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ON A FUNCTIONAL EQUATION CONNECTED WITH RAO'S QUADRATIC ENTROPY

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ABSTRACT. We determine the general solution of the functional equation

$$f\left(\frac{x+y}{2}\right)+f\left(\frac{x-y}{2}\right)=2f\left(\frac{x}{2}\right)+2f\left(\frac{y}{2}\right)+\lambda f(x)f(y),$$

for $f\colon[-1, 1]\to\mathbf{R}$. This equation was used by Lau in order to characterize Rao's quadratic entropies. The general solution is obtained here as a special case of a more general result for f mapping a neighborhood of 0 in a linear topological space into a field.

1. INTRODUCTION AND PRELIMINARIES

Let $\mathbf{R}$ and $\mathbf{C}$ denote the set of reals and the set of complex numbers, respectively. Let I denote the closed interval $[-1, 1]$, and let I° denote the open interval $]-1, 1[$. In connection with a characterization of quadratic entropies of Rao, Lau [3] obtained the solution of the functional equation

$$f\left(\frac{x+y}{2}\right)+f\left(\frac{x-y}{2}\right)=2f\left(\frac{x}{2}\right)+2f\left(\frac{y}{2}\right)+\lambda f(x)f(y) \tag{1}$$

under the conditions that $f\colon I\to\mathbf{R}$ is even, continuous, and nonnegative on I, infinitely differentiable on I°, and that $f(0)=0$, $f(1)=1$, and $\lambda\geq 0$. In this paper, we find the general solution of (1), that is, without any assumption concerning f or λ. This result is obtained as a corollary of a theorem in a more abstract setting.

To begin, we establish a simple result about subgroups.

Lemma. *Let* $\mathbf{X}$ *be a real linear space, and let* $\mathbf{S}$ *be a subgroup of* $(\mathbf{X}, +)$ *satisfying the following property*:

(P) *If* $x, y\in\mathbf{X}\backslash\mathbf{S}$, *then exactly one of* $x+y$, $x-y$ *belongs to* $\mathbf{S}$.

Then $\mathbf{S}=\mathbf{X}$.

Proof. Suppose there exists $x\in\mathbf{X}\backslash\mathbf{S}$. Then by (P), exactly one of $2x$, 0 belongs to $\mathbf{S}$. Since $\mathbf{S}$ is a subgroup, $0\in\mathbf{S}$ and hence $2x\in\mathbf{X}\backslash\mathbf{S}$. Now

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$2x$, $x \in \mathbf{X}\backslash\mathbf{S}$, so by (P) exactly one of $3x$, x belongs to $\mathbf{S}$. But $x \in \mathbf{X}\backslash\mathbf{S}$, so $3x \in \mathbf{S}$. Thus $3x \in \mathbf{S}$ for any $x \in \mathbf{X}\backslash\mathbf{S}$.

On the other hand, $3x \in \mathbf{S}$ for any $x \in \mathbf{S}$, since $\mathbf{S}$ is a subgroup. Therefore $3x \in \mathbf{S}$ for all $x \in \mathbf{X}$. Now, given any $y \in \mathbf{X}$, let $x = y/3 \in \mathbf{X}$; we conclude that $y = 3x \in \mathbf{S}$. Hence $\mathbf{X} = \mathbf{S}$.

Remark 1. The proof shows that the same result holds in any group in which division by 3 is possible.

2. Main results

Let $\mathbf{X}$ be a real topological linear space, and let $\mathbf{K}$ be a (commutative) field of characteristic different from 2.

Theorem. *Let* $\mathbf{U}$ *be a balanced, convex, open neighborhood of* 0 *in* $\mathbf{X}$. *The general solution* $f: \mathbf{U} \to \mathbf{K}$ *of the functional equation* (1) (*holding for all* $x, y \in \mathbf{U}$) *is given by*

$$f(x) = A(x, x) \quad \text{for all } x \in \mathbf{U}, \tag{2}$$

if $\lambda = 0$, *where* $A: \mathbf{X} \times \mathbf{X} \to \mathbf{K}$ *is an arbitrary symmetric biadditive map; or, if* $\lambda \neq 0$, *by*

$$f(x) = 0 \quad \text{for all } x \in \mathbf{U}, \tag{3}$$

or by

$$f(x) = -2\lambda^{-1} \quad \text{for all } x \in \mathbf{U}. \tag{4}$$

Proof. It is easy to check that (2)–(4) are solutions of (1) under the respective conditions.

For the converse, suppose first that $\lambda = 0$. Then (1) reduces to the well-known quadratic functional equation

$$f\left(\frac{x+y}{2}\right) + f\left(\frac{x-y}{2}\right) = 2f\left(\frac{x}{2}\right) + 2f\left(\frac{y}{2}\right) \tag{QE}$$

for all $x, y \in \mathbf{U}$. A careful examination of the proof of Theorem 2 in [1] reveals that its local version holds too. That is, we get $f(x) = A(x, x)$ for all $x \in \mathbf{U}$, where A is an arbitrary symmetric biadditive map from $\mathbf{U}^2$ into $\mathbf{K}$. Given any such $\mathbf{U}$, any biadditive map from $\mathbf{U}^2$ into $\mathbf{K}$ has a unique extension to a biadditive map on $\mathbf{X}^2$; the extension is symmetric if the original map is. Thus we obtain (2).

Henceforth, we assume that $\lambda \neq 0$. Putting $y = 0$ in (1), we obtain

$$f(0)\{2 + \lambda f(x)\} = 0 \quad \text{for all } x \in \mathbf{U}.$$

If $f(0) \neq 0$, then we get solution (4). Setting this aside, we assume now that

$$f(0) = 0. \tag{5}$$

Our goal is to show that $f = 0$. Interchanging x and y in (1) and comparing the result to (1), we find that $f(\frac{x-y}{2}) = f(\frac{y-x}{2})$; hence f is necessarily an even function on $\mathbf{U}$. Next, we define $F: \mathbf{U}^2 \to \mathbf{K}$ by

$$F(x, y) := f\left(\frac{x+y}{2}\right) + f\left(\frac{x-y}{2}\right) - 2f\left(\frac{x}{2}\right) - 2f\left(\frac{y}{2}\right) \tag{6}$$

for all $x, y \in \mathbf{U}$. Then clearly F satisfies

(7) $F(x+y, z)+F(x-y, z)+2F(y, z) = F(x, y+z)+F(x, y-z)-2F(x, y)$

for all $x, y, z \in \frac{1}{2}\mathbf{U}$. By (6) and (1), we also have

$$F(x, y) = \lambda f(x)f(y) \quad \text{for all } x, y \in \mathbf{U}. \tag{8}$$

Substituting this into (7) and using the fact that $\lambda \neq 0$, we get

(9) $\{f(x+y)+f(x-y)-2f(y)\}f(z) = f(x)\{f(y+z)+f(y-z)-2f(y)\}$

for all $x, y, z \in \frac{1}{2}\mathbf{U}$. We consider two cases.

Case 1. Suppose $f(z_0) \neq 0$ for some $z_0 \in \frac{1}{2}\mathbf{U}$. We shall show that this case is impossible. Putting $z = z_0$ in (9), we get for appropriate g

$$f(x+y)+f(x-y)-2f(x)-2f(y) = f(x)g(y) \tag{10}$$

for all $x, y \in \frac{1}{2}\mathbf{U}$. Since f is even, the left side of (10) is symmetric in x and y. Therefore it can be rewritten as

$$f(x+y)+f(x-y)-2f(x)-2f(y) = \gamma f(x)f(y) \tag{11}$$

for all $x, y \in \frac{1}{2}\mathbf{U}$ and for some constant γ; or

$$f\left(\frac{x+y}{2}\right)+f\left(\frac{x-y}{2}\right)-2f\left(\frac{x}{2}\right)-2f\left(\frac{y}{2}\right) = \gamma f\left(\frac{x}{2}\right)f\left(\frac{y}{2}\right) \tag{12}$$

for all $x, y \in \mathbf{U}$. Comparing (12) with (1), we see that

$$\gamma f\left(\frac{x}{2}\right)f\left(\frac{y}{2}\right) = \lambda f(x)f(y) \quad \text{for all } x, y \in \mathbf{U}. \tag{13}$$

From (13), we deduce that $\gamma \neq 0$, since $\lambda \neq 0$ and $f \neq 0$. Furthermore, with $x = y = 2z_0$ we see that $f(2z_0) \neq 0$. So, putting $y = 2z_0$ in (13), we obtain

$$f(x) = \beta f\left(\frac{x}{2}\right) \quad \text{for all } x \in \mathbf{U}, \tag{14}$$

where $\beta := \lambda^{-1}\gamma f(z_0)f(2z_0)^{-1} \neq 0$. On the other hand, (1) with $y = x$ yields (recalling $f(0) = 0$ from (5))

$$f(x) = 4f\left(\frac{x}{2}\right) + \lambda f(x)^2 \quad \text{for all } x \in \mathbf{U}. \tag{15}$$

Eliminating $f(x)$ from the system (14), (15), we get

$$f\left(\frac{x}{2}\right)\left\{\beta - 4 - \lambda\beta^2 f\left(\frac{x}{2}\right)\right\} = 0 \quad \text{for all } x \in \mathbf{U}.$$

Thus for every $x \in \frac{1}{2}\mathbf{U}$, either $f(x) = 0$ or $f(x) = \lambda^{-1}\beta^{-2}(\beta - 4)$. Because of this alternative and since $f \neq 0$, now (14) implies that $\beta = 1$. Hence

$$f(x) = f\left(\frac{x}{2}\right) \quad \text{for all } x \in \mathbf{U}; \tag{16}$$

$$f(\tfrac{1}{2}\mathbf{U}) = f(\mathbf{U}) \subset \{0, -3\lambda^{-1}\}. \tag{17}$$

So we can partition $\mathbf{U}$ into two sets, say $\mathbf{N}$ and $\mathbf{S}$, where

$$\mathbf{N} = \{x \in \mathbf{U} | f(x) = -3\lambda^{-1}\} \quad \text{and} \quad \mathbf{S} = \{x \in \mathbf{U} | f(x) = 0\},$$

so that $0 \in \mathbf{S}$, $\mathbf{N} \cap \mathbf{S} = \varnothing$, and $\mathbf{N} \cup \mathbf{S} = \mathbf{U}$. We observe that both $\mathbf{N}$ and $\mathbf{S}$ are closed under division by 2, because of (16). Let us state some further consequences of (1) with respect to $\mathbf{N}$ and $\mathbf{S}$.

Suppose $x, y \in \mathbf{S}$. Then (1) and (16) imply that

$$f\left(\frac{x+y}{2}\right) + f\left(\frac{x-y}{2}\right) = 0.$$

By (17), we conclude that both $\frac{x+y}{2}$ and $\frac{x-y}{2}$ are in $\mathbf{S}$. If we extend $\mathbf{S}$ to $\overline{\mathbf{S}} = \bigcup_{n=0}^{\infty} 2^n\mathbf{S}$, this means that $\overline{\mathbf{S}}$ is a subgroup of $(\mathbf{X}, +)$.

Now suppose $x, y \in \mathbf{N}$. Then (1), (16) imply

$$f\left(\frac{x+y}{2}\right) + f\left(\frac{x-y}{2}\right) = -3\lambda^{-1}.$$

Then (17) requires that exactly one of $\frac{x+y}{2}$, $\frac{x-y}{2}$ belongs to $\mathbf{N}$, while the other is in $\mathbf{S}$. Extending $\mathbf{N}$ to

$$\overline{\mathbf{N}} = \bigcup_{n=0}^{\infty} 2^n\mathbf{N} = \mathbf{X}\backslash\overline{\mathbf{S}},$$

we see that $\overline{\mathbf{S}}$ is a subgroup of $\mathbf{X}$ fulfilling the property (P) of our lemma. Hence $\overline{\mathbf{S}} = \mathbf{X}$, and therefore $\mathbf{S} = \mathbf{U}$. But this contradicts the existence of z_0. Therefore this case cannot occur.

Case 2. Suppose $f(\frac{1}{2}\mathbf{U}) = \{0\}$. Let $\mathfrak{R}$ be an arbitrary ray $\{tz | t > 0\}$ for some $z \in \mathbf{X}\backslash\{0\}$. If we restrict y to $\mathfrak{R} \cap \frac{1}{2}\mathbf{U}$ and x to $\mathfrak{R} \cap \mathbf{U}$ in (1), then $\frac{x}{2}$, $\frac{y}{2}$, and $\frac{x-y}{2}$ all belong to $\frac{1}{2}\mathbf{U}$, so (1) yields $f(\frac{x+y}{2}) = 0$. Thus $f(\mathfrak{R} \cap \frac{3}{4}\mathbf{U}) = \{0\}$. Since the ray $\mathfrak{R}$ was arbitrary, we have $f(\frac{3}{4}\mathbf{U}) = \{0\}$. Similarly, by induction on n, we can prove that $f((1-2^{-n})\mathbf{U}) = \{0\}$ for all natural numbers n. That is, $f(\mathbf{U}) = \{0\}$, which is (3). This concludes the proof of the theorem.

The next result shows how much the zero solution (i.e., (3)) on $\mathbf{U}$ may differ from zero at the boundary.

Corollary 1. *Let $\overline{\mathbf{U}}$ be the closure of an open ball $\mathbf{U}$ centered at 0 in $\mathbf{R}^n$, and let $\mathbf{K}$ be a quadratically closed commutative field of characteristic different from 2. Then the general solution $f: \overline{\mathbf{U}} \to \mathbf{K}$ of (1) is given by*

$$f(x) = A^2(x) \quad \text{for all } x \in \overline{\mathbf{U}}, \tag{18}$$

if $\lambda = 0$, where A^2 is the diagonal of an arbitrary biadditive map $A: \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{K}$; or if $\lambda \neq 0$, by

$$f(x) = 0 \quad \text{for all } x \in \overline{\mathbf{U}}, \tag{19}$$

$$f(x) = -2\lambda^{-1} \quad \text{for all } x \in \overline{\mathbf{U}}, \tag{20}$$

or by

$$f(x) = \begin{cases} \lambda^{-1} & \text{if } x = \pm x_0, \\ 0 & \text{otherwise} \end{cases} \tag{21}$$

for some $x_0 \in \overline{\mathbf{U}}\backslash\mathbf{U}$.

Proof. The solution (18), in case $\lambda = 0$, follows as before.

If $\lambda \neq 0$, then by our Theorem we have either $f(x) = 0$ on $\mathbf{U}$ or $f(x) = -2\lambda^{-1}$ on $\mathbf{U}$.

First, consider the case $f(x) = 0$ on $\mathbf{U}$. Then (1) reduces to

$$f(\tfrac{1}{2}x + \tfrac{1}{2}y) + f(\tfrac{1}{2}x - \tfrac{1}{2}y) = \lambda f(x)f(y) \tag{22}$$

for all $x, y \in \overline{\mathbf{U}}\backslash\mathbf{U}$. Moreover, if $y \neq \pm x$, then both $\frac{x+y}{2}$ and $\frac{x-y}{2}$ are in $\mathbf{U}$. Hence (22) implies $\lambda f(x)f(y) = 0$. So at most one of the quantities $f(x)$, $f(y)$ can be nonzero. Clearly, $f(x) = 0$ on $\overline{\mathbf{U}}$ satisfies (1); this is solution (19). Suppose now that $f(x_0) \neq 0$ for some $x_0 \in \overline{\mathbf{U}}\backslash\mathbf{U}$. By the argument above, we have shown that $f(y) = 0$ for all $y \in \overline{\mathbf{U}}\backslash\mathbf{U}$, $y \neq \pm x_0$. Putting $x = x_0$, $y = \pm x_0$ in (22), we get

$$f(x_0) = \lambda f(x_0)^2, \qquad f(x_0) = \lambda f(x_0)f(-x_0).$$

Since $f(x_0) \neq 0$, we have $f(x_0) = \lambda^{-1} = f(-x_0)$. This is solution (21).

Finally, consider the case $f(x) = -2\lambda^{-1}$ on $\mathbf{U}$. Let $y \in \mathbf{U}$, $x \in \overline{\mathbf{U}}\backslash\mathbf{U}$ in (1), so that $\frac{x+y}{2}$ and $\frac{x-y}{2}$ are in $\mathbf{U}$. Then (1) reduces to

$$-4\lambda^{-1} = -8\lambda^{-1} - 2f(x),$$

which implies (20). This completes the proof.

Corollary 2. *Let $\varepsilon > 0$. The general solution $f\colon (-\varepsilon, \varepsilon) \to \mathbf{C}$ of equation* (1) *is given by*

$$f(x) = A^2(x) \quad \textit{for all } x \in (-\varepsilon, \varepsilon), \tag{23}$$

if $\lambda = 0$; or if $\lambda \neq 0$, by

$$f(x) = 0 \quad \textit{for all } x \in (-\varepsilon, \varepsilon), \tag{24}$$

or

$$f(x) = -2/\lambda \quad \textit{for all } x \in (-\varepsilon, \varepsilon). \tag{25}$$

Here A^2 is the diagonal of an arbitrary biadditive $A\colon \mathbf{R}^2 \to \mathbf{C}$.

Since $\mathbf{R} \subset \mathbf{C}$, this specializes also to $f\colon (-\varepsilon, \varepsilon) \to \mathbf{R}$, which is the situation considered by Lau [3]. Implicit in the proof of Theorem 2 in [1] is the fact that we can select a real-valued A. Moreover, if the diagonal $A^2(x)$ is bounded or measurable on a set of positive measure, then $A^2(x) = ax^2$ [4, Theorems 3.8, 3.9, p. 36]. So we obtain the following corollaries (cf. [3]).

Corollary 3. *Let $\varepsilon > 0$. The general solution $f\colon (-\varepsilon, \varepsilon) \to \mathbf{R}$ of* (1) *which is measurable on an arbitrary subset of positive measure is given by*

$$f(x) = ax^2 \quad \textit{for all } x \in (-\varepsilon, \varepsilon), \tag{26}$$

if $\lambda = 0$, for some arbitrary real constant a, and by (24) *and* (25) *if $\lambda \neq 0$.*

Corollary 4. *The general solution $f\colon I \to \mathbf{R}$ to* (1) *satisfying $f(1) = 1$, $f(0) = 0$ and measurable on a subset of positive measure is given by*

$$f(x) = x^2 \quad \textit{for all } x \in I, \tag{27}$$

if $\lambda = 0$; *and, if* $\lambda = 1$,

$$f(x) = \begin{cases} 0 & \text{for } x \in I^0, \\ 1 & \text{for } x = \pm 1. \end{cases} \tag{28}$$

Proof. Applying Corollary 3 with $\varepsilon = 1$, we first eliminate (25) because of the condition $f(0) = 0$.

Next, consider the case (26) with $\lambda = 0$. Here (1) takes the form

$$f(\tfrac{1}{2}x + \tfrac{1}{2}y) + f(\tfrac{1}{2}x - \tfrac{1}{2}y) = 2f(\tfrac{1}{2}x) + 2f(\tfrac{1}{2}y). \tag{29}$$

Putting $x = y = 1$ into (29) and using the condition $f(1) = 1$, we get $a = 1$. Furthermore, $x = y = -1$ in (29) yields also $f(-1) = 1$, and so we have (27).

Finally, consider the case (24) with $\lambda \neq 0$. Putting $x = y = 1$ into (1) in this case yields, again because of $f(1) = 1$, $\lambda = 1$. Moreover, (1) with $x = 1$ and $y = -1$ gives now $1 = f(-1)$. Thus (24) gives rise to solution (28), and this completes the proof of the corollary.

A similar argument shows the following.

Corollary 5. *The general solution* $f: I \to \mathbf{R}$ *of* (1), *continuous at* 1 *or* -1, *satisfying* $f(1) = 1$, $f(0) = 0$, *is given by*

$$f(x) = x^2 \quad \text{for all } x \in I.$$

For similar results on arbitrary groups, see [2].

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