, disjoint, elementwise orthogonal subsets, as the span of either subset would be a nontrivial reducing subspace of T.

Lemma 4. *Let w be in $\mathbf{C}^N$, and suppose that D_w is a contraction. Then either D_w is a scalar multiple of the identity or $|w_j| < 1$ for $1 \leq j \leq N$.*

Proof. Clearly, $|w_j| \leq 1$ for all j. Suppose that $|w_J| = 1$ for some J. By Lemma 3, the matrix

$$((1 - w_j \bar{w}_k)\langle v_j, v_k\rangle)_{j,k=1}^N \tag{2}$$

is positive semidefinite, so the same is true for any submatrix. Suppose that K is such that $\langle v_J, v_K\rangle \neq 0$. The submatrix of (2) formed by the Jth and Kth rows and columns is

$$\begin{pmatrix} 0 & (1 - w_J\bar{w}_K)\langle v_J, v_K\rangle \\ (1 - w_K\bar{w}_J)\langle v_K, v_J\rangle & (1 - |w_K|^2)\|v_K\|^2 \end{pmatrix}.$$

The determinant of this submatrix equals $-|(1 - w_J\bar{w}_K)\langle v_J, v_K\rangle|^2$ but must be nonnegative because of positive definiteness. Hence, $(1 - w_J\bar{w}_K)\langle v_J, v_K\rangle = 0$, so $1 - w_J\bar{w}_K = 0$ and $w_J = w_K$.

We may iterate this procedure with w_K in place of w_J. It follows that $w_J = w_K$ for any K such that v_J and v_K are linked by a sequence $v_J = v_{\alpha_0}, \dots, v_{\alpha_m} = v_K$ for which $\langle v_{\alpha_j}, v_{\alpha_{j+1}}\rangle \neq 0$. But any v_K can be linked to v_J by such a sequence, as the span of all such vectors is a nonempty, reducing subspace of T. The lemma is therefore proved. □

It follows from Lemma 4 that if D_w is a contraction, not a scalar multiple of the identity, then it is unitarily equivalent to its Sz.-Nagy-Foaiş model [9], which we now describe.

Let d denote the rank of the operator $I - D_w^* D_w$, and let H_d^2 denote the Hardy space of $\mathbf{C}^d$-valued functions that are holomorphic in $\mathbf{D}$ and have square-summable Taylor coefficients. Then D_w is unitarily equivalent to the restriction of the backward shift operator S^* to an invariant subspace K. (The backward shift is the adjoint of the shift operator S of "multiplication by z" on H_d^2.) Under this unitary equivalence, the basis V of eigenvectors is mapped to a basis of K consisting of eigenvectors for S^*. These are of the form $c_j k_{\bar{w}_j}$, where $c_j \neq 0$ is in $\mathbf{C}^d$ and $k_{\bar{w}_j}(z) = (1 - w_j z)^{-1}$ is the reproducing kernel function at $\bar{w}_j$ for scalar-valued H^2. We note that the model operator (and hence D_w) possesses an H^∞ functional calculus for which von Neumann's inequality is valid, that is, $\|f(S^*|K)\| \leq \|f\|_\infty$ for all f in H^∞. (It is easy to prove this last statement directly for D_w.) An easy computation shows that $f(D_w) = D_{f(w)}$ for f in H^∞, where $f(w) = (f(w_1), \dots, f(w_N))$.

If $d = 1$, then single-variable complex function theory can be brought into play. In this case, we can improve substantially on Lemma 3.

Theorem 5. *Suppose that there exists z in $\mathbf{C}^N$ such that D_z is a contraction and $I - D_z^* D_z$ has rank 1. The following are then equivalent for any w in $\mathbf{C}^N$:*

(i) *D_w is a contraction.*
(ii) *The matrix $((1 - w_j\bar{w}_k)/(1 - z_j\bar{z}_k))_{j,k=1}^N$ is positive semidefinite.*
(iii) *There is a function f in H^∞ with $\|f\|_\infty \le 1$ such that $f(z_j) = w_j$.*
(iv) *There is a function f in H^∞ with $\|f\|_\infty \le 1$ such that $f(D_z) = D_w$.*

Proof. If (i) holds, then the matrix

$$\left((1 - w_j\bar{w}_k)\langle v_j, v_k\rangle\right)_{j,k=1}^N$$

is positive semidefinite by Lemma 3. Using our model for D_z, we find that

$$\langle v_j, v_k\rangle = \langle c_j k_{\bar{w}_j}, c_k k_{\bar{w}_k}\rangle = c_j\bar{c}_k(1 - z_j\bar{z}_k)^{-1}$$

and we have

$$\begin{aligned}\left((1 - w_j\bar{w}_k)\langle v_j, v_k\rangle\right)_{j,k=1}^N &= \left(c_j\bar{c}_k\frac{1 - w_j\bar{w}_k}{1 - z_j\bar{z}_k}\right)_{j,k=1}^N \\ &= C\left(\frac{1 - w_j\bar{w}_k}{1 - z_j\bar{z}_k}\right)_{j,k=1}^N C^*,\end{aligned}$$

where C is the diagonal matrix whose diagonal is $c_1, \dots, c_N$. Since no c_j equals 0, C is invertible and (ii) follows.

That (ii) implies (iii) is Pick's Theorem [4, p. 8]. Now (iv) follows from (iii) by the remarks preceding the statement of the theorem, and (i) follows from (iv) by the H^∞ version of von Neumann's inequality for the contraction D_z. □

We note that Theorem 5 is a special case of the work in [8].

Corollary 6. *Suppose that there exists z in $\mathbf{C}^N$ such that D_z is a contraction and $I - D_z^*D_z$ has rank 1. Then von Neumann's inequality holds for any n-tuple of contractions of the form $(D_{w_1}, \dots, D_{w_n})$.*

Proof. By Theorem 5, there are functions $f_1, \dots, f_n$ in H^∞ such that $\|f_j\|_\infty \le 1$ and $f_j(D_z) = D_{w_j}$. Now if p is a polynomial in n variables, then

$$p(D_{w_1}, \dots, D_{w_n}) = [p \circ (f_1, \dots, f_n)](D_z).$$

Since $p \circ (f_1, \dots, f_n)$ is in H^∞ and $\|p \circ (f_1, \dots, f_n)\|_\infty \le \|p\|_\infty$, the corollary follows from the H^∞ version of von Neumann's inequality for D_z. □

Theorem 1 is a restatement of Corollary 6.

3. Applications to dimensions two and three

We now consider the cases where the dimension of the underlying Hilbert space is two or three.

Theorem 7. *If T is any n-tuple of commuting, diagonalizable operators on a two-dimensional space, then von Neumann's inequality holds for T.*

Proof. Recall that v_1 and v_2 are the common eigenvectors of the operators in T. If $\langle v_1, v_2\rangle = 0$, then Proposition 2 immediately yields von Neumann's inequality. If $\langle v_1, v_2\rangle \ne 0$, then T has no nontrivial reducing subspace. Let X be any operator, not a scalar multiple of the identity, such that v_1 and v_2 are eigenvectors of X and $\|X\| = 1$. Then $I - X^*X \ne 0$. But if v is any vector such that $\|Xv\| = \|v\|$, then $(I - X^*X)v = 0$. It follows that the rank of

$I - X^*X$ equals 1. The hypotheses of Theorem 1 being satisfied, we conclude that von Neumann's inequality holds for T. □

Now suppose that T is an n-tuple of commuting, diagonalizable operators acting on a three-dimensional space. If T has a nontrivial reducing subspace, then von Neumann's inequality holds for T by Proposition 2 and the preceding result for two dimensions. In the contrary case it is not always possible to apply Theorem 1. For the next theorem, we assume (without loss of generality) that the eigenvectors v_1, v_2, and v_3 have unit norm.

Theorem 8. *The hypotheses of Theorem* 1 *can be satisfied if and only if*

$$|\alpha\beta|^2 + |\alpha\gamma|^2 + |\beta\gamma|^2 = |\alpha\beta\gamma|^2 + 2\Re(\alpha\bar{\beta}\gamma), \tag{3}$$

where $\alpha = \langle v_1, v_2\rangle$, $\beta = \langle v_1, v_3\rangle$, *and* $\gamma = \langle v_2, v_3\rangle$.

Proof. By the discussion preceding Theorem 5, it is enough to determine that (3) is necessary and sufficient for v_1, v_2, v_3 to be unitarily equivalent to $\eta_1\tilde{k}_{\bar{w}_1}$, $\eta_2\tilde{k}_{\bar{w}_2}$, $\eta_3\tilde{k}_{\bar{w}_3}$, where $|\eta_j| = 1$, $|w_j| < 1$, and $\tilde{k}_w = (1 - |w|^2)^{1/2}k_w$ is the normalized reproducing kernel function.

Let $\varphi_a(z) = (a - z)/(1 - \bar{a}z)$ for $|a| < 1$. An easy (but tedious) calculation shows that

$$\langle \eta\tilde{k}_w, \eta'\tilde{k}_{w'}\rangle = \left\langle \eta\frac{|1 - a\bar{w}|}{1 - a\bar{w}}\tilde{k}_{\varphi_a(w)}, \eta'\frac{|1 - a\bar{w}'|}{1 - a\bar{w}'}\tilde{k}_{\varphi_a(w')}\right\rangle$$

whenever $|\eta| = |\eta'| = 1$ and $|w|, |w'| < 1$. In other words, changing the w's by φ_a changes the $\tilde{k}_w$'s unitarily if we allow coefficients to be multiplied by unimodular constants. An application of $\varphi_{\bar{w}_3}$ followed by multiplication by suitable unimodular constants shows that the triple $\eta_1\tilde{k}_{\bar{w}_1}$, $\eta_2\tilde{k}_{\bar{w}_2}$, $\eta_3\tilde{k}_{\bar{w}_3}$ is unitarily equivalent to a triple $\zeta_1\tilde{k}_{z_1}$, $\zeta_2\tilde{k}_{z_2}$, $\tilde{k}_0$, where $|\zeta_1| = |\zeta_2| = 1$ and $|z_1|, |z_2| < 1$.

Now if v_1, v_2, v_3 is unitarily equivalent to a triple of this form, we have

$$\alpha = \langle v_1, v_2\rangle = \zeta_1\bar{\zeta}_2(1 - |z_1|^2)^{1/2}(1 - |z_2|^2)^{1/2}/(1 - \bar{z}_1 z_2), \tag{4}$$

$$\beta = \langle v_1, v_3\rangle = \zeta_1(1 - |z_1|^2)^{1/2}, \tag{5}$$

$$\gamma = \langle v_2, v_3\rangle = \zeta_2(1 - |z_2|^2)^{1/2}. \tag{6}$$

It follows that

$$\bar{z}_1 z_2 = 1 - \beta\bar{\gamma}/\alpha, \tag{7}$$

so

$$(1 - |\beta|^2)(1 - |\gamma|^2) = |z_1 z_2|^2 = |1 - \beta\bar{\gamma}/\alpha|^2. \tag{8}$$

Expanding this last equation and rearranging gives (3).

On the other hand, suppose α, β, and γ satisfy (3). It will be enough to find ζ_1 and ζ_2 unimodular constants and z_1 and z_2 in $\mathbf{D}$ satisfying (4), (5), and (6). We claim that none of α, β, or γ can equal 0. For if, say, $\alpha = 0$, then either $\beta = 0$ or $\gamma = 0$ by (3). In either case, we can find a nontrivial reducing subspace for T. (If $\alpha = \beta = 0$, for example, then $\mathbf{C}v_1$ is such a subspace.) This contradicts our assumptions on T. Set $|z_1| = (1 - |\beta|^2)^{1/2}$

and $|z_2| = (1-|\gamma|^2)^{1/2}$; both of these are defined by Cauchy-Schwarz inequality and less than 1 by the above claim. Now ζ_1 and ζ_2 can be determined by (5) and (6), respectively. The assumption that (3) holds yields (8), so if we take z_1 to be positive we can then choose the argument of z_2 to make (7) hold. These choices give our unitary equivalence, and we are done. □

We have not been able to determine whether von Neumann's inequality holds for arbitrary n-tuples of commuting, diagonalizable contractions on a three-dimensional space. This question is important in light of the following result.

Theorem 9. *If von Neumann's inequality holds for all n-tuples of commuting, diagonalizable contractions on a three-dimensional space, then it holds for all n-tuples of commuting contractions on a three-dimensional space.*

Theorem 9 follows from the following lemma, which implies that a counterexample to von Neumann's inequality could be perturbed to a counterexample consisting of diagonalizables.

Lemma 10. *Any n-tuple of commuting operators on $\mathbf{C}^3$ can be perturbed to commuting diagonalizables.*

We note that an analogous statement holds in the two-dimensional case [3].

Proof. Let $T = (T_1, \dots, T_n)$ be an n-tuple of commuting operators on $\mathbf{C}^3$. If any T_j has two distinct eigenvalues, then the matrices of $T_1, \dots, T_n$ with respect to a basis that realizes the Jordan form of T_j are all block diagonal. We can then perturb each n-tuple of blocks to commuting diagonalizables using, if necessary, the two-dimensional result just mentioned. We thus obtain a perturbation of T consisting of commuting diagonalizables.

We may therefore assume that every T_j has a single eigenvalue. After subtracting a constant multiple of the identity from T_j (which does not affect the possibility of perturbation), we may assume that every T_j is nilpotent.

Now the only matrices that commute with

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

are polynomials of this matrix. Hence, if any T_j has this as its Jordan form, then $T_k = p_k(T_j)$ for some polynomial p_k. If we now perturb T_j to a diagonalizable T_j' and then replace T_k by $p_k(T_j')$, the resulting n-tuple will be the desired perturbation consisting of diagonalizables.

The other possibility is that $T_j^2 = 0$ for all j. Now the only nilpotent matrices of order two that commute with

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

are of one of the forms

$$\begin{pmatrix} 0 & a & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & a' & 0 \\ 0 & 0 & 0 \\ 0 & b' & 0 \end{pmatrix}.$$

Since these matrices do not commute with each other if $bb' \neq 0$, we must have that every T_j is of the same form, either the one on the left or the one on the right. In either case, every T_j is a polynomial in two fixed, commuting matrices. As any two commuting matrices can be perturbed to commuting diagonalizables [5], we can replace T_j by its polynomial in these perturbations as above to get the desired perturbation of commuting diagonalizables. The lemma is now proved. □

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