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VON NEUMANN'S INEQUALITY FOR COMMUTING, DIAGONALIZABLE CONTRACTIONS. II

B. A. LOTTO AND T. STEGER

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ABSTRACT. We construct a triple $T = (T_1, T_2, T_3)$ of commuting, diagonalizable contractions on $\mathbf{C}^5$ and a polynomial p in three variables for which $\|p(T)\| > \|p\|_\infty$, where $\|p\|_\infty$ denotes the supremum norm of p over the unit polydisk in $\mathbf{C}^3$.

1. INTRODUCTION

In part I [2], the first author showed that von Neumann's inequality

$$\|p(T)\| \le \|p\|_\infty \tag{1}$$

holds for all polynomials p in n variables, where T is an n-tuple of commuting, diagonalizable contractions on $\mathbf{C}^N$ that satisfies some additional hypotheses. Here $\|p(T)\|$ denotes the operator norm of $p(T)$ and $\|p\|_\infty$ denotes the supremum norm of p over the unit polydisk of $\mathbf{C}^n$. In the present work we present an example to show that the extra hypotheses cannot be removed. Our example is based on an example due to Kaijser and Varopoulos [3, addendum] that shows that (1) can fail with $n = 3$ and $N = 5$. This example consists of nilpotents; our example is obtained by perturbing their example to diagonalizables.

2. THE COUNTEREXAMPLE

Theorem. *There are three commuting, diagonalizable contractions T_1, T_2, and T_3 on $\mathbf{C}^5$ and a polynomial p in three variables such that $\|p(T_1, T_2, T_3)\| > \|p\|_\infty$.*

We start with the example of Kaijser and Varopoulos mentioned above. Use the standard inner product on $\mathbf{C}^5$. The three operators

$$A_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1/\sqrt{3} & -1/\sqrt{3} & -1/\sqrt{3} & 0 \end{pmatrix},$$

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$$A_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -1/\sqrt{3} & 1/\sqrt{3} & -1/\sqrt{3} & 0 \end{pmatrix},$$

and

$$A_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & -1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} & 0 \end{pmatrix}$$

on $\mathbf{C}^5$ are commuting contractions, and the polynomial

$$p(z_1, z_2, z_3) = z_1^2 + z_2^2 + z_3^2 - 2z_1z_2 - 2z_1z_3 - 2z_2z_3$$

satisfies $\|p\|_\infty = 5$ and $\|p(A_1, A_2, A_3)\| > 5$. We will produce, for every $\epsilon > 0$, perturbations $A_j^{(\epsilon)}$ of A_j ($j = 1, 2, 3$) that commute and are diagonalizable, with the additional property that $A_j^{(\epsilon)} \to A_j$ as $\epsilon \to 0$. Since then $\|A_j^{(\epsilon)}\| \to \|A_j\| = 1$ as $\epsilon \to 0$, we may replace $A_j^{(\epsilon)}$ by $A_j^{(\epsilon)}/\|A_j^{(\epsilon)}\|$ and assume that $A_j^{(\epsilon)}$ is a contraction. We will then have

$$\|p(A_1^{(\epsilon)}, A_2^{(\epsilon)}, A_3^{(\epsilon)})\| \to \|p(A_1, A_2, A_3)\| > 5$$

as $\epsilon \to 0$, so that for small enough ϵ we have $\|p(A_1^{(\epsilon)}, A_2^{(\epsilon)}, A_3^{(\epsilon)})\| > 5$. Setting $T_j = A_j^{(\epsilon)}$ for such an ϵ gives the theorem. We therefore need only construct the perturbations.

Let

$$X = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & -1 & 0 \\ 0 & -1 & 1 & -1 & 0 \\ 0 & -1 & -1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix},$$

and let Y be any matrix such that $Y^{-1} = Y^{t}X$, that is, such that $YY^{t} = X^{-1}$. We can produce one such Y as follows. Since X^{-1} is real symmetric, there is a real orthogonal matrix U and a diagonal matrix D such that $X^{-1} = UDU^{t}$. Let $\sqrt{D}$ be a diagonal matrix whose square is D and set $Y = U\sqrt{D}U^{t}$.

By replacing Y by YO, where O is a suitably chosen orthogonal matrix, we may assume that the fifth row of Y contains only nonzero entries. We denote the fifth row by y and think of it as an element of $\mathbf{C}^5$.

Consider the linear map L from $\mathbf{C}^5$ into operators on $\mathbf{C}^5$ defined by $La = YAY^{t}$, where A is the diagonal matrix whose diagonal is a. The fifth row of La is $(y * a)Y^{t}$, where $*$ denotes coordinatewise multiplication. Since Y is invertible and y has only nonzero entries, it follows that the linear map that sends a to the fifth row of La is invertible. Hence, we can find u_1, u_2, and u_3 in $\mathbf{C}^5$ such that the fifth rows of Lu_1, Lu_2, and Lu_3 are $(0, 1, 0, 0, 0)$,

$(0, 0, 1, 0, 0)$, and $(0, 0, 0, 1, 0)$, respectively. Since La is always symmetric, we have

$$Lu_1 = \begin{pmatrix} * & * & * & * & 0 \\ * & * & * & * & 1 \\ * & * & * & * & 0 \\ * & * & * & * & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix},$$

$$Lu_2 = \begin{pmatrix} * & * & * & * & 0 \\ * & * & * & * & 0 \\ * & * & * & * & 1 \\ * & * & * & * & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix},$$

and

$$Lu_3 = \begin{pmatrix} * & * & * & * & 0 \\ * & * & * & * & 0 \\ * & * & * & * & 0 \\ * & * & * & * & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

For $\epsilon > 0$ and $j = 1, 2, 3$, let $U_j^{(\epsilon)}$ denote the diagonal matrix whose diagonal is ϵu_j. We have

$$YU_1^{(\epsilon)}Y^{-1} = YU_1^{(\epsilon)}Y^{t}X = \epsilon(Lu_1)X = \begin{pmatrix} 0 & * & * & * & * \\ \epsilon & * & * & * & * \\ 0 & * & * & * & * \\ 0 & * & * & * & * \\ 0 & \epsilon & -\epsilon & -\epsilon & 0 \end{pmatrix},$$

where the missing entries are constant multiples of ϵ. Let D_ϵ be the diagonal matrix with diagonal entries $(1, \epsilon^{-1}, \epsilon^{-1}, \epsilon^{-1}, \epsilon^{-2}/\sqrt{3})$. Then

$$(2) \qquad (D_\epsilon Y)U_1^{(\epsilon)}(Y^{-1}D_\epsilon^{-1}) = \begin{pmatrix} 0 & * & * & * & * \\ 1 & * & * & * & * \\ 0 & * & * & * & * \\ 0 & * & * & * & * \\ 0 & 1/\sqrt{3} & -1/\sqrt{3} & -1/\sqrt{3} & 0 \end{pmatrix},$$

where the missing entries are constant multiples of ϵ, ϵ^2, or ϵ^3. This is the perturbation $A_1^{(\epsilon)}$. Letting $Z_\epsilon = D_\epsilon Y$, we similarly set

$$(3) \qquad A_2^{(\epsilon)} = Z_\epsilon U_2^{(\epsilon)} Z_\epsilon^{-1} = \begin{pmatrix} 0 & * & * & * & * \\ 0 & * & * & * & * \\ 1 & * & * & * & * \\ 0 & * & * & * & * \\ 0 & -1/\sqrt{3} & 1/\sqrt{3} & -1/\sqrt{3} & 0 \end{pmatrix}$$

and

$$A_3^{(\epsilon)} = Z_\epsilon U_3^{(\epsilon)} Z_\epsilon^{-1} = \begin{pmatrix} 0 & * & * & * & * \\ 0 & * & * & * & * \\ 0 & * & * & * & * \\ 1 & * & * & * & * \\ 0 & -1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} & 0 \end{pmatrix}. \tag{4}$$

Clearly, $A_1^{(\epsilon)}$, $A_2^{(\epsilon)}$, and $A_3^{(\epsilon)}$ commute and are diagonalizable. The missing entries of (3) and (4), like those of (2), are constant multiples of ϵ, ϵ^2, and ϵ^3, so $A_j^{(\epsilon)} \to A_j$ as $\epsilon \to 0$ for $j = 1, 2, 3$.

We have found our perturbations and the theorem is proved.

3. Questions

The above proof involves perturbing a triple of commuting matrices to commuting, diagonalizable ones.

Question 1. Can any triple of commuting operators on a finite-dimensional space be perturbed to become commuting and diagonalizable?

It is known that any pair of commuting matrices can be perturbed to commuting diagonalizables [1]. The natural generalization to four matrices fails, as the following example shows.

Let $\{e_1, \dots, e_n\}$ be the standard basis for $\mathbf{C}^n$. We define n by n matrices that act as follows:

$$T_1 e_j = e_{j+1} \quad \text{for } 1 \le j \le n-3,$$
$$T_2 e_{n-1} = e_n, \quad T_3 e_1 = e_n, \quad T_4 e_{n-1} = e_{n-2},$$

where $T_k e_j = 0$ in all other cases. The product of two distinct T_k is zero, so these matrices commute. The algebra with identity generated by the T_k's has $(T_1^j)_{j=0}^{n-3}$, T_2, T_3, T_4 as a basis and so has dimension $n+1$. If we could perturb the T_k's to commuting diagonalizables, the algebra generated by the perturbations would be an $(n+1)$-dimensional commutative algebra of diagonalizable n by n matrices. But no such algebra exists.

This argument by dimension suggests the following questions.

Question 2. Consider a subalgebra of the n by n matrices, commutative and with identity. If this subalgebra has dimension no greater than n, can it be perturbed to a commutative subalgebra of diagonal matrices?

and

Question 3. Can a triple of commuting n by n matrices generate an algebra with identity of dimension greater than n?

Question 2 also suggests stabilizing an arbitrary collection of commuting n by n matrices.

Question 4. Given a finite collection of commuting matrices, can their direct sums with a large enough zero matrix be perturbed to commuting diagonalizables?

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Department of Mathematics, University of California, Berkeley, California 94720
E-mail address: lotto@math.berkeley.edu
Current address: Department of Mathematics, Vassar College, Poughkeepsie, New York, 12601
E-mail address: BeLotto@vassar.edu

Department of Mathematics, University of Chicago, Chicago, Illinois 60637
E-mail address: steger@zaphod.uchicago.edu
Current address: Department of Mathematics, University of Georgia, Boyd Graduate Studies, Athens, Georgia 30602-7403
E-mail address: steger@joe.math.uga.edu