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LINEAR MONOTONE OPERATORS AND WEIGHTED BMO

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ABSTRACT. In this paper, the linear monotone operators, which include the well-known Hardy operator and Riemann-Liouville fractional integrals, are introduced. A necessary and sufficient condition for them to be bounded from a Banach function space into a weighted BMO is given, and their compactness in some particular cases is studied. Meanwhile, the embedding properties concerning the weighted BMO are investigated.

1. INTRODUCTION

In order to generalize the Hardy operator defined by

$$Tf(x) = \int_0^x f(t)\,dt, \qquad x > 0, \tag{1}$$

many authors introduce some classes of Volterra operators and study their weighted norm inequalities. For example, in [2] Bloom and Kerman consider the operators of Hardy type defined by

$$Tf(x) = \int_0^x \phi(x, y) f(y)\,dy, \tag{2}$$

where kernels $\phi(x, y)$ on $R^+ \times R^+$ with the following properties:

(i) $\phi(x, y) > 0$ if $x > y$,
(ii) $\phi(x, y)$ is nondecreasing in x and nonincreasing in y,
(iii) $\phi(x, y) \approx \phi(x, z) + \phi(z, y)$ if $y < z < x$.

The symbol "$\approx$" means the ratio of the two sides is bounded between absolute positive constants. Simultaneously, in [10] Stepanov investigate the Volterra convolution operator K given by

$$Kf(x) = \int_0^x k(x - y) f(y)\,dy, \tag{3}$$

where the kernels k satisfy the conditions:

(i) $k(x) \geq 0$ is nondecreasing on $(0, \infty)$,
(ii) $k(x + y) \leq D(k(x) + k(y))$ for all $x, y \in (0, \infty)$.

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The weighted norm inequalities for (2), (3) have been characterized on weighted Lebesgue spaces. In this note, we introduce a more general operator including the T and K above, give a necessary and sufficient condition for its boundedness from general Banach function spaces into weighted BMO, and study its compactness in some particular cases. Meanwhile, we discuss the embedding properties concerning the weighted BMO which are intrinsically interesting.

2. Boundedness and compactness

We start with some necessary notation. Given an interval (A, B) in real line R^1 (finite or infinite), let (X, v) be a Banach function space on the measure space $((A, B),\ v\,dx)$, where $v(x)$ is a weight function on (A, B), i.e., nonnegative locally integrable with $0 < v(x) < \infty$ a.e. We omit mention of the interval (A, B) concerned, since it does not cause ambiguity. The definition of Banach function spaces and their general properties can be found in [1, pp. 1–30], however, we would like to point out that it includes common Lebesgue spaces $L^p 1 \le p \le \infty$, Lorentz spaces, and Orlicz spaces with or without weights.

For given (X, v), its associate space (X', v) is given by

$$(X', v) = \Big\{ g\colon g \text{ is measurable,}$$

$$\|g\|_{X',v} = \sup\left[\int_A^B |fg| v\,dx \colon f \in X,\ \|f\|_{X,v} \le 1\right] \tag{4}$$

$$= \sup\left[\left|\int_A^B fgv\,dx\right| \colon f \in X,\ \|f\|_{X,v} \le 1\right] < \infty \Big\};$$

(X', v) is also a Banach function space. The following properties of Banach function spaces (X, v) are needed for our argument:

$$0 \le f \le g \quad \text{implies} \quad \|f\|_{X,v} \le \|g\|_{X,v}, \tag{5}$$

$$\|\,|f|\,\|_{X,v} = \|f\|_{X,v}; \tag{6}$$

$$\int_A^B |fg| v\,dx \le \|f\|_{X,v} \|g\|_{X',v} \tag{7}$$

for every $f \in (X, v)$ and $g \in (X', v)$.

Suppose $w(x)$ is a weight function. The weighted BMO (see [6]) is defined by

(8)

$$\mathrm{BMO}_w = \Big\{ f\colon f \text{ is locally integrable,}$$

$$\|f\|_{*,w} = \sup_{[a,b]\subset(A,B)} \frac{1}{b-a} \int_a^b |f(x) - (f)_{w,(a,b)}| w(x)\,dx < \infty \Big\},$$

where $(f)_{w,(a,b)} = (1/w(a, b)) \int_a^b fw\,dx$ with $w(a, b) = \int_a^b w\,dx$. For a measurable set E, $|E|$ is its Lebesgue measure, $w(E) = \int_E w\,dx$, and χ_E denotes its characteristic function. When $w(x) \equiv 1$, we shall omit the index w in the notation BMO_w and its norm.

Let T be a operator which maps the appropriate functions on (A, B) into the functions on (A, B). We say T is monotone if $Tf(x)$ is monotone whenever $f \geq 0$. For a given operator T, its adjoint operator T^* is defined by

$$\int_A^B Tf(x)g(x)\,dx = \int_A^B f(x)T^*g(x)\,dx \tag{9}$$

for all integrable $g(x)$ with compact support in (A, B).

Theorem 1. *Let w, v be a pair of weight functions and (X, v) be a Banach function space. Suppose T is a linear monotone operator, and for $[a, b] \subset (A, B)$ set*

$$G_{w,(a,b)}(x) = \frac{\int_a^x w - \int_x^b w}{(b-a)w(a, b)} w(x)\chi_{[a,b]}(x). \tag{10}$$

Then

$$\|Tf\|_{*,w} \leq C\|f\|_{X,v} \tag{11}$$

holds for all f in (X, v) if and only if

$$\sup_{[a,b]\subset(A,B)} \left\| \frac{T^*G_{w,(a,b)}}{v} \right\|_{X',v} = J < \infty. \tag{12}$$

Furthermore, for the best constant C in (8), *we have*

$$J \leq C \leq 4J. \tag{13}$$

Proof. Sufficiency. For any function g and $[a, b] \subset (A, B)$, it is obvious that

$$\begin{aligned} &\frac{1}{b-a}\int_a^b |g(x) - (g)_{w,(a,b)}|w(x)\,dx \\ &\leq \frac{1}{(b-a)w(a, b)}\int_a^b\int_a^b |g(x) - g(y)|w(y)w(x)\,dy\,dx \\ &\leq \frac{2}{b-a}\int_a^b |g(x) - (g)_{w,(a,b)}|w(x)\,dx. \end{aligned} \tag{14}$$

If $g(x)$ is monotone, say increasing, then we have

$$\begin{aligned} &\int_a^b\int_a^b |g(x) - g(y)|w(y)w(x)\,dy\,dx \\ &= \int_a^b\int_a^x [g(x) - g(y)]w(y)w(x)\,dy\,dx \\ &\quad + \int_a^b\int_x^b [g(y) - g(x)]w(y)w(x)\,dy\,dx \\ &= \int_a^b g(x)w(x)\left[\int_a^x w - \int_x^b w\right]dx \\ &\quad + \int_a^b \left[\int_x^b gw - \int_a^w gw\right] w(x)\,dx \\ &= 2\int_a^b g(x)w(x)\left[\int_a^x w - \int_x^b w\right]dx. \end{aligned} \tag{15}$$

Similarly, we get

$$\begin{aligned}(16)\qquad &\int_a^b\int_a^b |g(x)-g(y)|w(y)w(x)\,dy\,dx\\ &\qquad = -2\int_a^b g(x)w(x)\left[\int_a^x w - \int_x^b w\right]dx\end{aligned}$$

when g is decreasing.

Combining (14)–(16) and (10), we have, for $f \geq 0$,

$$\begin{aligned}(17)\qquad &\frac{1}{b-a}\int_a^b |Tf(x)-(Tf)_{w,(a,b)}|w(x)\,dx\\ &\leq 2\left|\int_A^B Tf(x)G_{w,(a,b)}(x)\,dx\right| = 2\left|\int_A^B f(x)T^*G_{w,(a,b)}(x)\,dx\right|\\ &\leq \frac{2}{b-a}\int_a^b |Tf(x)-(Tf)_{w,(a,b)}|w(x).\end{aligned}$$

It follows from the Hölder inequality (7) that

$$\left|\int_A^B f(x)T^*G_{w,(a,b)}(x)\,dx\right| \leq \|f\|_{X,v}\left\|\frac{T^*G_{w,(a,b)}}{v}\right\|_{X',v} \leq J\|f\|_{X,v}.$$

Therefore, for $f \geq 0$, $f \in (X, v)$, we obtain

$$\|Tf\|_{*,w} \leq 2J\|f\|_{X',v}.$$

For general $f \in (X, v)$, write $f = f^+ - f^-$, where $f^+(x) = \max\{0, f(x)\}$. It follows that

$$\begin{aligned}\|Tf\|_{*,w} &\leq \|T(f^+)\|_{*,w} + \|T(f^-)\|_{*,w}\\ &\leq 2J(\|f^+\|_{X,v} + \|f^-\|_{X,v}) \leq 4J\|f\|_{X,v}.\end{aligned}$$

Conversely, assume $\|Tf\|_{*,w} \leq C\|f\|_{X,v}$. For any $[a, b] \subset (A, B)$, it follows from (17) that

$$\left|\int_A^B f(x)T^*G_{w,(a,b)}(x)\,dx\right| \leq C\|f\|_{X,v}$$

for all $f \geq 0$, $f \in (X, v)$. Taking supremum over all $f \geq 0$, $\|f\|_{X,v} \leq 1$, one obtains

$$\left\|\frac{T^*G_{w,(a,b)}}{v}\right\|_{X',v} \leq C;$$

hence $J \leq C$. The proof of Theorem 1 is complete.

It is obvious that the operators given by (2) and (3) both are linear monotone operators, and the special case of Theorem 1, when T is the Hardy operator defined by (1), has been obtained in [4].

Theorem 2. *Suppose T is a linear monotone operator satisfying that*

(18) $\qquad Tf(x) \equiv 0$ *on* $(A, R]$ *whenever* $\operatorname{supp} f \subset [R, B) \qquad (R \in (A, B))$.

If the operator $P_R f = T(f\chi_{(A,R]})$ is compact from (X, v) into BMO_w for every $R \in (A, B)$, then T is compact from (X, v) into BMO_w if and only if both (12) *and*

$$J_R = \sup_{[a,b]\subset(A,B)} \left\| \frac{T^* G_{w,(a,b)}}{v} \chi_{(R,B)} \right\|_{X',v} \to 0 \quad \text{as } R \to B. \tag{19}$$

hold.

Proof. Sufficiency. Let $T_R f = T(f\chi_{(R,B)})$. Then $T = P_R + T_R$. Since also T_R is a linear monotone operator from (X, v) into BMO_w and $T_R^* g = \chi_{(R,B)} T^* g$, it follows from Theorem 1 that

$$\|T_R\| \le 4 J_R \to 0 \quad \text{as } R \to B.$$

Therefore, T is compact from (X, v) into BMO_w.

Necessity. We only need to verify (19), if (19) is not true, then there exist an $\varepsilon > 0$, $R_n \to B$, and $[a_n, b_n]$, such that

$$\left\| \frac{T^* G_{w,(a_n,b_n)}}{v} \chi_{(R_n,B)} \right\|_{X',v} \ge \varepsilon;$$

therefore, for every n, there is a $g_n \ge 0$ with $\|g_n\|_{X,v} \le 1$ such that

$$\left| \int_{a_n}^{b_n} T^* G_{w,(a_n,b_n)}(x) \chi_{(R_n,B)}(x) g_n(x)\, dx \right| > \frac{\varepsilon}{2}. \tag{20}$$

Let $f_n(x) = g_n(x)\chi_{[a_n,b_n]}(x)\chi_{(R_n,B)}(x)$. Then $\|f_n\|_{X,v} \le 1$. Select a subsequence from $\{f_n\}$ by induction: $n_1 = 1$, and n_{k+1} is chosen such that $R_{n_{k+1}} > \max\{R_{n_k}, b_{n_k}\}$ $(k = 1, 2, \dots)$.

For every pair $k < s$, observing that $R_{n_s} > b_{n_k}$ and $f_{n_s} \equiv 0$ on (A, R_{n_s}), it follows from (18), (17), and (20) that

$$\begin{aligned} \|Tf_{n_k} - Tf_{n_s}\|_{*,w} &\ge \frac{1}{b_{n_k} - a_{n_k}} \int_{a_{n_k}}^{b_{n_k}} |Tf_{n_k}(x) - (Tf_{n_k})_{w,(a_{n_k},b_{n_k})}| w(x)\, dx \\ &\ge \left| \int_{a_{n_k}}^{b_{n_k}} T^* G_{w,(a_{n_k},b_{n_k})}(x) f_{n_k}(x)\, dx \right| > \frac{\varepsilon}{2}, \end{aligned}$$

which implies that T is not compact from (X, v) into BMO_w. This completes Theorem 2.

The first important example of linear monotone operators is the Riemann-Liouville fractional integrals (see [5, 9]) defined by

$$T_\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)\, dt, \qquad x > 0,\ \alpha \ge 1, \tag{21}$$

and then

$$T_\alpha^* g(x) = \frac{1}{\Gamma(\alpha)} \int_x^\infty (t-x)^{\alpha-1} g(t)\, dt.$$

When $\alpha = n$ is a positive integer, we have

$$T_1^* G_{w,(a,b)}(x) = G_{w,(a,b)}^1(x) = \frac{w(a,x)w(x,b)}{(b-a)w(a,b)} \chi_{[a,b]}(x) \tag{22}$$

and

$$(23)\qquad T_n^* G_{w,(a,b)}(x) = G_{w,(a,b)}^n(x) = \int_x^b G_{w,(a,b)}^{n-1}(t)\,dt \qquad (n \geq 2).$$

Furthermore, if $w(x) \equiv 1$, a simple calculation shows that

(24)

$$G_{1,(a,b)}^n(x) = \begin{cases} 0, & x \in (b,\infty), \\ \dfrac{(b-x)^n[2x-(n+1)a+(n-1)b]}{(n+1)!(b-a)^2}, & x \in [a,b], \\ \dfrac{1}{(n+1)!(b-a)^2}\{(b-x)^n[2x-(n+1)a+(n-1)b] \\ \qquad -(a-x)^n[2x+(n-1)a-(n+1)b]\}, & x \in (0,a). \end{cases}$$

In this case, Theorem 1 can be rewritten as follows (cf. [7]).

Theorem 1′. *Let $AC^{(n-1)}(A, B) = \{f: f^{(n-1)}$ is absolutely continuous in (A, B), and $f(A) = f'(A) = \cdots = f^{(n-1)}(A) = 0\}$. Suppose the set $\{f^{(n)}: f \in AC^{(n-1)}\}$ is dense in (X, v). Then*

$$\|f\|_{*,w} \leq C\|f^{(n)}\|_{X,v}$$

holds for all $f \in AC^{(n-1)}$ if and only if

$$\sup_{[a,b]\subset(A,B)} \left\| \frac{G_{w,(a,b)}^n}{v} \right\|_{X',v} = J < \infty.$$

Moreover, for the best constant C, there is the estimate $J \leq C \leq 4J$.

It is obvious that Theorem 1′ gives some sufficient conditions such that the weighted Sobolev space concerning (X, v) is continuously embedded into BMO_w. We shall omit it.

The other example is a Volterra operator given by

$$Tf(x) = \delta(x)\int_0^x f(t)\sigma(t)\,dt, \qquad x > 0,$$

where $\delta(x)$ and $\sigma(x)$ are prescribed functions satisfying local integrability and $\frac{\sigma}{v} \in (X', v)$ respectively (see [3]). It is obvious that

$$T^* g(x) = \sigma(x)\int_x^\infty g(t)\delta(t)\,dt.$$

In addition, we assume δ and σ are weight functions and δ is increasing. Then T is a linear monotone operator. Applying Theorem 2 to this operator, we obtain a criterion for its compactness.

A Banach function space (or a normed function space) (X, v) is said to have absolutely continuous norm if, for every $f \in (X, v)$ and any sequence of measurable sets $\{E_n\}$ with $E_n \to \varnothing$ a.e., the equation

$$\lim_{n\to\infty} \|f\chi_{E_n}\|_{X,v} = 0$$

holds.

Theorem 3. *Suppose T is defined as above and δ is bounded. Assume (X', v) has absolutely continuous norm and $\|w\|_\infty < \infty$. Then T is compact from (X, v) into BMO_w if and only if*

(i)

$$J = \sup_{[a,b]\subset(0,\infty)} \left\| \frac{T^* G_{w,(a,b)}}{v} \right\|_{X',v} < \infty$$

and

(ii)

$$J_R = \sup_{[a,b]\subset(0,\infty)} \left\| \frac{T^* G_{w,(a,b)}}{v} \chi_{(R,\infty)} \right\|_{X',v} \to 0, \quad \text{as } R \to \infty.$$

Proof. According to Theorem 2, it is sufficient to prove that $T_R f = T(f\chi_{(0,R]})$ is compact from (X, v) into BMO_w for every given $R > 0$. Indeed, suppose $\{f_n\}$ is a sequence in the closed unit ball of (X, v), we have

$$|T_R f_n(x)| \le \delta(R) \left\| \frac{\sigma}{v} \right\|_{X',v} \quad \text{for any } n \text{ and } x \in [0, R]$$

and

$$|T_R f_n(x) - T_R f_n(y)| \le \delta(R) \left\| \frac{\sigma}{v} \chi_{[x,y]} \right\|_{X',v}$$

for any n and $0 \le x < y \le R$. Since $\frac{\sigma}{v} \in (X', v)$ and (X', v) has absolutely continuous norm, $\{T_R f_n\}$ is uniformly bounded and equicontinuous. By use of the Arzela-Ascoli theorem it contains a subsequence which is uniformly convergent on $[0, R]$ and, furthermore, on $[0, \infty)$, because $T_R f_n(x) = \delta(x) \int_0^R f_n(t)\sigma(t)\,dt$ on (R, ∞). Note that

$$\|f\|_{*,w} \le 2\|w\|_\infty \|f\|_\infty, \tag{25}$$

if $\|w\|_\infty < \infty$; the corresponding subsequence is convergent in BMO_w. The proof of Theorem 3 is complete.

When $\delta(x) = \sigma(x) = 1$, the previous T is the Hardy operator defined in (1). For the Hardy operator, we can remove the prescribed condition $\sigma/v = 1/v \in (X', v)$ in Theorem 3. Indeed, we have

Theorem 4. *Suppose T is the Hardy operator defined in (1). Assume (X', v) has absolutely continuous norm and $\|w\|_\infty < \infty$. Then, T is compact from (X, v) into BMO_w if and only if*

(i)

$$J = \sup_{[a,b]\subset(0,\infty)} \left\| \frac{G^1_{w,(a,b)}}{v} \right\|_{X',v} < \infty,$$

(ii)

$$J_R = \sup_{[a,b]\subset(0,\infty)} \left\| \frac{G^1_{w,(a,b)}}{v} \chi_{(R,\infty)} \right\|_{X',v} \to 0, \quad \text{as } R \to \infty,$$

and

(iii)

$$J_r = \sup_{[a,b]\subset(0,\infty)} \left\| \frac{G^1_{w,(a,b)}}{v} \chi_{(0,r)} \right\|_{X',v} \to 0, \quad \text{as } r \to 0,$$

where $G^1_{w,(a,b)}(x)$ is defined in (22).

Proof. Sufficiency. It is enough to prove that, for given $R > 0$, T_R is compact. For any $0 < r < R$, write $T_R = P_{R,r} + P_r$ with $P_r f = T(f\chi_{(0,r)})$ and $P_{R,r}f = T(f\chi_{[r,R]})$. Since $J < \infty$ implies $\chi_{[r,R]}/v \in (X', v)$ for any $0 < r < R < \infty$, the argument similar to that used in Theorem 3 shows $P_{R,r}$ is compact. On the other hand we have $\|P_r\| \le 4J_r$; therefore, T_R is compact.

Necessity. We only need to verify (iii). The procedure is nearly the same as that in Theorem 2 for proving that the compactness of T implies (19). We shall only mention the difference. In this case, we select subsequence by $n_1 = 1$, and n_{k+1} is chosen such that $r_{n_{k+1}} < \min\{r_{n_k}, a_{n_k}\}$ $(k = 1, 2, \dots)$. Then, for every $k < s$, $Tf_{n_s}(x) \equiv C_{n_s} = \int_0^{b_{n_s}} f_{n_s}(t)\,dt$ on $[a_{n_k}, b_{n_k}]$. Hence, it follows that

$$\begin{aligned}
&\|Tf_{n_k} - Tf_{n_s}\|_{*,w} \\
&\quad \ge \frac{1}{b_{n_k} - a_{n_k}} \int_{a_{n_k}}^{b_{n_k}} |T(f_{n_k} - f_{n_s})(x) - (T(f_{n_k} - f_{n_s}))_{w,(a_{n_k}, b_{n_k})}| w(x)\,dx \\
&\quad = \frac{1}{b_{n_k} - a_{n_k}} \int_{a_{n_k}}^{b_{n_k}} |Tf_{n_k}(x) - (Tf_{n_k})_{w,(a_{n_k}, b_{n_k})}| w(x)\,dx \\
&\quad \ge \left| \int_{a_{n_k}}^{b_{n_k}} T^* G_{w,(a_{n_k}, b_{n_k})}(x) f_{n_k}(x)\,dx \right| > \frac{\varepsilon}{2}.
\end{aligned}$$

The proof of Theorem 4 is complete.

A function f in a normed function space X is said to have the continuous norm if for every $x \in (A, B)$ and $\varepsilon > 0$ given there exists a $\delta > 0$ such that $\|f\chi_{[x-r,x+r]}\|_X < \varepsilon$, whenever $0 < r < \delta$.

In [4] we proved that, if Hardy operator is compact from a Banach function space (X, v) into BMO, then for every given $0 < r < R < \infty$ the function $\chi_{[r,R]}/v$ has continuous norm in (X', v). Thus, the argument in Theorem 4 shows:

Theorem 4′. *Suppose (X, v) is a Banach function space. Then the Hardy operator is compact from (X, v) into BMO if and only if:*

(i)

$$J = \sup_{[a,b]\subset(0,\infty)} \left\| \frac{G^1_{1,(a,b)}}{v} \right\|_{X',v} < \infty;$$

(ii) *for every given $0 < r < R < \infty$, the function $\chi_{[r,R]}/v$ has continuous norm in (X', v);*

(iii)

$$J_r = \sup_{[a,b]\subset(0,\infty)} \left\| \frac{G^1_{1,(a,b)}}{v} \chi_{(0,r)} \right\|_{X',v} \to 0, \quad \text{as } r \to 0,$$

and

$$J_R = \sup_{[a,b]\subset(0,\infty)} \left\| \frac{G^1_{1,(a,b)}}{v} \chi_{(R,\infty)} \right\|_{X',v} \to 0, \quad \text{as } R \to \infty,$$

where $G^1_{1,(a,b)}(x) = (b - x)(x - a)\chi_{[a,b]}(x)/(b - a)^2$ is defined in (22) with $w(x) \equiv 1$.

Theorems 4 and 4′ are the improved versions of our previous compactness criteria in [4].

3. Embedding properties concerning weighted BMO

In Theorems 3 and 4, the condition $\|w\|_\infty < \infty$ just guarantee the inequality (25). Hence, when we try to generalize this condition, a natural problem arises, that is, whether $\|w\|_\infty < \infty$ is necessary for $\|f\|_{*,w} \le C\|f\|_\infty$. More generally, given a function space X, under which condition is X is continuously embedded into BMO_w? And how about the inverse embedding? The following theorems treat these questions.

Theorem 5. *The inequality $\|f\|_{*,w} \le C\|f\|_\infty$ holds for all $f \in L_\infty$, if and only if $\|w\|_\infty < \infty$. Furthermore, for the best constant C, $\|w\|_\infty = C$.*

Proof. Sufficiency. Suppose $\|w\|_\infty < \infty$. We shall prove $\|f\|_{*,w} \le \|w\|_\infty$ for all $f \in L_\infty$ with $\|f\|_\infty \le 1$.

For every interval $[a, b] \subset (A, B)$ given, without loss of generality, we may assume $(f)_{w,(a,b)} \ge 0$. Otherwise, we consider $-f$. Then $(-f)_{w,(a,b)} \ge 0$ and

$$\int_a^b |f(x) - (f)_{w,(a,b)}|w(x)\,dx = \int_a^b |-f(x) - (-f)_{w,(a,b)}|w(x)\,dx.$$

Write $f = f^+ - f^-$, where $f^+ = \max\{f(x), 0\}$. Let $E_1 = \{x \in [a, b]: f(x) \ge (f)_{w,(a,b)}\}$, $E_2 = \{x \in [a, b]: 0 \le f(x) \le (f)_{w,(a,b)}\}$, and $E_3 = \{x \in [a, b]: f(x) < 0\}$. Then

$$\begin{aligned}(26)\qquad &\int_a^b |f(x) - (f)_{w,(a,b)}|w(x)\,dx \\ &= \int_{E_1} (f^+ - (f)_{w,(a,b)})w\,dx \\ &\quad + \int_{E_2} ((f)_{w,(a,b)} - f^+)w\,dx + \int_{E_3} (f^- + (f)_{w,(a,b)})w\,dx.\end{aligned}$$

If $w(E_2) + w(E_3) \le w(E_1)$, it follows from (26) that

$$\begin{aligned}(27)\qquad &\int_a^b |f(x) - (f)_{w,(a,b)}|w(x)\,dx \\ &= \int_{E_1} f^+w\,dx + \int_{E_2} f^+w\,dx + \int_{E_3} f^-w\,dx \\ &\quad + (w(E_2) + w(E_3) - w(E_1))(f)_{w,(a,b)} - 2\int_{E_2} f^+w\,dx \\ &\le \int_{E_1} f^+w\,dx + \int_{E_2} f^+w\,dx + \int_{E_3} f^-w\,dx \\ &= \int_a^b |f(x)|w(x)\,dx \le \|w\|_\infty(b - a).\end{aligned}$$

When $w(E_2)+w(E_3)>w(E_1)$, the right side of (26) is

$$\begin{aligned}(28)\qquad &\int_{E_1} f^+w\,dx-\int_{E_2} f^+w\,dx\\ &\quad+\int_{E_3} f^-w\,dx+w(a\,,\,b)(f)_{w\,,(a\,,b)}-2w(E_1)(f)_{w\,,(a\,,b)}\\ &\le\int_{E_1} f^+w\,dx-\int_{E_2} f^+w\,dx+\int_{E_3} f^-w\,dx+\int_a^b fw\,dx\\ &=2\int_{E_1} f^+w\,dx\le 2w(E_1)\le w(a\,,\,b)\le\|w\|_\infty(b-a).\end{aligned}$$

Combining (27) and (28), we have $\|f\|_{*,w}\le\|w\|_\infty$, whenever $\|f\|_\infty\le 1$.

Necessity. We need to prove that $\|f\|_{*,w}\le C\|f\|_\infty$ implies $\|w\|_\infty\le C$. Assume $\|w\|_\infty>C$. Then there exists $\varepsilon>0$ such that

$$|E|=|\{x\in(A\,,\,B)\colon w(x)>(1+2\varepsilon)C\}|>0.$$

On the other hand, there is a $[a\,,\,b]\subset(A\,,\,B)$ satisfying

$$|[a\,,\,b]\cap E|>\frac{b-a}{1+\varepsilon}.$$

We can divide $[a\,,\,b]\cap E$ into two disjoint subsets E_i $(i=1\,,\,2)$ with $w(E_1)=w(E_2)$. Let

$$f(x)=\chi_{E_1}(x)-\chi_{E_2}(x).$$

Then $f_{w\,,(a\,,b)}=0$, and

$$\begin{aligned}\|f\|_{*,w}&\ge\frac{1}{b-a}\int_a^b|f(x)-(f)_{w\,,(a\,,b)}|w(x)\,dx\\ &=\frac{w(E_1\cup E_2)}{b-a}\ge\frac{1+2\varepsilon}{1+\varepsilon}C>C\|f\|_\infty.\end{aligned}$$

This is a contradiction. The proof of Theorem 5 is complete.

Theorem 5 gives not only a necessary and sufficient condition for L_∞ to be continuously embedded into BMO_w but also the exact estimate for the best constant C, which improves the well-known estimate (25). Replacing L_∞ by BMO, we have

Theorem 6. *The inequality $\|f\|_{*,w}\le C\|f\|_*$ holds for all $f\in\mathrm{BMO}$, if and only if $\|w\|_\infty<\infty$. Moreover, for the best constant C, $\|w\|_\infty\le C\le 2\|w\|_\infty$.*

Proof. It is sufficient to prove that $\|f\|_{*,w}\le C\|f\|_*$ implies $\|w\|_\infty\le C$. Assume $\|w\|_\infty>C$. Then there exists $0<\varepsilon<1$ such that

$$|E|=|\{x\in(A\,,\,B)\colon w(x)>C/(1-\varepsilon)\}|>0.$$

Meanwhile, there is a $[a\,,\,b]\subset(A\,,\,B)$ satisfying

$$|[a\,,\,b]\cap E|>(1-\varepsilon)(b-a).$$

We can divide $[a\,,\,b]\cap E$ into two disjoint subsets E_i $(i=1\,,\,2)$ with equal measure. If $w(E_1)\le w(E_2)$, let $f(x)=\chi_{E_1}(x)$. Then, noting that $\|\chi_E\|_*=\frac{1}{2}$

for any measurable set $E \subsetneq (A, B)$ with positive measure (see [8]), it follows that

$$\begin{aligned}\|f\|_{*,w} &\geq \frac{1}{b-a}\int_{E_1\cup E_2}|f(x)-(f)_{w,(a,b)}|w(x)\,dx\\ &= \frac{w(E_1)}{b-a}+\frac{w(E_1)}{(b-a)w(a,b)}(w(E_2)-w(E_1))\\ &\geq \frac{w(E_1)}{b-a} > C\|f\|_*.\end{aligned}$$

If $w(E_1) > w(E_2)$, let $f(x) = \chi_{E_2}(x)$. A similar argument shows that

$$\|f\|_{*,w} \geq \frac{w(E_2)}{b-a}+\frac{w(E_2)}{(b-a)w(a,b)}(w(E_1)-w(E_2)) > C\|f\|_*.$$

This is a contradiction. The proof of Theorem 6 is complete.

Conversely, we have

Theorem 7. *The inequality* $\|f\|_* \leq C\|f\|_{*,w}$ *holds for all* $f \in \mathrm{BMO}_w$, *if and only if* $\|\frac{1}{w}\|_\infty < \infty$. *And, for the best constant* C,

$$\frac{1}{4}\left\|\frac{1}{w}\right\|_\infty \leq C \leq 2\left\|\frac{1}{w}\right\|_\infty.$$

Proof. It is sufficient to prove that $\|f\|_* \leq C\|f\|_{*,w}$ implies $\frac{1}{4}\|\frac{1}{w}\|_\infty \leq C$. Assume it is not true. Then there exists an $\varepsilon > 0$ such that

$$|E| = |\{x \in (A, B): 1/w(x) > 4C+\varepsilon\}| > 0.$$

Let $f(x) = \chi_E(x)$. Then

$$\begin{aligned}&\frac{1}{b-a}\int_a^b |f(x)-(f)_{w,(a,b)}|w(x)\,dx\\ &\quad= \frac{2w([a,b]\cap E)w([a,b]\cap E^c)}{(b-a)w(a,b)} \leq \frac{2w([a,b]\cap E)}{(b-a)} \leq \frac{2}{4C+\varepsilon}.\end{aligned}$$

Therefore,

$$\frac{1}{2} = \|f\|_* \leq C\|f\|_{*,w} \leq \frac{2C}{4C+\varepsilon}.$$

This contradiction completes the proof of Theorem 7.

Combining Theorems 6 and 7 shows that BMO is equivalent to BMO_w if and only if the weight function is bounded above and bounded away from zero. However, usually, BMO_w cannot be continuously embedded into another function space, if $\|1\|_X > 0$. Conversely, we have

Theorem 8. *Suppose* X *is a linear normed space which consists of functions on* (A, B) *and contains all simple functions. If* X *has absolutely continuous norm* (*see the definition before Theorem* 3), *then* X *is not continuously embedded into* BMO_w *except* $w(x) \equiv 0$ *a.e.*

Proof. Suppose $w(x) \not\equiv 0$. Let $I(x, r) = [x-r, x+r]$. According to Lebesgue differential theorem, there exists $x_0 \in (A, B)$, such that

$$\lim_{r\to 0}\frac{1}{|I(x_0, r)|}\int_{I(x_0, r)} w(x)\,dx = w(x_0) > 0.$$

For every $I(x_0, r)$, choose c_r satisfying $w(x_0 - r, c_r) = w(c_r, x_0 + r) = \frac{1}{2}w(I(x_0, r))$. Let

$$f_r(x) = \chi_{[x_0-r, c_r]}(x) - \chi_{(c_r, x_0+r]}(x).$$

Then $\|f_r\|_X \to 0$ as $r \to 0$. However, since $(f_r)_{w, I(x_0, r)} = 0$, we have

$$\begin{aligned}\|f_r\|_{*,w} &\geq \frac{1}{|I(x_0, r)|}\int_{I(x_0, r)} |f_r(x) - (f_r)_{w, I(x_0, r)}|w(x)\,dx \\ &= \frac{1}{I(x_0, r)|}\int_{I(x_0, r)} w(x)\,dx \to w(x_0) > 0 \quad (\text{as } r \to 0).\end{aligned}$$

The proof of Theorem 8 is complete.

Remark. The other weighted BMO is defined by

$$\mathrm{BMO}^w = \left\{ f\colon f \text{ is locally integrable,} \right.$$

$$\left. \|f\|_*^w = \sup_{[a,b]\subset(A,B)} \frac{1}{w(a, b)}\int_a^b |f(x) - (f)_{(a,b)}|\,dx < \infty \right\},$$

which corresponds to the weighted L_∞ defined by

$$L_\infty^w = \{f\colon f \text{ is measurable}, \|f\|_\infty^w = \|f/w\|_\infty < \infty\}.$$

All arguments in this paper are still available if BMO_w is replaced by BMO^w. However, we would like to point out that the estimates for the best constants in Theorems 6 and 7 will be improved. That is, we have

Theorem 6′. *The inequality* $\|f\|_*^w \leq C\|f\|_*$ *holds for all* $f \in \mathrm{BMO}$, *if and only if* $\|1/w\|_\infty < \infty$. *Moreover, the best constant* C *is equal to* $\|1/w\|_\infty$.

Theorem 7′. *The inequality* $\|f\|_* \leq C\|f\|_*^w$ *holds for all* $f \in \mathrm{BMO}^w$, *if and only if* $\|w\|_\infty < \infty$. *And, for the best constant* C, $\frac{1}{4}\|w\|_\infty \leq C \leq \|w\|_\infty$.

On the other hand, all theorems in §3 hold also in more general cases, for example, in high-dimension Euclidean spaces.

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