A NONSPECTRAL DENSE BANACH SUBALGEBRA
OF THE IRRATIONAL ROTATION ALGEBRA

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Abstract. We give an example of a dense, simple, unital Banach subalgebra \( A \)
of the irrational rotation \( C^* \)-algebra \( B \), such that \( A \) is not a spectral subalgebra of \( B \). This answers a question posed by T. W. Palmer (Spectral algebras, Rocky Mountain J. Math. 22 (1992), 293–328).

If \( A \) is a subalgebra of an algebra \( B \) (both algebras over the complex numbers), we say that \( A \) is a spectral subalgebra of \( B \) if the quasi-invertible elements of \( A \) are precisely the quasi-invertible elements of \( B \) which lie in \( A \). In the language of [3], this is equivalent to saying that \( A \) is a spectral invariant subalgebra of \( B \).

There are many known examples of dense unital Banach subalgebras of \( C^* \)-algebras which are not spectral. For example, see Example 3.1 of [3]. The example we give here is of interest because the Banach algebra is simple and, thus, answers Question 5.12 of [1] in the negative.

Recall that the irrational rotation algebra associated with an irrational real number \( \theta \) is the \( C^* \)-crossed product of the integers \( \mathbb{Z} \) with the commutative \( C^* \)-algebra of continuous functions on the circle \( C(\mathbb{T}) \), where \( n \in \mathbb{Z} \) acts via \( \alpha_n(\phi)(z) = \phi(z - n\theta) \), for \( \phi \in C(\mathbb{T}) \) and \( z \in \mathbb{T} \). Let \( B = \mathbb{Z} \times C(\mathbb{T}) \) denote this crossed product.

Let \( A \) be the set of functions \( F \) from \( \mathbb{Z} \) to \( C(\mathbb{T}) \) which satisfy the integrability condition
\[
\|F\|_A = \sum_{n \in \mathbb{Z}} e^{\|n\|} \|F(n)\|_{C(\mathbb{T})} < \infty,
\]
where \( \| \|_\infty \) denotes the sup norm on \( C(\mathbb{T}) \). Then \( A \) is complete for the norm \( \|
\|_A \) and is a Banach algebra. The algebra \( A \) is contained in \( L^1(\mathbb{Z}, C(\mathbb{T})) \) with dense and continuous inclusion and, hence, is contained in \( B \) with dense and continuous inclusion. Recall that the multiplication (in both \( A \) and \( B \)) is given by
\[
F * G(n, z) = \sum_{m \in \mathbb{Z}} F(n, z)G(n - m, z - m\theta), \quad F, G \in A, n \in \mathbb{Z}, z \in \mathbb{T}.
\]
Let \( u_n = \delta_n \otimes 1 \in A \) denote the delta function at \( n \in \mathbb{Z} \) tensored with the identity in \( C(\mathbb{T}) \). Then \( u_0 \) is the unit in both \( A \) and \( B \).

**Theorem 1.** The Banach algebra \( A \) is simple.

**Proof.** We imitate the argument of [2]. Define a continuous linear map \( P: A \to C(\mathbb{T}) \subseteq A \) by \( P(F) = F(0) \). Note that \( \|P(F)\|_A \leq \|F\|_A \) for \( F \in A \). Let \( J \) be a closed two-sided ideal in \( A \), which is not equal to \( A \). Since \( \mathbb{Z} \) acts ergodically on \( \mathbb{T} \), we know that \( C(\mathbb{T}) \) has no nontrivial closed \( \mathbb{Z} \)-invariant ideals. Hence, \( J \cap C(\mathbb{T}) = 0 \).

We show that \( P(J) = 0 \). It suffices to show that \( P(J) \subseteq J \). Let \( \epsilon > 0 \) and \( F \in A \). Let \( N \) be a sufficiently large integer for which

\[
\sum_{|n| > N} e^{\epsilon n} \|F(n)\|_\infty < \epsilon.
\]

Define \( F_1 \in A \) by \( F_1(n) = 0 \) if \( |n| > N \), and \( F_1(n) = F(n) \) if \( |n| \leq N \). By the proof of Lemma 6 of [2], there exists unimodular functions \( \theta_1, \ldots, \theta_M \in C(\mathbb{T}) \) such that

\[
\sum_{i=1}^{M} \theta_i^n F_1 \theta_n.
\]

(Here unimodular means that \( |\theta_i(z)| = 1 \) for each \( z \in \mathbb{T} \) and \( i = 1, \ldots, M \).) Hence,

\[
\left\|P(F) - \frac{1}{M} \sum_{n=1}^{M} \theta_n^* F_1 \theta_n\right\|_A \leq \|P(F - F_1)\|_A + \|F - F_1\|_A < 2\epsilon.
\]

Now if \( F \in J \), \( (*) \) shows that \( P(F) \) can be approximated arbitrarily closely by elements of \( J \). Since \( J \) is closed, this shows that \( P(F) \in J \). Hence, \( P(J) \subseteq J \) and \( P(J) = 0 \).

If \( P(F u_n) = 0 \) for all \( n \), then \( F(n) = 0 \) for all \( n \) and so \( F = 0 \). Since \( J \) is a two-sided ideal and \( P(J) = 0 \), we have \( P(J u_n) = 0 \) for all \( n \). Hence, \( J = 0 \) and \( A \) is simple. \( \square \)

**Theorem 2.** The Banach algebra \( A \) is not a spectral subalgebra of \( B \).

**Proof.** We construct an algebraically irreducible \( A \)-module which is not contained in any \( * \)-representation of \( B \) on a Hilbert space. By Corollary 1.5 of [3], it will follow that \( A \) is not a spectral subalgebra of \( B \).

Let \( E \) be the Banach \( A \)-module \( C(\mathbb{T}) \) with sup norm and with (continuous) action of \( A \) given by

\[
(F \varphi)(z) = \sum_n F(n, z) e^{n \varphi(z - n\theta)}, \quad \varphi \in E, F \in A, z \in \mathbb{T}.
\]

We show that \( E \) is in fact algebraically irreducible. Let \( \varphi \in E \) be not identically equal to zero. Since the complex conjugate of \( \varphi \) is in \( A \), the algebraic span \( A \varphi \) contains \( |\varphi|^2 \), which we denote by \( \psi \). Note \( u_n \psi(z) = e^{n \psi}(z - n\theta) \). Since \( \theta \) is irrational and \( \mathbb{T} \) is compact, there exists finitely many \( n_1, \ldots, n_k \in \mathbb{Z} \) such that the sum of \( u_n \psi \) from \( i = 1 \) to \( k \) never vanishes on \( \mathbb{T} \). If \( \chi \) is this sum, then \( 1/\chi \) is in \( C(\mathbb{T}) \subseteq A \), so \( 1 \in A \varphi \) and, hence, \( E = A \varphi \). This proves that \( E \) is algebraically irreducible.
It remains to show that no $^*$-representation of $B$ on a Hilbert space contains $E$. But the action of $Z$ on $1 \in E$ is given by $u^n 1 = e^n 1$. Clearly the Hilbert space could not have a unitary, or even isometric, action of $Z$. □

REFERENCES

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