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ISOSPECTRAL CONVEX DOMAINS IN THE HYPERBOLIC PLANE

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ABSTRACT. We construct pairs of nonisometric convex polygons in the hyperbolic plane for which the Laplacians are both Dirichlet and Neumann isospectral. We also give examples of pairs of isospectral potentials for the Schrödinger operator on certain convex hyperbolic polygons.

Given a bounded domain Ω with piecewise-smooth boundary in a Riemannian manifold M, denote by $\mathrm{Spec}_{\mathrm{D}}(\Omega)$ (respectively, $\mathrm{Spec}_{\mathrm{N}}(\Omega)$) the eigenvalue spectrum of the Laplace-Beltrami operator acting on smooth functions on Ω with Dirichlet (respectively, Neumann) boundary conditions. A pair of domains Ω_1 and Ω_2 in M is *Dirichlet isospectral* if $\mathrm{Spec}_{\mathrm{D}}(\Omega_1) = \mathrm{Spec}_{\mathrm{D}}(\Omega_2)$; Neumann isospectrality is similarly defined.

Mark Kac's question "Can one hear the shape of a drum?" [K] asks whether Dirichlet isospectral domains in the Euclidean plane must be isometric. Recently, the authors and Wolpert [GWW1, 2] answered Kac's question negatively by exhibiting a pair of nonisometric domains in the Euclidean plane which are Dirichlet and Neumann isospectral; the construction also yields isospectral domains in the round 2-sphere and in the hyperbolic plane. Using similar methods, Buser et al. [BCDS] constructed other examples. In all cases, however, the domains are nonconvex. Thus Kac's question for convex plane domains remains open.

The purpose of this note is to exhibit pairs of convex domains in the hyperbolic plane which are both Dirichlet and Neumann isospectral. These domains are obtained from those of [GWW1, 2] by modifying the shape of the fundamental tile used in the construction, as in [BCDS]; Bérard's extension [B1] of Sunada's Theorem [S] facilitates the proof of isospectrality by "transplantation" of eigenfunctions from one domain to the other, as in [BCDS, B2].

Let T be a hyperbolic triangle with vertex angles α, β, and γ, and let $\Omega_{\alpha,\beta,\gamma}$ be the hyperbolic polygon composed of seven copies of the tile T glued together as in Figure 1 on the next page; whenever two triangles share an edge, each is the reflection of the other about their common edge. The domain depicted in Figure 1 is constructed from the triangle with angles $\alpha = \gamma = \pi/3$, $\beta = \pi/4$, which results in a convex quadrilateral; other choices of angles lead to other polygons. Figure 2 depicts $\Omega_{\pi/4,\pi/4,\pi/3}$ and $\Omega_{\pi/3,\pi/4,\pi/4}$.

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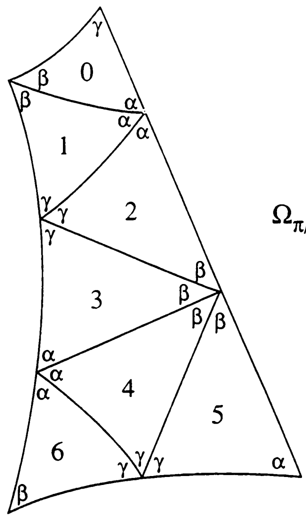
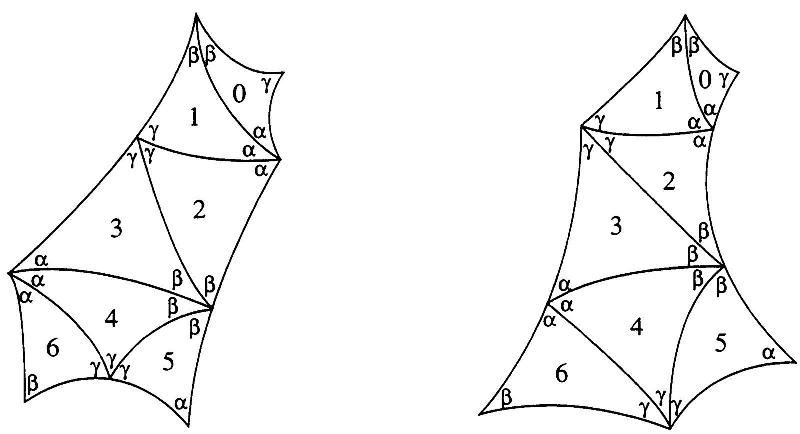


$\Omega_{\pi/3,\,\pi/4,\,\pi/3}$

FIGURE 1



$\Omega_{\pi/4,\,\pi/4,\,\pi/3}$ $\Omega_{\pi/3,\,\pi/4,\,\pi/4}$

FIGURE 2

Theorem. *Let $\alpha, \gamma \leq \pi/3$ and $\beta \leq \pi/4$. Then the hyperbolic domains $\Omega_{\alpha,\beta,\gamma}$ and $\Omega_{\gamma,\beta,\alpha}$ are convex polygons which are both Dirichlet and Neumann isospectral. They are isometric if and only if $\alpha = \gamma$.*

Proof. If $\alpha = \gamma$, then the domains are easily seen to be isometric. Conversely, if α and γ are unequal, then a comparison of the ordering of the vertex angles as one traverses the boundaries of the two domains shows that $\Omega_{\alpha,\beta,\gamma}$ and $\Omega_{\gamma,\beta,\alpha}$ are not isometric.

The inequalities on α, β, and γ obviously ensure convexity of the domains.

Before proving isospectrality, we note that the same combinatorial gluing pattern was used with a different shaped fundamental tile in [GWW1, 2] and [BCDS] to produce nonconvex isospectral plane domains. The isospectrality proof is the same; we briefly review it. The proof uses Berard's extension of Sunada's Theorem to provide an explicit "transplantation" isomorphism which carries a λ-eigenfunction on $\Omega_{\alpha,\beta,\gamma}$ to a λ-eigenfunction on $\Omega_{\gamma,\beta,\alpha}$, as follows. Given $f \in L^2(\Omega_{\alpha,\beta,\gamma})$, let $f_0, f_1, \dots, f_6$ be the restrictions of f to the tiles labelled $0, 1, \dots, 6$ in $\Omega_{\alpha,\beta,\gamma}$ (see Figure 1). Let $[f] = {}^t(f_0, f_1, \dots, f_6)$, a column vector of functions on T. Let S_{D} be the matrix

$$\begin{bmatrix} a & -a & a & -b & b & -a & -b \\ -a & a & -b & a & -a & b & b \\ a & -b & a & -a & b & -b & -a \\ -b & a & -a & b & -a & b & a \\ b & -a & b & -a & b & -a & -a \\ -a & b & -b & b & -a & a & a \\ -b & b & -a & a & -a & a & b \end{bmatrix},$$

and let S_{N} be the same matrix with all minus signs replaced by plus signs, where a and b are chosen so that S_{D} and S_{N} are orthogonal. Then the transplantation of Dirichlet eigenfunctions sends an eigenfunction f on $\Omega_{\alpha,\beta,\gamma}$ to the function g on $\Omega_{\gamma,\beta,\alpha}$ determined by $[g] = S_{\mathrm{D}}[f]$. One easily verifies that g is a Dirichlet eigenfunction on $\Omega_{\gamma,\beta,\alpha}$ by inspection. The Neumann transplantation is defined similarly, using S_{N}.

Remark. When $\alpha = \gamma$, a construction analogous to that in [GWW2, §5] produces pairs of isospectral potentials for the Schrödinger operator on $\Omega_{\alpha,\beta,\gamma}$. In particular, for $\alpha = \gamma = \pi/3$ and $\beta = \pi/4$ as in Figure 1, this yields isospectral Schrödinger operators on a convex quadrilateral in the hyperbolic plane.

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