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ON CERTAIN PAIRS OF FUNCTIONS OF SEMIPRIME RINGS

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ABSTRACT. Let f and g be functions of a set S into a semiprime ring R. A necessary and sufficient condition for f and g to satisfy $f(s)xg(t) = g(s)xf(t)$ for all $s, t \in S$, $x \in R$ is given. As an application, biderivations and commuting additive mappings of semiprime rings are characterized.

1. INTRODUCTION

A well-known result of Martindale states that if R is a prime ring and $a, b \in R$ satisfy

$$(1) \qquad axb = bxa \quad \text{for all } x \in R$$

then a and b are linearly dependent over C, the extended centroid of R (this was first proved for primitive rings by Amitsur [1, Lemma 6] and subsequently generalized to prime rings by Martindale [12, Theorem 1]). This result has proved to be a very useful tool in the study of prime rings. Therefore, we believe that it would be of interest to consider the identity (1) in some other rings. In this paper we extend Martindale's result to semiprime rings R by showing that there is the following relation between the elements $a, b \in R$ satisfying (1): there exist mutually orthogonal idempotents $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in C$ with sum 1 such that $\varepsilon_1 a = \lambda\varepsilon_1 b$ for some invertible $\lambda \in C$, $\varepsilon_2 b = 0$, and $\varepsilon_3 a = 0$. In fact, we prove a result of this kind for a more general situation where f and g are mappings of an arbitrary set S into a semiprime ring R such that

$$(2) \qquad f(s)xg(t) = g(s)xf(t) \quad \text{for all } s, t \in S,\ x \in R.$$

This is a worthwhile extension, as we shall see in §4.

Let R be a ring. A biadditive mapping $B: R \times R \to R$ is called a *biderivation* if for every $u \in R$ the mappings $x \to B(x, u)$ and $x \to B(u, x)$ are derivations of R. In [10] it was shown that every biderivation B of a noncommutative prime ring R is of the form $B(x, y) = \lambda[x, y]$ for some $\lambda \in C$, where $[u, v]$ denotes the commutator $uv - vu$. In this paper, using our theorem concerning identity (2), we shall extend this result to semiprime rings.

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It turns out that the notion of biderivations is closely connected with the notion of commuting additive mapping. A mapping f of a ring R into itself is said to be *commuting* if $[f(x), x] = 0$ for all $x \in R$. The study of such mappings was initiated a number of years ago by Posner [13]. Over the last twenty years, numerous papers concerning commuting and some related mappings have been published (see [8] for references). In [8] the present author showed that every commuting additive mapping f of a prime ring R is of the form $f(x) = \lambda x + \zeta(x)$ where $\lambda \in C$ and ζ is an additive mapping of R into C (see also [7, 9, 10] where some related results are presented). Ara and Mathieu [4] have generalized this theorem to semiprime rings. Applying our theorem on biderivations, we will obtain a brief proof of the result of Ara and Mathieu.

2. Preliminaries

In this section we recall some facts concerning semiprime rings and their extended centroids.

Let R be a semiprime ring. The (two-sided) annihilator of an ideal A of R will be denoted by $A^\perp$, and note that $A^\perp$ coincides with the left and right annihilators of A. Note also that $A \cap A^\perp = (0)$ and that $A \oplus A^\perp$ is an essential ideal of R.

For any semiprime ring R one can construct the *ring of quotients* Q of R [2]. As R can be embedded isomorphically in Q, we consider R as a subring of Q. If the element $q \in Q$ commutes with every element in R then q belongs to C, the center of Q. C contains the centroid of R, and it is called the *extended centroid* of R. In general, C is a von Neumann regular ring, and C is a field if and only if R is prime [2, Theorem 5]. The subring CR of Q is called the *central closure* of R. R is said to be *centrally closed* if it is equal to its central closure, or, equivalently, its centroid coincides with its extended centroid. The central closure CR is a centrally closed semiprime ring [5, Theorem 3.2] (thus, the extended centroid of CR is C).

Let E be an essential ideal of R. It is a basic fact that if $\varphi: E \to R$ is an R-bimodule homomorphism then there exists $\lambda \in C$ such that $\varphi(u) = \lambda u$ for any $u \in E$. Another useful fact is that $qE = (0)$ where $q \in Q$ implies that $q = 0$.

Amitsur [2] called an ideal B of R *closed* if $B = (B^\perp)^\perp$. The annihilator $A^\perp$ of any ideal A of R is a closed ideal (namely, we clearly have $A^\perp \subseteq ((A^\perp)^\perp)^\perp$; conversely, since $A \subseteq (A^\perp)^\perp$, it follows that $((A^\perp)^\perp)^\perp \subseteq A^\perp$). If B is a closed ideal then there exists an idempotent $\varepsilon \in C$ such that $B = R \cap \varepsilon Q$ [2, Corollary 9]. Now suppose that R is centrally closed. We claim that, in this case, we have $B = \varepsilon R$. Indeed, if $x \in B$ then $x \in \varepsilon Q$, so $x = \varepsilon x$; thus, $B \subseteq \varepsilon R$. Conversely, since R is centrally closed, we have $\varepsilon R \subseteq R$, and so $\varepsilon R \subseteq B$. We summarize these observations into the following statement: If R is a centrally closed semiprime ring and A is an ideal of R, then there exists an idempotent $\varepsilon \in C$ such that $A^\perp = \varepsilon R$.

3. The main result

The main goal of this section is to prove the following theorem, which generalizes [6, Lemma].

Theorem 3.1. *Let S be a set and R be a semiprime ring. If functions f and g of S into R satisfy* (2) *then there exist idempotents $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in C$ and an invertible element $\lambda \in C$ such that $\varepsilon_i\varepsilon_j = 0$, for $i \neq j$, $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 1$, and*

$$\varepsilon_1 f(s) = \lambda\varepsilon_1 g(s), \qquad \varepsilon_2 g(s) = 0, \qquad \varepsilon_3 f(s) = 0$$

holds for all $s \in S$.

Proof. Obviously, the identity (2) also holds in case x is an element from CR. Thus, there is no loss of generality in assuming that R is centrally closed.

Let $A = Rf(S)R$ and $B = Rg(S)R$. We have $A^\perp = \omega R$ and $B^\perp = \delta R$ for some idempotents $\omega, \delta \in C$. We set $\varepsilon_1 = (1-\omega)(1-\delta)$, $\varepsilon_2 = (1-\omega)\delta$, and $\varepsilon_3 = \omega$. Clearly, the ε_i's are mutually orthogonal idempotents with sum 1. Since $\delta g(s) \in B^\perp$, $s \in S$, we have $\delta g(s)R\delta g(s) = (0)$, which yields $\delta g(s) = 0$. Hence, $\varepsilon_2 g(s) = 0$, $s \in S$. Similarly we see that $\varepsilon_3 f(s) = 0$, $s \in S$.

Note that $(\varepsilon_1 A)^\perp = (\varepsilon_1 B)^\perp = (1-\varepsilon_1)R$. Hence, $E = \varepsilon_1 A \oplus (1-\varepsilon_1)R$ is an essential ideal of R. Define $\varphi\colon E \to R$ by

$$\varphi\left(\varepsilon_1\left(\sum_{i=1}^n x_i f(s_i) y_i\right) + (1-\varepsilon_1)r\right) = \varepsilon_1\left(\sum_{i=1}^n x_i g(s_i) y_i\right) + (1-\varepsilon_1)r.$$

In order to show that φ is well defined, we suppose that

$$\varepsilon_1\left(\sum_{i=1}^n x_i f(s_i) y_i\right) = 0.$$

Consequently, $\varepsilon_1(\sum_{i=1}^n x_i f(s_i) y_i) z g(t) = 0$ holds for all $z \in R$, $t \in S$. Since, by (2), we have $f(s_i) y_i z g(t) = g(s_i) y_i z f(t)$, it follows that

$$\varepsilon_1\left(\sum_{i=1}^n x_i g(s_i) y_i\right) z f(t) = 0$$

for all $z \in R$, $t \in S$. Thus, the element $\varepsilon_1(\sum_{i=1}^n x_i g(s_i) y_i)$ lies in $A^\perp$. Since $A^\perp = \omega R$ and $\varepsilon_1 = (1-\omega)(1-\delta)$, it follows that $\varepsilon_1(\sum_{i=1}^n x_i g(s_i) y_i) = 0$. This proves that φ is well defined.

Clearly, φ is an R-bimodule homomorphism. Thus, there exists $\lambda \in C$ such that $\varphi(u) = \lambda u$ holds for every $u \in E$. Hence, $\varepsilon_1 f(s) = \lambda\varepsilon_1 g(s)$ for all $s \in S$. It remains to prove that λ is invertible. Note that $\lambda E = \varepsilon_1 B \oplus (1-\varepsilon_1)R$. Since $\varepsilon_1 B \oplus (1-\varepsilon_1)R$ is an essential ideal (namely, $(\varepsilon_1 B)^\perp = (1-\varepsilon_1)R$), λ cannot be a divisor of zero. Consequently, as C is a von Neumann regular ring, λ is invertible. The proof of the theorem is thus completed.

Corollary 3.2. *If elements a and b of a semiprime ring R satisfy* (1) *then there exist idempotents $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in C$ such that: $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 1$, $\varepsilon_i\varepsilon_j = 0$ for $i \neq j$, $\varepsilon_2 b = 0$, $\varepsilon_3 a = 0$, and $\varepsilon_1 a = \lambda\varepsilon_1 b$ where λ is an invertible element in C.*

Proof. Define mappings f and g of R by $f(r) = a$, $g(r) = b$; now apply Theorem 3.1.

Theorem 3.1 and Corollary 3.2 extend [4, Lemma 2.5] and [4, Lemma 2.4], respectively. We remark that Ara [3] recently considered the situation where the elements a_i, b_i in a semiprime ring R satisfy $\sum a_i x b_i = 0$ for every $x \in R$.

4. Applications

The following result is a generalization of [10, Theorem 3.3].

Theorem 4.1. *Let R be a semiprime ring, and let $B: R \times R \to R$ be a biderivation. Then there exist an idempotent $\varepsilon \in C$ and an element $\mu \in C$ such that the algebra $(1-\varepsilon)R$ is commutative and $\varepsilon B(x, y) = \mu\varepsilon[x, y]$ for all $x, y \in R$.*

Proof. In [10, Lemma 3.1] it was proved that any biderivation B (of an arbitrary ring R) satisfies the relation

$$B(x, y)z[u, v] = [x, y]zB(u, v)$$

for all $x, y, z, u, v \in R$. (Let us sketch the proof: We compute $B(xu, yv)$ in two different ways. Using the fact that B is a derivation in the first argument, we get $B(xu, yv) = B(x, yv)u + xB(u, yv)$; since B is a derivation in the second argument, it follows that $B(xu, yv) = B(x, y)vu + yB(x, v)u + xB(u, y)v + xyB(u, v)$. Now compute $B(xu, yv)$ in a different way so that we first use the fact that B is a derivation in the second argument. Comparing both computations, one gets $B(x, y)[u, v] = [x, y]B(u, v)$. Replacing u by zu, one completes the proof.)

Now, let $S = R \times R$ and define $A: S \to R$ by $A(x, y) = [x, y]$. Note that the mappings $A, B: S \to R$ satisfy all the requirements of Theorem 3.1. Thus, there exist mutually orthogonal idempotents $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in C$ with sum 1 such that for all $x, y \in R$ we have $\varepsilon_2[x, y] = 0$, $\varepsilon_3 B(x, y) = 0$, and $\varepsilon_1 B(x, y) = \lambda\varepsilon_1[x, y]$ for some $\lambda \in C$. We set $\varepsilon = \varepsilon_1 + \varepsilon_3$, $\mu = \lambda\varepsilon_1$, and note that ε and μ have desired properties.

It is easy to construct nonzero biderivations in commutative rings. For instance, if d and g are nonzero derivations of a commutative integral domain, then the mapping $(x, y) \to d(x)g(y)$ is a nonzero biderivation.

Corollary 4.2. *Let R be a semiprime ring, and let $f: R \to R$ be a commuting additive mapping. Then there exist $\lambda \in C$ and an additive mapping $\zeta: R \to C$ such that $f(x) = \lambda x + \zeta(x)$ for all $x \in R$.*

Proof. Linearizing $[f(x), x] = 0$ we obtain $[f(x), y] = [x, f(y)]$. Hence, we see that the mapping $(x, y) \to [f(x), y]$ is a biderivation. By Theorem 4.1 there exist an idempotent $\varepsilon \in C$ and an element $\mu \in C$ such that the algebra $(1-\varepsilon)R$ is commutative (hence, $(1-\varepsilon)R \subseteq C$), and $\varepsilon[f(x), y] = \mu\varepsilon[x, y]$ holds for all $x, y \in R$. Thus, $\varepsilon f(x) - \mu\varepsilon x$ commutes with every element in R, so that $\varepsilon f(x) - \mu\varepsilon x \in C$. Now, let $\lambda = \mu\varepsilon$ and define a mapping ζ by $\zeta(x) = (\varepsilon f(x) - \lambda x) + (1-\varepsilon)f(x)$. Note that ζ maps in C and that $f(x) = \lambda x + \zeta(x)$ holds for every $x \in R$.

A mapping f of a ring R into itself is said to be *centralizing* if $[f(x), x]$ lies in the center of R for every x in R. In [8, Proposition 3.1] it was shown that every centralizing additive mapping of a two-torsionfree semiprime ring is in fact commuting. Therefore, as an immediate consequence of Corollary 4.2 we obtain the following result which was with a different proof also obtained by Ara and Mathieu [4].

Corollary 4.3. *Let R be a two-torsionfree semiprime ring, and let $f: R \to R$ be a centralizing additive mapping. Then there exist $\lambda \in C$ and an additive mapping $\zeta: R \to C$ such that $f(x) = \lambda x + \zeta(x)$ for all $x \in R$.*

For the case when R is a prime ring, the identity (2) was treated in [6, Lemma]. This result was needed for showing that every semiderivation f of a prime ring R is either a derivation, or it is of the form $f(x) = \lambda(1-g)(x)$ where $\lambda \in C$ and g is an endomorphism of R [6, Theorem] (see also [11]). As a test how Theorem 3.1 can be applied we now prove an analogous result for semiprime rings.

Proposition 4.4. *Let R be a semiprime ring, and let g be a ring endomorphism of R. If an additive mapping $f: R \to R$ satisfies*

$$f(xy) = f(x)g(y) + xf(y) = f(x)y + g(x)f(y)$$

for all $x, y \in R$, then there exists an idempotent $\varepsilon \in C$ and an invertible element $\lambda \in C$ such that the mapping $x \to \varepsilon f(x)$ is a derivation and $(1-\varepsilon)f(x) = \lambda(1-\varepsilon)(1-g)(x)$ holds for all $x \in R$.

Proof. Repeating the arguments given in the proof of [6, Theorem] we see that

$$(1-g)(x)yf(z) = f(x)y(1-g)(z)$$

for all $x, y, z \in R$. Thus, the mappings f and $1-g$ satisfy the requirements of Theorem 3.1. Therefore, there exist mutually orthogonal idempotents $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in C$ such that for every $x \in R$ we have $\varepsilon_2(1-g)(x) = 0$, $\varepsilon_3 f(x) = 0$, and $\varepsilon_1 f(x) = \lambda\varepsilon_1(1-g)(x)$ where λ is an invertible element in C. Let $\varepsilon = \varepsilon_2 + \varepsilon_3$, and note that ε satisfies the assertion of the theorem.

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