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DISCREPANCY OF BEHAVIOR OF PERTURBED SEQUENCES IN L^p SPACES

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ABSTRACT. Given $p \in [1, \infty)$, examples of sequences $\{n_k\}_{k\subset \mathbb{N}}$ such that for any ergodic dynamical system (X, β, m, T) the averages

$$A_N f(x) = \frac{1}{N}\sum_{k=1}^{N} f(T^{n_k}x)$$

converge almost everywhere in all $L^q(X)$, $q > p$, but fail to have a finite limit for some function in $L^p(X)$ are shown. Also, sequences such that for all ergodic dynamical systems the averages $A_N f(x)$ do not converge for some function $f \in L^p(X)$ for all $1 \le p < \infty$ but do converge for all functions in $L^\infty(X)$ are shown.

0. INTRODUCTION

In this paper, sequences of natural numbers with the property that the averages along them diverge in some L^p spaces and converge in others are given. Emerson [7] developed a useful tool to produce such examples and Bellow used it in [1] to construct sequences $\{n_k\}_{k\subset \mathbb{N}}$ that, for a fixed p, $1 < p < \infty$, are universally bad in L^q for $1 \le q < p$ but universally good for all $q \ge p$. The question of whether it is possible to produce sequences that are bad in L^1 but good in all L^p for $p > 1$ and sequences which are bad in all L^p for all $p < \infty$ but good in L^∞ was not answered. The former is relevant when one considers, for example, the sequence of squares, which is a good universal sequence in all L^p, $p > 1$, but its behavior in L^1 remains unknown to us at the moment. Emerson's tool can further be used to produce such examples. The existence of three types of squences are proved:

(1) for any p, $1 \le p < \infty$, there are sequences $\{n_k\}_{k\subset \mathbb{N}}$ which are universally bad on L^p but universally good in L^q, $q > p$;
(2) there are sequences $\{n_k\}_{k\subset \mathbb{N}}$ which are universally bad in L^p for all $1 \le p < \infty$ but universally good in L^∞;
(3) for fixed p, $1 \le p < \infty$, there are sequences $\{n_k\}_{k\subset \mathbb{N}}$ which are universally bad on L^q, $1 \le q \le p$, but universally good on L^q, $q > p$, so that $\lim_{k\to\infty} n_{k+1} - n_k = \infty$.

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From (1) one obtains sequences which are universally bad on L^1 but universally good in L^q for all $q > 1$. Example (2) is the most curious because one can get convergence of the averages for bounded functions but not for nonbounded ones. The first two examples plus that of Bellow were constructed by starting with a sequence of blocks of consecutive numbers which is universally good in L^1 and perturbing it by the right amount to obtain the desired result. Example (3) shows that the block averages are not essential to the construction. Moreover, the sequence can be chosen so that the gap between consecutive terms tends to infinity.

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1. Preliminaries

Denote by (X, β, m) a probability space and $T: X \to X$ a measure-preserving point transformation. The tuple (X, β, m, T) is called a dynamical system. The point transformation T induces an operator on measurable functions on (X, β, m) defined by $Tf(x) = f(Tx)$. By abuse of notation, both of them are denoted in the same way.

Recall that a dynamical system (X, β, m, T) is "aperiodic" if the set of periodic points has measure zero and the system is "ergodic" if T is an ergodic transformation with respect to the measure m.

Definition 1.1. A sequence $\{n_k\}_{k\subset\mathbf{N}} \in \mathbf{N}$ is called *universally good in L^p* if for all aperiodic (ergodic) dynamical systems (X, β, m, T) the averages

$$A_k f(x) = \frac{1}{k}\sum_{j=1}^{k} T^{n_j} f(x)$$

converge a.e. for all $f \in L^p(X)$, and is called *universally bad in L^p* if for all aperiodic (ergodic) dynamical systems the averages $A_k f$ do not converge a.e. for some $f \in L^p(X)$.

The definitions of good and bad universal are related to the Conze's Principle [6]. Fix a sequence $\mathbf{n} = \{n_k\}_{k\subset\mathbf{N}}$ and $1 \le p < \infty$. By Sawyer's extension fo the Banach Principle, for any dynamical system (X, β, m, T), where T commutes with a mixing family of transformations (see [8]), if the averages $A_k f$ converge a.e. in $L^p(X)$, $p \ge 1$, there exists a finite constant C, such that

$$m\left(\left\{x \in X\colon \sup_{N\in\mathbf{N}}\left|\frac{1}{N}\sum_{k=0}^{N-1} T^{n_k} f(x)\right| > \lambda\right\}\right) \le C\frac{\|f\|_p^p}{\lambda^p}$$

for all $f \in L^p(X)$. Therefore, one can associate to $\mathbf{n}$ a minimal constant $0 < C(\mathbf{n}, p) \le \infty$, such that

$$m\left(\left\{x \in X\colon \sup_{N\in\mathbf{N}}\left|\frac{1}{N}\sum_{k=0}^{N-1} T^{n_k} f(x)\right| > \lambda\right\}\right) \le C(\mathbf{n}, p)\frac{\|f\|_p^p}{\lambda^p}$$

for all $f \in L^p(X)$, $\lambda > 0$, and all aperiodic (ergodic) dynamical systems (X, β, m, T).

Theorem 1.2 (Conze's Principle). *For any given sequence* $\mathbf{n} = \{n_k\}_{k \subset \mathbb{N}}$, *the associated minimal constant* $C(\mathbf{n}, 1)$ *is finite if and only if there exists an aperiodic (ergodic) dynamical system* (X, β, m, T) *for which*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} T^{n_k} f(x)$$

exists a.e. for each $f \in L^1(X)$.

We will use the same notation as in Bellow [1]. Let θ be an irrational number, and define $T_\theta\colon [0, 1) \mapsto [0, 1)$ by $T_\theta(x) = x + \theta \bmod 1$. Define

$$p_k = \sum_{j=1}^{k} \frac{1}{j} \bmod 1 \quad \text{and} \quad J_k = [p_k, p_{k+1}] \bmod 1. \tag{1.a}$$

Because of the divergence of the series $\sum_{j=1}^{\infty} 1/j$, for every $x \in [0, 1)$, there exist infinitely many k's such that $x \in J_k$.

Given a family of pairwise disjoint blocks B_k, consider the elements of $\bigcup_{k=1}^{\infty} B_k$ arranged in increasing order without repetitions. This set can be thought of as a sequence and will be referred to as the *block sequence* $\bigcup_{k=1}^{\infty} B_k$. Suppose $\{D_k\}$ is another sequence of pairwise disjoint blocks and disjoint from the B_k's such that both sequences intertwine:

$$\cdots \underbrace{[\cdots\cdots\cdots]}_{B_k} \cdots \underbrace{[\cdots]}_{D_k} \cdots \underbrace{[\cdots\cdots\cdots]}_{B_{k+1}} \cdots .$$

One can form the sequence $\bigcup_{k=1}^{\infty}(B_k \cup D_k)$ which is called a *perturbation* of the original one.

Let $\{B_k\}$ and $\{D_k\}$ be as above, and let $l_k = |B_k|$ and $d_k = |D_k|$.

Theorem 1.3 (Bellow [1]). *Let* B_k *and* D_k *be as above. If the sequence* $\bigcup_{k=1}^{\infty} B_k$ *is a good universal sequence in* L^p *and*

$$\sum_{k=1}^{\infty} \left(\frac{d_1 + \cdots + d_k}{l_1 + \cdots + l_k}\right)^p < \infty,$$

then the sequence $\bigcup_{k=1}^{\infty}(B_k \cup D_k)$ *is also good universal in* L^p.

This theorem illustrates how much a sequence can be perturbed such that the resulting sequence has the same behavior. From it follows

Proposition 1.4. *Let* B_k *and* D_k *be as above. Let* $l_k = |B_k|$ *and* $d_k = |D_k|$. *Suppose* $l_1 + \cdots + l_k \le C l_{k+1}$ *for all* k, $d_k = c_k l_k$ *such that they also satisfy* $\sum_{k=1}^{\infty}(l_k/l_{k+1})^p < \infty$ *and* $\sum_{k=1}^{\infty} c_k^p < \infty$. *If* $\bigcup_k B_k$ *is a good universal sequence in* L^p *then the perturbed sequence* $\bigcup_{k=1}^{\infty}(B_k \cup D_k)$ *is also good universal in* L^p.

Proof. This proposition is an application of the previous theorem. Since $\sum_{k=1}^{\infty} c_k^p < \infty$, $\lim_{k\to\infty} c_k = 0$. Choose k_0 so that $c_k < 1$ for all $k \ge k_0$. Then

$$\begin{aligned}\frac{d_1 + \cdots + d_k}{l_1 + \cdots + l_k} &\le \frac{d_1 + \cdots + d_{k_0-1}}{l_1 + \cdots + l_k} + \frac{d_{k_0} + \cdots + d_{k-2}}{l_1 + \cdots + l_k} + \frac{d_{k-1} + d_k}{l_1 + \cdots + l_k} \\ &\le \frac{C_0}{l_k} + \frac{C l_{k-1}}{l_k} + \frac{d_{k-1}}{l_{k-1}} + \frac{d_k}{l_k} = \frac{C_0}{l_k} + C\frac{l_{k-1}}{l_k} + c_{k-1} + c_k .\end{aligned}$$

Since $\sum_{k=1}^{\infty}(l_k/l_{k+1})^p < \infty$ and $\sum_{k=1}^{\infty} c_k^p < \infty$,

$$\sum_{k=1}^{\infty}\left(\frac{d_1+\cdots+d_k}{l_1+\cdots+l_k}\right)^p < \infty .$$

The result follows from Theorem 1.3. □

Theorem 1.3 has an analogous version for L^{∞}.

Theorem 1.5. *Let $\{B_k\}$ be a sequence of pairwise disjoint, consecutive blocks such that the sequence they define is good universal in L^{∞}. Let $\{D_k\}$ be a perturbation of the B_k's, and denote $l_k = |B_k|$, $d_k = |D_k|$. If*

$$\lim_{k\to\infty}\frac{d_1+\cdots+d_k}{l_1+\cdots+l_k} = 0$$

then the perturbed sequence is also good universal in L^{∞}.

Proof. Let (X, β, m, T) be an aperiodic dynamical system and $f \in L^{\infty}(X)$. Let $C = \bigcup_{k=1}^{\infty}(B_k \cup D_k)$, $b_n = |\bigcup_{k=1}^{\infty} B_k \cap [0, n]|$, and $c_n = |\bigcup_{k=1}^{\infty} D_k \cap [0, n]|$. The averages

$$A_n f(x) = \frac{1}{|C\cap[0, n]|}\sum_{u\in C\cap[0, n]} T^u f(x)$$

can be written as a convex combination of averages on the B_k's and on the D_k's, particularly,

$$\begin{aligned} A_n f(x) &= \frac{b_n}{b_n+c_n}\left[\frac{1}{b_n}\sum_{u\in\bigcup_{k=1}^{\infty} B_k\cap[0, n]} T^u f(x)\right] \\ &\quad + \frac{c_n}{b_n+c_n}\left[\frac{1}{c_n}\sum_{u\in\bigcup_{k=1}^{\infty} D_k\cap[0, n]} T^u f(x)\right] \qquad (1.b)\\ &= \frac{b_n}{b_n+c_n}A_n^B f(x) + \frac{c_n}{b_n+c_n}A_n^D f(x). \end{aligned}$$

Observe that

$$\frac{c_n}{b_n} = \begin{cases} \dfrac{d_1+\cdots+d_{k-1}}{l_1+\cdots+l_{k-1}+s_k} & \text{if } k \text{ is the smallest integer such that } B_k \text{ is not contained in } [0, n] \text{ and } B_k\cap[0, n]\neq\varnothing, \\ \dfrac{d_1+\cdots+d_{k-2}+r_{k-1}}{l_1+\cdots+l_{k-1}} & \text{if } k \text{ is the smallest integer such that } B_k \text{ is not contained in } [0, n],\ B_k\cap[0, n]=\varnothing, \text{ and } B_{k-1}\subset[0, n], \end{cases}$$

where $0 \le r_{k-1} \le d_{k-1}$ and $0 \le s_k \le l_k$. In the first situation, one has the picture:

$$\cdots\underbrace{[\cdots]}_{D_{k-2}}\cdots\underbrace{[\cdots\cdots]}_{B_{k-1}}\cdots\underbrace{[\cdots]}_{D_{k-1}}\cdots\underbrace{[\overbrace{\cdots\cdots\mathbf{n}}^{s_k}\cdots\cdots]}_{B_k}\cdots\underbrace{[\cdots]}_{D_k}\cdots$$

and in the second:

$$\cdots \underbrace{[\cdots]}_{D_{k-2}} \cdots \underbrace{[\cdots\cdots]}_{B_{k-1}} \cdots \underbrace{[\overbrace{\cdots}^{r_{k-1}}\mathbf{n}\cdots]}_{D_{k-1}} \cdots \underbrace{[\cdots\cdots\cdots]}_{B_k} \cdots \underbrace{[\cdots]}_{D_k} \cdots$$

In both cases one has,

$$\frac{c_n}{b_n} \leq \frac{d_1 + \cdots + d_{k-1}}{l_1 + \cdots + l_{k-1}} \to 0 \tag{1.c}$$

by the hypothesis of the theorem. Therefore,

$$\lim_{k\to\infty} \frac{b_n}{b_n + c_n} = 1 \quad \text{and} \quad \lim_{k\to\infty} \frac{c_n}{c_n + b_n} = 0 .$$

Hence, since $\lim_{n\to\infty} A_n^B f(x)$ exists a.e. for all $f \in L^\infty(X)$ and $\lim_{n\to\infty}(c_n A_n^D f(x)/(c_n + b_n)) = 0$ for all $f \in L^\infty(X)$, $\lim_{n\to\infty} A_n f(x)$ also exists a.e. for all $f \in L^\infty(X)$. □

Remark 1.6. The argument used in Theorem 1.5 gives another proof of Theorem 1.3. Indeed, from (1.b),

$$|A_n f(x)| \leq \frac{b_n}{b_n + c_n} |A_n^B f(x)| + \frac{c_n}{b_n + c_n} |A_n^D f(x)| .$$

Since the sequence $\bigcup_{k=1}^\infty B_k$ is universally good in $L^p(X)$ and $b_n/(b_n+c_n) \leq 1$, it follows that $\sup_{n\in\mathbf{N}} |A_n^B f|$ is weak (p, p). It suffices, then, to check that $\sup_{n\in\mathbf{N}}(c_n|A_n^D f|/(b_n + c_n))$ satisfies a maximal inequality because, by (1.c), the averages $A_n f(x)$ converge a.e. for all functions in L^∞. But

$$\begin{aligned} \frac{c_n}{b_n + c_n}|A_n^D f(x)| &\leq \frac{1}{b_n + c_n} \sum_{u\in\bigcup_{i=1}^\infty D_i\cap[o\,,\,n]} |T^u f(x)| \\ &\leq \frac{1}{l_1 + \cdots + l_{k-1}} \sum_{u\in\bigcup_{i=1}^{k-1} D_i\cap[0\,,\,n]} |T^u f(x)| \quad (\text{if } B_{k-1} \subset [0\,,\,n] \text{ but } B_k \not\subset [0\,,\,n]) \\ &= \left(\frac{d_1 + \cdots + d_{k-1}}{l_1 + \cdots + l_{k-1}}\right) \frac{1}{d_1 + \cdots + d_{k-1}} \sum_{u\in\bigcup_{i=1}^{k-1} D_i} |T^u f(x)| . \end{aligned}$$

Therefore,

$$\left\| \sup_{n\in\mathbf{N}} \frac{c_n}{b_n + c_n} A_n^D f \right\|_p^p \leq \sum_{k=1}^\infty \left(\frac{d_1 + \cdots + d_k}{l_1 + \cdots + l_k}\right)^p \|f\|_p^p .$$

Let $\{n_k\}$ be a sequence of positive integers. Reorganize the sequence into blocks B_k and C_k such that $|B_k| = b_k$, $|C_k| = d_k$, the block B_k is to the right of C_{k-1} and the block C_k is to the right of B_k:

$$\cdots \underbrace{[\cdots\cdots\cdots]}_{B_k} \cdots \underbrace{[\cdots\cdots]}_{C_k} \cdots \underbrace{[\cdots\cdots\cdots\cdots]}_{B_{k+1}} \cdots$$

Then $\{n_k\} = \bigcup_{k=1}^\infty (B_k \cup C_k)$. Suppose there are blocks D_k of integers such that $D_k \cap C_k = \varnothing$, $|D_k| = d_k$ and, if $C_k = \{n_{i_k}, n_{i_k+1}, \ldots, n_{i_k+d_k-1}\}$ then, $D_k \subset (n_{i_k}, n_{i_k+d_k})$.

Corollary 1.7. *With the above notation, assume that* $\{n_k\}$ *is a universally good sequence in* $L^p(X)$, $p \geq 1$, *and that*

$$\sum_{k=1}^{\infty}\left(\frac{d_1+\cdots+d_k}{b_1+\cdots+b_k}\right)^p < \infty \quad \text{if } p < \infty,$$

or

$$\lim_{k\to\infty}\frac{d_1+\cdots+d_k}{b_1+\cdots+d_k} = 0 \quad \text{if } p = \infty.$$

Then, the perturbed sequence $\bigcup_{k=1}^{\infty}(B_k \cup C_k \cup D_k)$ *is also a universally good sequence in* $L^p(X)$.

We leave it to the reader to check that the same arguments outlined in Theorem 1.5 and Remark 1.6 work in this context as well.

Lemma 1.8. *Let* $B_k = [n_k, n_k+1, \ldots, n_k+l_k-1]$ *be blocks of consecutive integers such that there exists* r *with* $l_k \geq n_{k-r}$ *for all big enough values of* k. *Then, the sequence* $\bigcup_{k=1}^{\infty} B_k$ *is universally good in* L^1.

This lemma gives the basic construction of a good universal sequence in L^1. See Bellow-Jones-Rosenblatt [3, Corollary 3] and Bellow-Losert [2, Corollary 3.3]. The next proposition gives a criterion by which to perturb a good universal sequence in L^p defined by blocks B_k to obtain a bad universal sequence in L^p.

Proposition 1.9. *Let* $p \geq 1$, *and let* $\{B_k\}$ *be a family of blocks such that the sequence defined by them is good universal in* L^p. *Let* $l_k = |B_k|$, *and assume that* $l_1 + \cdots + l_{k-1} \leq Cl_k$. *Suppose that in between the blocks* B_k*'s one can insert blocks* D_k*'s of length* d_k *such that* $d_k = c_k l_k$, *where* $c_k < 1$ *for all* k *but* $\lim_{k\to\infty} c_k(k/\log^2 k)^{1/p} = \infty$, *and such that, for all* $u \in D_k$, $T_\theta^u(p_k) < 1/k$ *for some* θ *irrational. Then the perturbed sequence* $\bigcup_{k=1}^{\infty}(B_k \cup D_k)$ *is bad universal in* L^p.

Proof. By Conze's Principle, it is enough to consider the dynamical system (X, β, m, T) where $X = [0, 1)$ with the Lebesgue measure and $T(x) = x + \theta \bmod 1$, θ irrational. Let

$$f(x) = \left(\frac{1}{x\log^2(x/2)}\right)^{1/p}\chi_{(0,1)}(x).$$

Then $f \in L^p(X)$. By construction of the J_k's and p_k's in (1.a), if $x \in J_k$ and $u \in D_k$, then $T_\theta^u(x) < 2/k$, so

$$T_\theta^u f(x) \geq \left(\frac{k}{2\log^2 k}\right)^{1/p}.$$

Since $d_k < l_k$ and $l_1 + \cdots + l_{k-1} \leq Cl_k$, one has

$$l_1 + \cdots + l_k + d_1 + \cdots + d_k \leq 2(l_1 + \cdots + l_k) \leq 2(C+1)l_k = C'l_k.$$

Then

$$\frac{1}{(l_1+\cdots+l_k+d_1+\cdots+d_k)}\sum_{m\in((B_1\cup D_1)\cup\cdots\cup(B_k\cup D_k))} T_\theta^m f(x)$$

$$> \frac{1}{C'l_k}\sum_{m\in D_k} T_\theta^m f(x) \geq c\frac{d_k}{l_k}\left(\frac{k}{2\log^2 k}\right)^{1/p} = cc_k\left(\frac{k}{\log^2 k}\right)^{1/p}.$$

Since $\lim_{k\to\infty} c_k(k/\log^2 k)^{1/p} = \infty$ and every point in $(0, 1)$ is in infinitely many J_k's,

$$\sup_{k\in\mathbb{N}} \frac{1}{(l_1 + \cdots + l_k + d_1 + \cdots + d_k)} \sum_{m\in((B_1\cup D_1)\cup\cdots\cup(B_k\cup D_k))} T_\theta^m f(x) = \infty$$

for every x in $(0, 1)$. Thus, $C(\bigcup_{k=1}^\infty (B_k \cup D_k), p) = \infty$ in Conze's Principle and the sequence $\bigcup_{k=1}^\infty (B_k \cup D_k)$ is bad universal in L^p. $\square$

2. Examples

Basic construction. By employing Lemma 1.8, one can construct blocks B_k of consecutive numbers which are good universal in L^1. Then, one can perturb those blocks in the sense of Theorem 1.3 to obtain a sequence which preserves the good universal property in an appropriate L^p space. See Bellow [1] and Emerson [7].

Fix $q > 1$ and a sequence $\{c_k\}$ with $\sum_{k=1}^\infty c_k^q < \infty$. Then construct the blocks B_k and D_k as follows: let $n_1 = 1$, $l_1 = 1$, $u_1^1 = 1$, $d_1 = 1$, and proceed by induction. Suppose $n_1, \dots, n_{k-1}$; $l_1, \dots, l_{k-1}$; $D_1, \dots, D_{k-1}$; $d_1, \dots, d_{k-1}$ have already been defined. The block B_k will consist of l_k consecutive numbers, $B_k = [n_k, n_k + 1, \dots, n_k + l_k - 1]$, where n_k is chosen to the right of D_{k-1} and l_k is taken so large that it satisfies:

(a) $l_k \geq n_{k-1}$,
(b) $l_k \geq k \cdot l_{k-1} \geq l_1 + \cdots + l_{k-1}$.

Condition (a) guarantees that the sequence defined by the blocks B_k is good universal in L^1. As in the Proposition 1.4, set $d_k = c_k l_k$, and let D_k consist of d_k numbers to the right of $n_k + l_k$. Then, the sequence $\bigcup_{k=1}^\infty (B_k \cup D_k)$ is good universal in L^q.

Example 1. In the basic construction above, let the blocks D_k consist of the first d_k numbers to the right of B_k such that $T_\theta^u(p_k) < 1/k$, for a fixed irrational θ. For a fixed $p \geq 1$, take $c_k = (\log^3 k/k)^{1/p}$. Then, for all $q > p$, $\sum_{k=1}^\infty c_k^q < \infty$. Therefore, $\bigcup_{k=1}^\infty (B_k \cup D_k)$ is a good universal sequence in L^q. But since

$$\lim_{k\to\infty} c_k(k/\log^2 k)^{1/p} = \infty,$$

the new sequence is bad universal in L^p by Proposition 1.9.

Example 2. Build blocks B_k's and D_k's as in Example 1, but now let $d_k = l_k/\log k$. Then, $\lim_{k\to\infty}(d_k/l_k) = 0$. So, by Theorem 1.5, the sequence $\bigcup_{k=1}^\infty (B_k \cup D_k)$ is good universal in L^∞. However, for any $\alpha > 0$,

$$\lim_{k\to\infty} \frac{d_k}{(l_1 + \cdots + l_k + d_1 + \cdots + d_k)} k^\alpha = \infty.$$

This forces the sequence to be universally bad in all L^p, $p < \infty$. Indeed, let $0 < \alpha < 1$ and

$$f(x) = \begin{cases} 1/x^\alpha & \text{if } x \in (0, 1], \\ 0 & \text{if } x = 0. \end{cases}$$

Then $f \in L^p(X)$ for all $1 \leq p < 1/\alpha$. By the construction of the blocks D_k, $f(T^u(x)) \geq (k/2)^\alpha$ for all $x \in J_k$, $u \in D_k$. Therefore,

$$\frac{1}{l_1 + \cdots + l_k + d_1 + \cdots + d_k} \sum_{u \in \bigcup_{j=1}^k (B_j \cup D_j)} f(T^u(x)) \geq \frac{d_k}{(l_1 + \cdots + l_k + d_1 + \cdots + d_k)} \left(\frac{k}{2}\right)^\alpha .$$

And by the above, this diverges as $k \to \infty$. Consequently the maximal function for f diverges a.e.

From Theorem 1.3 it is clear that the blocks B_k need not consist of consecutive numbers. However, the sequence $\bigcup_{k \in \mathbb{N}} B_k$ should be good in some L^p space. Following this idea, one can construct sequences with gaps going to infinity with good behavior in some $L^p(X)$ spaces and bad behavior in others. Máte Wierdl has conjectured that, if $\{n_k\}_{k \subset \mathbb{N}}$ is a sequence whose gaps increase to infinity, then the sequence is bad universal in L^1. This example is interesting in connection with this conjecture.

Let $[x]$ denote the integer part of x, i.e., $[x] \in \mathbb{Z}$ and $[x] \leq x < [x] + 1$. Also, let $\{x\}$ denote the fractional part of x, i.e., $\{x\} = x - [x]$. The following proposition gives a crude estimate on the gaps of a certain type of sequences. The argument is due to Rosenblatt.

Proposition 2.1. *Let $\theta \in (0, 1)$ be irrational. Then there is a constant $C(k) \geq 1$, depending on k and θ, such that the gaps of the sequence*

$$\{m_j\}_{j \in \mathbb{N}} = \{n \in \mathbb{N}\colon \{n\theta\} < 1/k\}$$

are bounded by $C(k)k$, that is, $m_{j+1} - m_j \leq C(k)k$ for all j.

Proof. Let $M \in \mathbb{N}$ be the first number so that $\{M\theta\} \in (1/2k, 1/k)$, and denote $M_0 = -[M\theta] \in \mathbb{Z}$. We will estimate $m_{j+1} - m_j$. Fix j and let $N = -[m_j\theta]$. Then

$$M_0 + M\theta \in (1/2k, 1/k) \quad \text{and} \quad N + m_j\theta \in (0, 1/k).$$

Then, for any $R \in \mathbb{N}$,

$$R/2k < N + m_j\theta + RM_0 + RM\theta < (R+1)/k .$$

If $R < k - 1$, then $N + m_j\theta + RM_0 + RM\theta < 1$. But if $R > 2k$, then $N + m_j\theta + RM_0 + RM\theta > 1$. Let R_0 be the first integer number in $[k-1, 2k]$ such that $N + m_j\theta + R_0M_0 + R_0M\theta > 1$. But then

$$\begin{aligned} 1 &< N + m_j\theta + R_0M_0 + R_0M\theta \\ &= N + m_j\theta + (R_0 - 1)M_0 + (R_0 - 1)M\theta + M_0 + M\theta \\ &< 1 + M_0 + M\theta < 1 + 1/k . \end{aligned}$$

This means that $m_j + R_0M$ is an element of the sequence (m_s). Therefore, $m_{j+1} - m_j \leq R_0M \leq 2Mk$. Put $C(k) = 2M$. □

Example 3. Let $\{n_k\}$ be a universally good sequence in $L^p(X)$, $p > 1$, with gaps tending to infinity, i.e., $n_{j+1} - n_j \to \infty$ as $n \to \infty$. Let $\varphi_k = C(k) \cdot k$ be the bound for the gaps of the sequence $\{n \in \mathbb{N}\colon \{n\theta\} \in (0, 1/k)\}$ as in the above proposition. With the notation preceding Corollary 1.7, organize the sequence $\{n_k\}$ into blocks B_k and C_k with $|B_k| = b_k$ and $|C_k| = d_k$. These blocks are constructed inductively in such a way that

$$b_k \geq kb_{k-1} \quad \text{and} \quad d_k = b_k\alpha_k \tag{2.a}$$

and such that, if $n_i \in C_k$, then

$$n_{i+1} - n_i \geq 4\varphi_k . \tag{2.b}$$

Choose the blocks D_k in the following way: for each $n_i \in C_k$, choose an integer $u \in (n_i, n_{i+1})$ such that $T^u(p_k) \in (0, 1/k)$ and $d(u, \{n_k\}) > k$. This can be done since the sequence $\{n: \{n\theta + p_k\} \in (0, 1/k)\}$ has gaps bounded by φ_k, the length of the interval (n_i, n_{i+1}) is at least $4\varphi_k$, and $\varphi_k \geq k$ by Proposition 2.1. As in the previous examples, the sequence $\bigcup_{k=1}^{\infty}(B_k \cup C_k \cup D_k)$ is universally good in $L^q(X)$, $q \geq p$, if $\sum_{k=1}^{\infty} \alpha_k^q < \infty$. Also, it is universally bad in $L^q(X)$ if $\lim_{k\to\infty} \alpha_k(k/2\log^2 k)^{1/q} = \infty$ because the same arguments in Proposition 1.9 apply to this case.

Taking $\alpha_k = (\log^3 k/k)^{1/p}$, the sequence $\bigcup_{k=1}^{\infty}(B_k \cup C_k \cup D_k)$ is universally good in $L^q(X)$ for all $q > p$ and universally bad in $L^p(X)$. Also, if $\{n_k\}$ is universally good in all $L^p(X)$, $p > 1$, and $\alpha_k = \log^3 k/k$, the sequence $\bigcup_{k=1}^{\infty}(B_k \cup C_k \cup D_k)$ is universally good in all $L^p(X)$, $p > 1$, but universally bad in L^1.

Consider the sequence $\{n^2\}_{n=1}^{\infty}$. This sequence was shown by Bourgain [4, 5] to be universally good in all L^p, $p > 1$. However, its behavior in L^1 is still unknown. The above example gives a method of perturbing any sequence with gaps going to infinity and which is universally good in all L^p spaces, $p > 1$, such that the perturbed one will also have gaps going to infinity and be universally good in L^p for all $p > 1$ but universally bad in L^1. This result was also shown by Wierdl [11] by a different type of perturbation.

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