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$(\mathbf{M}, \varepsilon, \delta)$-MINIMAL CURVE REGULARITY

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ABSTRACT. $(\mathbf{M}, cr^\alpha, \delta)$-minimal sets are embedded $C^{1,\alpha/2}$ curves meeting in threes at $120°$ angles.

1. INTRODUCTION

$(\mathbf{M}, \varepsilon, \delta)$-minimal sets model physical surfaces minimizing area subject to constraints. For example, soap bubble clusters tend to minimize the area required to enclose and separate given volumes of air (see Figure 1.1 on the next page). $\mathbf{M}$ stands for "mass" or area. The effect of the constraints goes to 0 at a rate specified by the function ε, typically $\varepsilon(r) = cr^\alpha$, $0 < \alpha \le 1$, at a scale $r \le \delta$. By definition, a k-dimensional $(\mathbf{M}, \varepsilon, \delta)$-set S in $\mathbf{R}^n$, inside a ball of diameter $r \le \delta$, roughly satisfies

$$\operatorname{area}(S) \le (1 + \varepsilon(r)) \operatorname{area}(S')$$

for any Lipschitz deformation S' of S, which need not respect the constraint. (See 2.1 for a complete definition.)

Almgren [A] (cf. [M1, 11.3]) introduced m-dimensional $(\mathbf{M}, \varepsilon, \delta)$-minimal sets and for $m \ge 2$ proved them regular almost everywhere [A, IV.13(6) and IV.3(1)]. Taylor [T] showed that two-dimensional $(\mathbf{M}, cr^\alpha, \delta)$-minimal sets have precisely the singularities which long had been observed in soap bubble clusters. Taylor's basic methods do not work for $m = 1$; in particular, the proof of [T, Proposition IV.2, Part (2), p. 525] yields a summable geometric series only if $m \ge 2$.

This paper fills in the missing regularity for one-dimensional $(\mathbf{M}, cr^\alpha, \delta)$-minimal sets. Theorem 3.8 proves that they consist of $C^{1,\alpha/2}$ curves meeting in threes at equal $120°$ angles at isolated points. The proof illustrates the standard regularity theory, simplified by arguments special to the one-dimensional case.

Examples of $(\mathbf{M}, cr, \delta)$-minimal curves include planar soap bubbles which provide the least-perimeter way to enclose and separate prescribed areas. The difficulty in proving their $(\mathbf{M}, cr, \delta)$-minimality is that small Lipschitz deformations distort areas, which must then be carefully adjusted elsewhere. Actually

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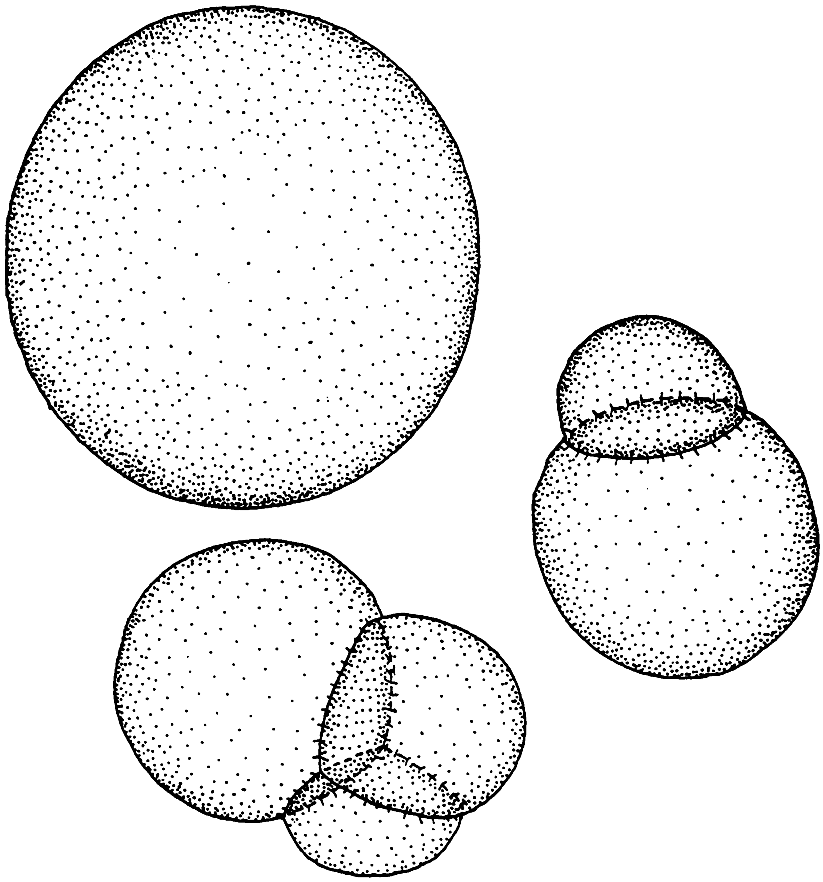

FIGURE 1.1. Soap bubble clusters tend to minimize the area required to enclose and separate given volumes of air. (Drawings by Jim Bredt from [M4].)

once this adjustment is accomplished, regularity follows quickly without needing $(\mathbf{M}, cr, \delta)$-minimality. See [M2].

2. $(\mathbf{M}, \varepsilon, \delta)$-MINIMAL SETS

2.1. **Definition** ([A, III.1, II.1], cf. [M1, 11.3]). Let B be a closed subset of $\mathbf{R}^n$. Fix $\delta, c, \alpha > 0$. A nonempty, bounded subset $S \subset \mathbf{R}^n - B$ of finite m-dimensional Hausdorff measure is $(\mathbf{M}, cr^\alpha, \delta)$-*minimal* if $S = \operatorname{spt}(\mathscr{H}^m \lfloor S) - B$ and

$$\mathscr{H}^m(S \cap W) \leq (1 + cr^\alpha)\mathscr{H}^m(\varphi(S \cap W))$$

whenever

(a) $\varphi\colon \mathbf{R}^n \to \mathbf{R}^n$ is lipschitzian,
(b) $W = \mathbf{R}^n \cap \{z : \varphi(z) \neq z\}$,
(c) $\operatorname{diam}(W \cup \varphi(W)) = r < \delta$,
(d) $\operatorname{dist}(W \cup \varphi(W), B) > 0$.

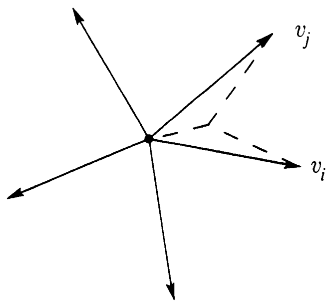

FIGURE 2.2. If the angle between vectors v_i and v_j is less than $120°$, the network can be shortened by pinching them together.

The following proposition gives some of the standard properties of $(\mathbf{M}, cr^\alpha, \delta)$-minimal sets.

2.2. **Proposition.** *Let B be a closed subset of $\mathbf{R}^n$. Fix $\delta, c, \alpha > 0$. Let $S \subset \mathbf{R}^n$ be $(\mathbf{M}, cr^\alpha, \delta)$-minimal, $p \in S$.*

(1) *Rectifiability* [A, II.3(9)]. *S is $(\mathscr{H}^m, m)$-rectifiable.*

(2) *Monotonicity* [T, II.1]. *For $0 < r < \min\{\delta/2, \operatorname{dist}(p, B)\}$,*

$$r^{-m}\mathscr{H}^m(S \cap \mathbf{B}^n(p, r))e^{mcr^\alpha/\alpha}$$

is monotonically nondecreasing.

(3) *Tangent cones* [T, II.2]. *S has a weak tangent cone C at p, with density $\Theta(C, 0) = \Theta(S, p)$, which is $(\mathbf{M}, 0, \infty)$-minimal. For $m = 1$, C is either a straight line or three rays meeting at equal $120°$ angles.*

(4) *Lower mass bound. For $0 < r < \min\{\delta/2, \operatorname{dist}(p, B)\}$, if $m = 1$,*

$$\mathscr{H}^1(S \cap \mathbf{B}^n(p, r)) \geq 2re^{-cr^\alpha/\alpha}.$$

Proof. (1) is proved in [A, II.3(9)]. (2) and the first sentence of (3) are proved in [T, II.1 and II.2]. For $m = 1$, the characterization of C has the following easy standard proof. C consists of k unit vectors $v_1, \dots, v_k$. By minimality, $k \geq 2$, and if $k = 2$, C is a straight line.

Suppose $k \geq 3$. As a small multiple tu of any unit vector u is added and the tails of any two vectors v_i, v_j are moved to the head of tu as in Figure 2.2, the rate of change of length must be nonnegative, that is,

$$1 - v_i \cdot u - v_j \cdot u \geq 0,$$

so that $|v_i + v_j| \leq 1$, i.e., $1 \geq |v_i + v_j|^2 = 2 + 2v_i \cdot v_j$, and $v_i \cdot v_j \leq -\frac{1}{2}$, i.e., the angle between v_i and v_j is at least $120°$. Hence C is three rays meeting at $120°$.

To prove (4), note that by (3) and (2)

$$\begin{aligned} 1 \leq \Theta(S, p) &= \lim_{s \to 0}(2s)^{-1}\mathscr{H}^1(S \cap \mathbf{B}^n(p, s)) \\ &\leq (2r)^{-1}\mathscr{H}^1(S \cap \mathbf{B}^n(p, r))e^{cr^\alpha/\alpha}. \end{aligned}$$

The following lemma is very handy for showing the $(\mathbf{M}, cr^\alpha, \delta)$-minimality of sets that arise as solutions to variational problems.

2.3. **Lemma** (cf. [M3, Lemma 2.5]). *The condition in the definition of an $(\mathbf{M}, cr^\alpha, \delta)$-minimal set S that $S = \operatorname{spt}(\mathscr{H}^m \lfloor S) - B$ may be replaced by the condition that S be rectifiable (technically $(\mathscr{H}^m, m)$-rectifiable and $\mathscr{H}^m$-measurable as in* [M1, 3.10]), *with the understanding that S may be altered on a set of $\mathscr{H}^m$ measure 0.*

Proof. The proof is exactly as for $(\mathbf{M}, 0, \delta)$-minimal sets in [M3, Lemma 2.5], using monotonicity 2.2(2). The proof of monotonicity in [T, II.1] uses rectifiability but does not require that $S = \operatorname{spt}(\mathscr{H}^m \lfloor S) - B$ or that S be closed.

3. Regularity of $(\mathbf{M}, cr^\alpha, \delta)$-minimal curves

The hard part of establishing the main regularity Theorem 3.8 is proving that the $(\mathbf{M}, cr^\alpha, \delta)$-minimal set is a smooth curve at all points of density one (Corollary 3.7). The analysis begins by characterizing in very general contexts minimal networks connecting three points (3.1 and 3.2).

The following proposition provides the familiar shortest networks connecting three given points in $\mathbf{R}^2$ in the general context of *sets* rather than *curves*, using the method of [LM, 1.1].

3.1. **Proposition.** *Consider a triangle in $\mathbf{R}^2$ with largest angle θ. There is a unique smallest subset S of the triangular region that separates its sides, in the sense that any curve inside the triangle from one side to another intersects S, namely:*

(a) *if $\theta \geq 120°$, the " V " consisting of the two shortest sides of the triangle;*
(b) *if $\theta < 120°$, the " Y " consisting of segments from the three vertices meeting at 120°.*

(*By "unique smallest" we mean any other S' satisfies either $\mathscr{H}^1(S') > \mathscr{H}^1(S)$, or $\mathscr{H}^1(S') = \mathscr{H}^1(S)$ and $S' \supset S$.*)

Proof. Assume $\mathscr{H}^1(S) \leq \mathscr{H}^1(V)$ if $\theta \geq 120°$ and $\mathscr{H}^1(S) \leq \mathscr{H}^1(Y)$ if $\theta < 120°$. Label the vertices of the triangle p_i so that $p_1p_2 \geq p_2p_3 \geq p_3p_1$. We may assume that the vertices lie on the unit circle $\mathbf{S}^1(0, 1)$. Let C_1, C_2, C_3 denote the oriented open arcs of the circle from p_3 to p_1, from p_1 to p_2, and from p_2 to p_3.

Let R_1 be the component of $\mathbf{B}^2(0, 1) - S$ containing C_1; let R_2 be the component of $\mathbf{B}^2(0, 1) - S$ containing C_2; let R_3 be $\mathbf{B}^2(0, 1) - S - R_1 - R_2$.

Since the topological boundary

$$\operatorname{Bdry} R_i \subset \mathbf{S}^1(0, 1) \cup S$$

has finite $\mathscr{H}^1$ measure, it follows from [F, 4.5.12 and 2.10.6] that the current boundary $T_i = \partial(\mathbf{E}^2 \lfloor R_i)$ is representable by integration. Hence by the Gauss-Green-Federer Theorem ([M, 12.2] or [F,4.5.6]), at almost every point in the essential boundary there is an "exterior normal". It follows that there are oriented rectifiable curves C_1, C_2, C_3 (of multiplicity one almost everywhere) such that the oriented essential boundary of R_1 is $\gamma_1 + C_1 - \gamma_2$, of R_2 is $\gamma_2 + C_2 - \gamma_3$, and of R_3 is $\gamma_3 + C_3 - \gamma_1$, as in Figure 3.1.

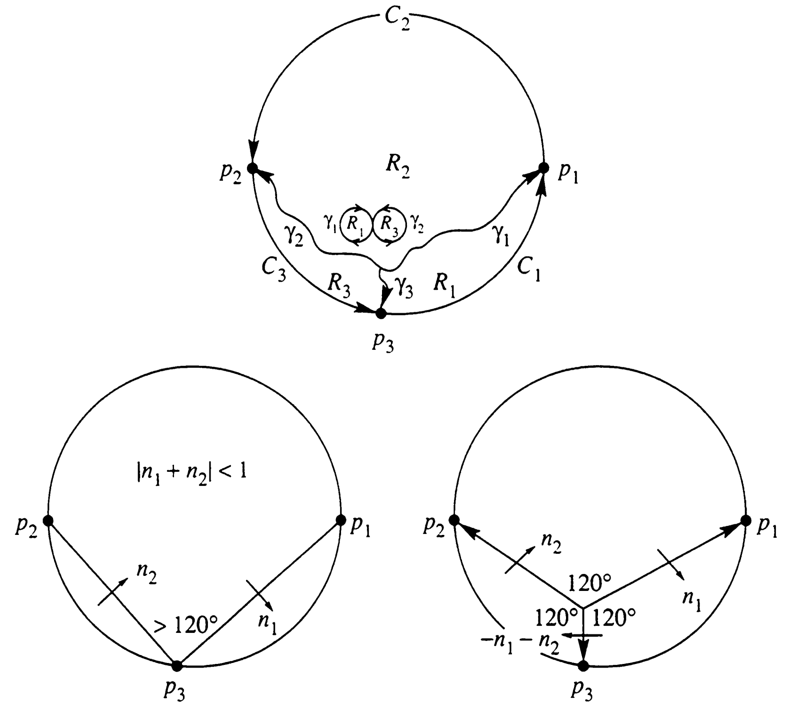

FIGURE 3.1. Rectifiable curves γ_i separate the regions R_i. n_1, n_2 are the unit normals to the segments of V or Y to p_1 and p_2, the end points of the longest side of triangle $p_1p_2p_3$.

Let n_1 be the oriented unit normal to the segment of V or Y to p_1; let n_2 be the oriented unit normal to the segment of V or Y to p_2. Note that in case (a), when $\theta \geq 120°$, $|n_1 + n_2| < 1$; in case (b), when $\theta < 120°$, $-n_1 - n_2$ is the oriented unit normal to the segment of Y to p_3. Now

$$\begin{aligned}
&\mathscr{H}^1(\gamma_1) + \mathscr{H}^1(\gamma_2) + \mathscr{H}^1(\gamma_3) \\
&\qquad \geq \text{Flux of } n_1 \text{ thru } \gamma_1 + \text{ Flux of } n_2 \text{ thru } \gamma_2 + \text{ Flux of } (-n_1 - n_2) \text{ thru } \gamma_3 \\
&\qquad = \text{Flux of } n_1 + n_2 \text{ thru } C_1 + \text{ Flux of } n_2 \text{ thru } C_2.
\end{aligned}$$

The inequality holds because the three vector fields have length at most one. The equality holds by the divergence theorem applied to the R_i. The fixed sum is a constant independent of S.

In case (a), when $\theta \geq 120°$ and $|n_1 + n_2| < 1$, equality holds if and only if γ_3 is absent and n_1, n_2 are normal to γ_1, γ_2; i.e., $\bigcup \gamma_i = V$. In case (b), when $\theta = 120°$ and $-n_1 - n_2$ is the oriented unit normal to the third segment of Y, equality holds if and only if n_1, n_2, $-n_1 - n_2$ are normal to γ_1, γ_2, γ_3; i.e., $\bigcup \gamma_i = Y$.

Since $S \supset \bigcup \operatorname{Bdry} R_i \supset \bigcup \gamma_i$, it follows that V or Y is the unique smallest subset separating the sides of the triangle.

The following proposition certifies the familiar shortest network connecting three given points in $\mathbf{R}^n$ as uniquely minimizing in the context of $(\mathbf{M}, \varepsilon r^\alpha, \delta)$-minimal sets.

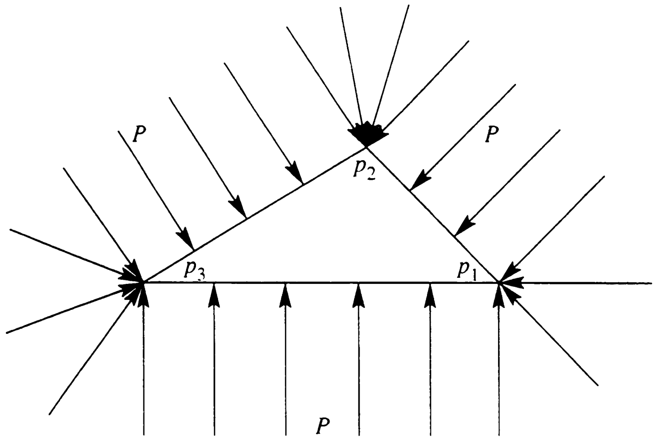

FIGURE 3.2. P denotes projection onto $\mathbf{R}^2$ followed by projection onto the triangular region.

3.2. **Proposition.** *Fix* $c, \alpha, \delta > 0$. *Let* $B = \{p_1, p_2, p_3\}$ *be the vertices of a triangle in the open* δ*-ball* $\mathbf{U}^2(0, \delta) \subset \mathbf{R}^2 \subset \mathbf{R}^n$, *with largest angle* θ. *There is a unique smallest* $(\mathbf{M}, cr^\alpha, \delta)$*-minimal set in* $\mathbf{U}^n(0, \delta)$ *with closure containing* B, *namely*:

(a) *if* $\theta \geq 120°$, *the two shortest sides of the triangle, called* V;
(b) *if* $\theta < 120°$, *segments from three vertices meeting at* $120°$, *called* Y.

(*Here by "unique smallest" we mean any other such* $(\mathbf{M}, cr^\alpha, \delta)$*-minimal set* S *has larger one-dimensional Hausdorff measure.*)

Proof. Let S be $(\mathbf{M}, cr^\alpha, \delta)$-minimal with respect to B. Let P denote orthogonal projection of $\mathbf{R}^n$ onto $\mathbf{R}^2$, followed by projection onto the triangular region by orthogonal projection of much of the exterior onto the sides and mapping the rest of the exterior onto the vertices, as in Figure 3.2.

First of all note that $\mathscr{H}^1(P(S)) \leq \mathscr{H}^1(S)$. Second we claim that $P(S)$ separates the sides of the triangle, in the sense that any curve inside the triangle from one side to another must intersect $P(S)$. Otherwise such a curve would isolate one vertex p_i and a portion of $P(S)$. Its inverse image under the projections would isolate a corresponding portion of S, which could be deformed to p_i, reducing its $\mathscr{H}^1$-measure from something positive to 0 and contradicting $(\mathbf{M}, cr^\alpha, \delta)$-minimality.

By Proposition 3.1, $\mathscr{H}^1(V) \leq \mathscr{H}^1(P(S)) \leq \mathscr{H}^1(S)$ if $\theta \geq 120°$, and $\mathscr{H}^1(Y) \leq \mathscr{H}^1(P(S)) \leq \mathscr{H}^1(S)$ if $\theta < 120°$.

Finally we prove uniqueness. First consider the case $\theta \geq 120°$. Suppose $\mathscr{H}^1(V) = \mathscr{H}^1(P(S)) = \mathscr{H}^1(S)$, and hence $V \subset P(S)$ by Proposition 3.1.

We claim that P is injective on S. Suppose $P(s_1) = P(s_2) = p$. Then, for small ε, $S \cap \mathbf{B}(s_1, \varepsilon)$ and $S \cap \mathbf{B}(s_2, \varepsilon)$ are disjoint and both map into $P(S) \cap \mathbf{B}(p, \varepsilon)$, which has length 2ε. Since by monotonicity (2.2(4))

$$\mathscr{H}^1(S \cap \mathbf{B}(s_i, \varepsilon)) \geq 2\varepsilon e^{-c\varepsilon^\alpha/\alpha} \geq 1.5\varepsilon$$

for small ε, it follows that $\mathscr{H}^1(S) > \mathscr{H}^1(P(S))$, the desired contradiction.

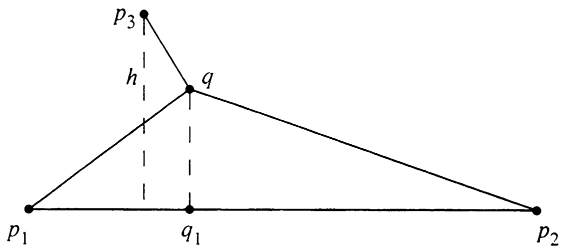

FIGURE 3.3. The distance h from p_3 to the line p_1p_2 satisfies $h \le q_1q + qp_3 \le 2\sqrt{c}a^{1+\alpha/2}$.

We may assume that V consists of the two line segments $\overline{p_3p_1}$ and $\overline{p_2p_3}$. Let $S_1 = \{x \in S_i : P(x) \in \overline{p_3p_1}\}$. Then $\mathscr{H}^1(S_1) = p_3p_1$. For any $q \in S_1$, $S_1 - q$ is $(\mathbf{M}, cr^\alpha, \delta)$-minimal with respect to $B_1 = \{p_1, p_3, q\}$. By the portion of this Proposition 3.2 already proved, if $q \notin \overline{p_3p_1}$, $\mathscr{H}^1(S_1) > p_3p_1$—a contradiction. Therefore, $S_1 \subset \overline{p_3p_1}$. Similarly, $S_2 = \{x \in S_i : P(x) \in \overline{p_1p_2}\} \subset \overline{p_1p_2}$. We conclude that $S = V$.

Second consider the case $\theta < 120°$. Y consists of line segments $\overline{0p_1}$, $\overline{0p_2}$, $\overline{0p_3}$. Let $S_1 = \{x \in S : P(x) \in \overline{0p_1}\}$. Then $\mathscr{H}^1(S_1) = 0p_1$. For any $q \in S_1$, $S_1 - q$ is $(\mathbf{M}, cr^\alpha, \delta)$-minimal with respect to $B_1 = \{p_1, P^{-1}(0), q\}$. By the portion of this Proposition 3.2 already proved, if $P^{-1}(0) \neq 0$ or if $q \notin \overline{0p_1}$, $\mathscr{H}^1(S_1) > 0p_1$—a contradiction. We conclude $S = Y$.

The next three lemmas estimate the curving of $(\mathbf{M}, cr^\alpha, \delta)$-minimal sets and lead to smooth regularity at points of density 1 (Corollary 3.7).

3.3. **Lemma.** *Fix $\delta, c, \alpha > 0$. Suppose $a < \delta$ and $a^{\alpha/2} < 1/12\sqrt{c}$. Let $B = \{p_1, p_2\}$ be two distinct points. Let $S \subset \mathbf{B}^n(0, a)$ be an $(\mathbf{M}, cr^\alpha, \delta)$-minimal set. Suppose p_1, p_2, p_3 lie in the closure of S. Then the distance h from p_3 to the line p_1p_2 satisfies $h \le 3\sqrt{c}a^{1+\alpha/2}$.*

Proof. Since S is $(\mathbf{M}, cr^\alpha, \delta)$-minimal,

$$\mathscr{H}^1(S) \le (1 + ca^\alpha)p_1p_2. \tag{1}$$

On the other hand, by 3.2, there is a point $q \in \mathbf{B}(0, a)$ (the vertex of the V or Y) such that

$$\mathscr{H}^1(S) \ge \sum p_iq. \tag{2}$$

Let q_1 be the orthogonal projection of q onto the line p_1p_2, so that $h \le q_1q + qp_3$. See Figure 3.3.

By (1) and (2),

$$\sum p_iq \le (1 + ca^\alpha)p_1p_2 \le p_1p_2 + ca^\alpha p_1p_2 \le p_1q_1 + q_1p_2 + 2ca^{1+\alpha}.$$

Since $p_2q \ge q_1p_2$,

$$p_1q + p_3q \le p_1q_1 + 2ca^{1+\alpha}, \qquad p_3q + (p_1q - p_1q_1) \le 2ca^{1+\alpha}.$$

Hence $p_3q \le 2ca^{1+\alpha}$, $(p_1q - p_1q_1) \le 2ca^{1+\alpha}$, and

$$(q_1q)^2 = (p_1q - p_1q_1)(p_1q + p_1q_1) \le (2ca^{1+\alpha})(4a) \le 8ca^{2+\alpha}.$$

Therefore,

$$h \leq q_1 q + p_3 q \leq \sqrt{8c}a^{1+\alpha/2} + 2ca^{1+\alpha} \leq 3\sqrt{c}a^{1+\alpha/2}$$

by the initial hypothesis on $a^{\alpha/2}$.

3.4. **Lemma.** *Fix $\delta, c, \alpha > 0$. Suppose $a < \delta$ and $a^{\alpha/2} < 1/48\sqrt{c}$. Let $B = \{p_1, p_2\} \subset S^{n-1}(0, a)$ be two distinct points. Let $S \subset \mathbf{B}^n(0, a)$ be an $(\mathbf{M}, cr^\alpha, \delta)$-minimal set containing the origin 0. If $p \in S$ and $|p| \geq a/4$, then the angle θ between the lines $0p$ and $0p_1$ satisfies $\theta \leq 26\sqrt{c}a^{\alpha/2}$. If $\mathscr{H}^1(S) < \frac{5}{2}ae^{-ca^\alpha/\alpha}$, then, for any $p \in S$,*

$$\theta \leq 26\sqrt{c}a^{\alpha/2}/(1 - (\tfrac{1}{2})^{\alpha/2}).$$

Proof. By Lemma 3.3, 0 and p are both within $3\sqrt{c}a^{1+\alpha/2}$ of the line p_1p_2. Hence,

$$\sin\theta \leq 6\sqrt{c}a^{1+\alpha/2}/|p| \leq 24\sqrt{c}a^{\alpha/2} \leq \tfrac{1}{2}$$

and

$$\theta \leq \tfrac{\pi}{3}\sin\theta \leq 26\sqrt{c}a^{\alpha/2}.$$

Finally suppose $\mathscr{H}^1(S)e^{ca^\alpha/\alpha} < \frac{5}{2}a$. By monotonicity 2.1(2), for all $0 < a' \leq a$, $\mathscr{H}^1(S \llcorner \mathbf{B}(0, a')) < \frac{5}{2}a'$. By the minimality of S, for all $0 < a' < a$, the sphere $\mathbf{S}^{n-1}(0, a')$ intersects S in at least two points. First taking $a' = a/2$ yields an a_2, $a'/2 \leq a_2 \leq a'$, such that $\mathbf{S}^{n-1}(0, a')$ intersects S in exactly two points. Continuing yields a sequence $a_1 = a, a_2, \ldots, a_m$ such that $\mathbf{S}^{n-1}(0, a_i)$ intersects S in exactly two points,

$$a_i/4 \leq a_{i+1} \leq a_i/2 \quad \text{and} \quad a_m/4 \leq |p| \leq a_m.$$

Applying the first part of this Lemma 3.4 m times yields that the angle θ between p and p_1 or p_2 satisfies

$$\begin{aligned}\theta &\leq \sum_{i=1}^{m} 26\sqrt{c}a_i^{\alpha/2} \leq 26\sqrt{c}a^{\alpha/2}\sum_{i=1}^{\infty}(1/2)^{i\alpha/2}\\ &\leq 26\sqrt{c}a^{\alpha/2}/(1 - (1/2)^{\alpha/2}).\end{aligned}$$

3.5. **Lemma.** *Fix $\delta, c, \alpha > 0$. Let B be a closed subset of $\mathbf{R}^n$. Let S be $(\mathbf{M}, cr^\alpha, \delta)$-minimal. Suppose $a < \delta$ and $a^{\alpha/2} < 1/48\sqrt{c}$. Let q_1, q_2, q_3 be points of S with $q_2, q_3 \in \mathbf{B}(q_1, \frac{3}{4}a)$. Suppose $\mathbf{B}(q_1, a)$ is disjoint from B and satisfies*

$$\mathscr{H}^1(S \cap \mathbf{B}(q_1, a)) < \tfrac{5}{2}ae^{-ca^\alpha/\alpha}.$$

Then the angle θ between the lines q_1q_2 and q_1q_3 satisfies $\theta \leq c_1a^{\alpha/2}$, where $c_1 = 52\sqrt{c}/(1 - (\frac{1}{2})^{\alpha/2})$.

Proof. By the minimality of S_1 for all $r \leq a$, the sphere $\mathbf{S}^{n-1}(q_1, r)$ intersects S in at least two points. Hence, for some $\frac{3}{4}a \leq a' \leq a$, $\mathbf{S}^{n-1}(q_1, a')$ intersects S in exactly two points, namely, $B' = \{p_1, p_2\}$. $S' = S \cap \mathbf{B}(q_1, a') - B'$ is $(\mathbf{M}, cr^\alpha, \delta)$-minimal with respect to B'. By monotonicity 2.1(2), $\mathscr{H}^1(S') < \frac{5}{2}a'$. By Lemma 3.4, the angle between the lines q_1p_1 and q_1q_2 and the angle between the lines q_1p_1 and q_1q_3 are each at most $26\sqrt{c}(a')^{\alpha/2}/(1 - (\frac{1}{2})^{\alpha/2})$. Therefore,

$$\theta \leq 52\sqrt{c}a^{\alpha/2}/(1 - (\tfrac{1}{2})^{\alpha/2}) \leq c_1a^{\alpha/2}.$$

3.6. **Proposition.** *Fix $\delta, c, \alpha > 0$. Let B be a closed subset of $\mathbf{R}^n$. Let S be $(\mathbf{M}, cr^\alpha, \delta)$-minimal. Suppose $e^{cA^\alpha/\alpha} < 1.05$, $\mathbf{B}(p_1, A) \cap B = \varnothing$, and*

$$\mathscr{H}^1(\mathbf{B}(p_1, A) \cap S) < 2.1A.$$

Suppose $0 < a < \min\{\delta, A/22\}$, $a^{\alpha/2} < 1/48\sqrt{c}$, and $p_2, p_3, p_4 \in \mathbf{B}(p_1, \frac{3}{4}a)$. Then the angle θ between the lines p_1p_2 and p_3p_4 satisfies $\theta \le c_2 a^{\alpha/2}$, where $c_2 = 104\sqrt{c}/(1 - (\frac{1}{2})^{\alpha/2})$.

Proof. By monotonicity 2.1(2),

$$\mathscr{H}^1(S \cap \mathbf{B}(p_1, a)) < 2.1ae^{cA^\alpha/\alpha} < 2.5ae^{-ca^\alpha/\alpha}.$$

By Lemma 3.5, the angle between the lines p_1p_2 and p_1p_3 is at most $c_1a^{\alpha/2}$. To estimate $\mathscr{H}^1(S \cap \mathbf{B}(p_3, a))$, note that

$$\mathscr{H}^1(S \cap \mathbf{B}(p_3, A-a)) \le \mathscr{H}^1(S \cap \mathbf{B}(p_1, A)) < 2.1A \le 2.2(A-a)$$

because $a < A/22$. By monotonicity,

$$\mathscr{H}^1(S \cap \mathbf{B}(p_3, a)) < 2.2ae^{cA^\alpha/\alpha} < 2.5ae^{-cA^\alpha/\alpha}.$$

By Lemma 3.5, the angle between the lines p_1p_3 and p_3p_4 is at most $c_1a^{\alpha/2}$. Therefore, $\theta \le 2c_1a^{\alpha/2} = c_2a^{\alpha/2}$.

3.7. **Corollary.** *A one-dimensional $(\mathbf{M}, cr^\alpha, \delta)$-minimal set in $\mathbf{R}^n$ is a $C^{1,\alpha/2}$ curve at all points of density one (with a uniform Hölder constant).*

The following theorem is the main result of this paper.

3.8. **Regularity Theorem.** *Fix $\delta, c, \alpha > 0$. Let B be a closed subset of $\mathbf{R}^n$. Let S be a one-dimensional $(\mathbf{M}, cr^\alpha, \delta)$-minimal set with respect to B. Then S consists of $C^{1,\alpha/2}$ curves, with Hölder constant at most $104\sqrt{c}/(1 - (\frac{1}{2})^{\alpha/2})$, meeting in threes at equal $120°$ angles at isolated points.*

Proof. Let $x_0 \in S$. If a weak tangent cone C to S at x_0 is a line and $\Theta(S, x_0) = 1$, then by Corollary 3.7, S is a $C^{1,\alpha/2}$ curve at x_0 with the Hölder constant asserted. Otherwise by 2.2(3), a weak tangent cone C to S at x_0 consists of three rays meeting at $120°$ and $\Theta(S, x_0) = \frac{3}{2}$. For convenience we assume $x_0 = 0$.

We claim that $\Theta(S, x) = 1$ in a deleted neighborhood of 0. Otherwise there is a sequence $x_i \to 0$ in $S - \{0\}$ with $\Theta(S, x_i) = \frac{3}{2}$. By taking a subsequence if necessary, we may assume that the homothetic expansions $S_i = \mu(1/|x_i|)(S \cap \mathbf{B}(0, 2|x_i|))$ converge weakly to the three planar vectors $2, 2e^{2\pi i/3}, 2e^{4\pi i/3}$. By monotonicity 2.2(4), the point $x_i/|x_i|$ must be close to lying on one of these three vectors, say $x_i \approx 1$. There also must be points in S near $e^{2\pi i/3}$ and $e^{4\pi i/3}$, at which the density is at least one. Since $\mathbf{B}(1, \sqrt{3}/2)$, $\mathbf{B}(e^{2\pi i/3}, \sqrt{3}/2)$, and $\mathbf{B}(e^{4\pi i/3}, \sqrt{3}/2)$ have disjoint interiors, $\mathscr{H}^1(S_i)$ is greater than or approximately equal to

$$\frac{3}{2} \cdot 2 \cdot \frac{\sqrt{3}}{2} + 1 \cdot 2 \cdot \frac{\sqrt{3}}{2} + 1 \cdot 2 \cdot \frac{\sqrt{3}}{2} = \frac{7}{2}\sqrt{3} > 6.06.$$

For large i, $\mathscr{H}^1(S_i) > 6.06 > \frac{3}{2} \cdot 2 \cdot 2$, which contradicts $\Theta(S, 0) = \frac{3}{2}$. We conclude that $\Theta(S, x)$ must be 1 in a deleted neighborhood of 0.

By Corollary 3.7, in a deleted neighborhood of 0, S consists of $C^{1,\alpha/2}$ curves with a uniform Hölder constant. Since the tangent cone consists of three rays meeting at $120°$, at 0, S consists of three curves meeting at $120°$.

REFERENCES

[A] F. J. Almgren, Jr., *Existence and regularity almost everywhere of solutions to elliptic variational problems with constraints*, Mem. Amer. Math. Soc., vol. 4, Amer. Math. Soc., Providence, RI, 1976.

[F] Herbert Federer, *Geometric measure theory*, Springer-Verlag, New York, 1969.

[LM] Gary Lawlor and Frank Morgan, *Paired calibrations applied to soap films, immiscible fluids, and surfaces or networks minimizing other norms*, Pacific J. Math. (to appear).

[M1] Frank Morgan, *Geometric measure theory: a beginner's guide*, Academic Press, New York, 1988.

[M2] ——, *Soap bubbles in* $\mathbf{R}^2$ *and in surfaces*, Pacific J. Math. (to appear).

[M3] ——, *Size-minimizing rectifiable currents*, Invent. Math. **96** (1989), 333–348.

[M4] ——, *Soap bubbles and soap films*, Mathematical Vistas: New and Recent Publications in Mathematics from the New York Academy of Sciences (Joseph Malkevitch and Donald McCarthy, eds.), vol. 607, New York Acad. Sci., New York, 1990, pp. 98–106.

[T] Jean E. Taylor, *The structure of singularities in soap-bubble-like and soap-film-like minimal surfaces*, Ann. of Math. (2) **103** (1976), 489–539.

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