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A NONSPECTRAL DENSE BANACH SUBALGEBRA OF THE IRRATIONAL ROTATION ALGEBRA

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ABSTRACT. We give an example of a dense, simple, unital Banach subalgebra A of the irrational rotation C^*-algebra B, such that A is not a spectral subalgebra of B. This answers a question posed by T. W. Palmer (*Spectral algebras*, Rocky Mountain J. Math. **22** (1992), 293–328).

If A is a subalgebra of an algebra B (both algebras over the complex numbers), we say that A is a *spectral subalgebra* of B if the quasi-invertible elements of A are precisely the quasi-invertible elements of B which lie in A. In the language of [3], this is equivalent to saying that A is a *spectral invariant subalgebra* of B.

There are many known examples of dense unital Banach subalgebras of C^*-algebras which are not spectral. For example, see Example 3.1 of [3]. The example we give here is of interest because the Banach algebra is simple and, thus, answers Question 5.12 of [1] in the negative.

Recall that the irrational rotation algebra associated with an irrational real number θ is the C^*-crossed product of the integers $\mathbb{Z}$ with the commutative C^*-algebra of continuous functions on the circle $C(\mathbb{T})$, where $n \in \mathbb{Z}$ acts via $\alpha_n(\varphi)(z) = \varphi(z - n\theta)$, for $\varphi \in C(\mathbb{T})$ and $z \in \mathbb{T}$. Let $B = \mathbb{Z} \rtimes C(\mathbb{T})$ denote this crossed product.

Let A be the set of functions F from $\mathbb{Z}$ to $C(\mathbb{T})$ which satisfy the integrability condition

$$\|F\|_A = \sum_{n\in\mathbb{Z}} e^{|n|}\|F(n)\|_\infty < \infty,$$

where $\| \ \|_\infty$ denotes the sup norm on $C(\mathbb{T})$. Then A is complete for the norm $\| \ \|_A$ and is a Banach algebra. The algebra A is contained in $L^1(\mathbb{Z}, C(\mathbb{T}))$ with dense and continuous inclusion and, hence, is contained in B with dense and continuous inclusion. Recall that the multiplication (in both A and B) is given by

$$F * G(n, z) = \sum_{m\in\mathbb{Z}} F(n, z)G(n-m, z-m\theta), \qquad F, G \in A, n \in \mathbb{Z}, z \in \mathbb{T}.$$

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Let $u_n = \delta_n \otimes 1 \in A$ denote the delta function at $n \in \mathbb{Z}$ tensored with the identity in $C(\mathbb{T})$. Then u_0 is the unit in both A and B.

Theorem 1. *The Banach algebra A is simple.*

Proof. We imitate the argument of [2]. Define a continuous linear map $P: A \to C(\mathbb{T}) \subseteq A$ by $P(F) = F(0)$. Note that $\|P(F)\|_A \le \|F\|_A$ for $F \in A$. Let J be a closed two-sided ideal in A, which is not equal to A. Since $\mathbb{Z}$ acts ergodically on $\mathbb{T}$, we know that $C(\mathbb{T})$ has no nontrivial closed $\mathbb{Z}$-invariant ideals. Hence, $J \cap C(\mathbb{T}) = 0$.

We show that $P(J) = 0$. It suffices to show that $P(J) \subseteq J$. Let $\epsilon > 0$ and $F \in A$. Let N be a sufficiently large integer for which

$$\sum_{|n|>N} e^{|n|}\|F(n)\|_\infty < \epsilon.$$

Define $F_1 \in A$ by $F_1(n) = 0$ if $|n| > N$, and $F_1(n) = F(n)$ if $|n| \le N$. By the proof of Lemma 6 of [2], there exists unimodular functions $\theta_1, \dots, \theta_M \in C(\mathbb{T})$ such that

$$P(F_1) = \frac{1}{M}\sum_{n=1}^{M} \theta_n^* F_1 \theta_n.$$

(Here unimodular means that $|\theta_i(z)| = 1$ for each $z \in \mathbb{T}$ and $i = 1, \dots, M$.) Hence,

$$(*) \qquad \left\| P(F) - \frac{1}{M}\sum_{n=1}^{M} \theta_n^* F \theta_n \right\|_A \le \|P(F - F_1)\|_A + \|F - F_1\|_A < 2\epsilon.$$

Now if $F \in J$, $(*)$ shows that $P(F)$ can be approximated arbitrarily closely by elements of J. Since J is closed, this shows that $P(F) \in J$. Hence, $P(J) \subseteq J$ and $P(J) = 0$.

If $P(Fu_n) = 0$ for all n, then $F(n) = 0$ for all n and so $F = 0$. Since J is a two-sided ideal and $P(J) = 0$, we have $P(Ju_n) = 0$ for all n. Hence, $J = 0$ and A is simple. □

Theorem 2. *The Banach algebra A is not a spectral subalgebra of B.*

Proof. We construct an algebraically irreducible A-module which is not contained in any *-representation of B on a Hilbert space. By Corollary 1.5 of [3], it will follow that A is not a spectral subalgebra of B.

Let E be the Banach A-module $C(\mathbb{T})$ with sup norm and with (continuous) action of A given by

$$(F\varphi)(z) = \sum_n F(n, z)e^n \varphi(z - n\theta), \qquad \varphi \in E,\ F \in A,\ z \in \mathbb{T}.$$

We show that E is in fact algebraically irreducible. Let $\varphi \in E$ be not identically equal to zero. Since the complex conjugate of φ is in A, the algebraic span $A\varphi$ contains $|\varphi|^2$, which we denote by ψ. Note $u_n\psi(z) = e^n\psi(z - n\theta)$. Since θ is irrational and $\mathbb{T}$ is compact, there exists finitely many $n_1, \dots, n_k \in \mathbb{Z}$ such that the sum of $u_{n_i}\psi$ from $i = 1$ to k never vanishes on $\mathbb{T}$. If χ is this sum, then $1/\chi$ is in $C(\mathbb{T}) \subseteq A$, so $1 \in A\varphi$ and, hence, $E = A\varphi$. This proves that E is algebraically irreducible.

It remains to show that no *-representation of B on a Hilbert space contains E. But the action of $\mathbb{Z}$ on $1 \in E$ is given by $u_n 1 = e^n 1$. Clearly the Hilbert space could not have a unitary, or even isometric, action of $\mathbb{Z}$. $\square$

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