ON NONATOMIC BANACH LATTICES AND HARDY SPACES

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(Communicated by Dale Alspach)

Abstract. We are interested in the question when a Banach space $X$ with an unconditional basis is isomorphic (as a Banach space) to an order-continuous nonatomic Banach lattice. We show that this is the case if and only if $X$ is isomorphic as a Banach space with $X(\ell_2)$. This and results of Bourgain are used to show that spaces $H_1(T^\alpha)$ are not isomorphic to nonatomic Banach lattices. We also show that tent spaces introduced by Coifman, Meyer, and Stein are isomorphic to $\text{Rad } H_1$.

1. Introduction

There is a natural distinction between sequence spaces and function spaces in functional analysis; as an example, let us point out the subtitles of two volumes of [15] and [16]. In this paper we use the term sequence space to indicate a space with the structure of an atomic Banach lattice and the term function space to indicate a space with the structure of a nonatomic Banach lattice. Many classical function spaces (e.g., the spaces $L_p[0, 1]$ for $1 < p < \infty$ [22] or [16]) have unconditional bases and hence are isomorphic as Banach spaces to sequence spaces (atomic Banach lattices). On the other hand, $L_1[0, 1]$ has no unconditional basis ([22] or [16]) and in the other direction the sequence spaces $\ell_p$ for $p \neq 2$ are not isomorphic to any nonatomic Banach lattice [1]. In this note we discuss a general criterion for deciding whether a Banach space with an unconditional basis (i.e., a sequence space) can be isomorphic to a nonatomic Banach lattice (i.e., a function space). Our main result (Theorem 2.4) gives a simple necessary and sufficient condition for an atomic Banach lattice $X$ to be isomorphic to an order-continuous nonatomic Banach lattice; of course, if $X$ contains no copy of $c_0$, every Banach lattice structure on $X$ is order-continuous.

Our main motivation is to study the Hardy space $H_1(T)$. After the discovery that the space $H_1(T)$ has an unconditional basis [17] it becomes natural to
investigate if $H_1(T)$ is isomorphic to a nonatomic Banach lattice. Applying Theorem 2.4 to $H_1$ and using some previous results of Bourgain [2, 3] we show that $H_1$ is not isomorphic to any nonatomic Banach lattice and furthermore that $H_1(T^n)$ is not isomorphic to a nonatomic Banach lattice for any natural number $n$.

We conclude by showing that the space $\text{Rad} H_1$ or $H_1(\ell_2)$ is isomorphic to the tent spaces $T^1$ introduced by Coifman, Meyer, and Stein [4].

2. Lattices with unconditional bases

Our terminology about Banach lattices will agree with [16]; we also refer the reader to [9, 10] for the isomorphic theory of nonatomic Banach lattices.

A (real) Banach lattice $X$ is called order-continuous if every order-bounded increasing sequence of positive elements is norm convergent. Any Banach lattice not containing $c_0$ is automatically order-continuous.

For any order-continuous Banach lattice $X$ we can define an associated Banach lattice $X(\ell_2)$ (using the Krivine calculus [16, pp. 40–42]) as the space of sequences $(x_n)_{n=1}^\infty$ in $X$ such that $(\sum_{k=1}^\infty |x_k|^2)^{1/2}$ is order-bounded (and hence is a convergent sequence) in $X$. $X(\ell_2)$ becomes an order-continuous Banach lattice when normed by $\|\(x_n\)\| = \|(\sum_{n=1}^\infty |x_n|^2)^{1/2}\|$. If $X$ has nontrivial cotype then $X(\ell_2)$ is naturally isomorphic to the space $\text{Rad} X$ which is the subspace of $L_2([0, 1]; X)$ of functions of the form $\sum_{n=1}^\infty x_n r_n$ where $(r_n)$ is the sequence of Rademacher functions. The space $\text{Rad} X$ is clearly an isomorphic invariant of $X$; so if two Banach lattices $X$ and $Y$ with nontrivial cotype are isomorphic, it follows easily that $X(\ell_2)$ and $Y(\ell_2)$ are isomorphic. However, this result holds in general by a result of Krivine [13] or [16, Theorem 1.f.14].

**Theorem 2.1.** If $X$, $Y$ are order-continuous Banach lattices and $T : X \to Y$ is a bounded linear operator, then if $(x_n) \in X(\ell_2)$ we have $(Tx_n) \in Y(\ell_2)$ and

$$\|\(T(x_n)\)\|_{Y(\ell_2)} \leq K_G\|T\|\|(x_n)\|_{X(\ell_2)}.$$

Here, as usual, $K_G$ denotes the Grothendieck constant.

**Proof.** Essentially this is Krivine's theorem, but we do need to show that if $(x_n) \in X(\ell_2)$ then $(Tx_n) \in Y(\ell_2)$. To see this we show that $(\sum_{k=1}^\infty |Tx_k|^2)^{1/2}$ is norm-Cauchy. In fact, if $m > n$ then

$$\left\|\left(\sum_{k=1}^m |Tx_k|^2\right)^{1/2} - \left(\sum_{k=1}^n |Tx_k|^2\right)^{1/2}\right\|_Y \leq \left\|\left(\sum_{k=n+1}^m |Tx_k|^2\right)^{1/2}\right\|_Y \leq K_G\|T\|\left\|\left(\sum_{k=n+1}^\infty |x_k|^2\right)^{1/2}\right\|_X$$

which converges to zero as $n \to \infty$ by the order-continuity of $X$. □

**Corollary 2.2.** If two order-continuous Banach lattices $X$ and $Y$ are isomorphic as Banach spaces, then $X(\ell_2)$ and $Y(\ell_2)$ are isomorphic as Banach spaces.

If $X$ is a separable order-continuous nonatomic Banach lattice then $X$ can be represented as (i.e., is linearly and order isomorphic with) a Köthe function.
space on \([0, 1]\) in such a way that \(L_\infty[0, 1] \subset X \subset L_1[0, 1]\) and inclusions are continuous. It will then follow that \(L_\infty\) is dense in \(X\), and the dual of \(X\) can be represented as a space of functions, namely, \(X^* = \{f \in L_1 : \int |fg| \, dt < \infty \text{ for every } g \in X\}\).

Now we are ready to state our main result. Let us observe that for rearrangement invariant function spaces on \([0, 1]\) this result was proved in [9] (cf. also [16, 2.d]) by a quite different technique.

**Theorem 2.3.** Let \(X\) be an order-continuous, nonatomic Banach lattice with an unconditional basis. Then \(X\) is isomorphic as a Banach space to \(X(\ell_2)\).

**Proof.** We will represent \(X\) as a Köthe function space on \([0, 1]\) as described above. Suppose \((\phi_n)_{n=1}^\infty\) is a normalized unconditional basis of \(X\). Then there is an order-continuous atomic Banach lattice \(Y\) which we identify as a sequence space and operators \(U: X \to Y\) and \(V: Y \to X\) such that \(UV = I_X\), \(VU = I_Y\), and \(U(\phi_n) = e_n\) for \(n = 1, 2, \ldots\), where \(e_n\) denotes the canonical basis vectors in \(Y\). We can regard \(Y^*\) as a space of sequences and further suppose that \(\|e_n\|_{Y^*} = \|e_n\|_Y = 1\). We will identify \(Y(\ell_2)\) as a space of double sequences with canonical unconditional basis \((e_{mn})_{m,n=1}^\infty\); thus for any finitely nonzero sequence we have \(\|\sum a_{mn}e_{mn}\|_{Y(\ell_2)} = \|\sum_m(\sum_n |a_{mn}|^2)^{1/2}e_m\|_Y\).

Let \(r_n\) denote the Rademacher functions and for each fixed \(f \in X\) note that \((r_nf)\) converges weakly to zero, since for \(g \in X^*\) we have \(\lim_{n \to \infty} \int r_nfg \, dt = 0\). In particular, we have for each \(m \in \mathbb{N}\) that \((r_n\phi_m)\) converges weakly to zero. It follows by a standard gliding hump technique that if \(\eta = (2\|U\|\|V\|)^{-1}\) then we can find for each \((m, n) \in \mathbb{N}^2\) an integer \(k(m, n)\) and disjoint subsets \((A_{mn})\) of \(\mathbb{N}\) so that \(\|U(\phi_m r_k(m, n))_y A_{mn} - U(\phi_m r_k(m, n))y\| \leq \eta\).

Identifying \(Y^*\) as a sequence space, we let \(\psi_m = U^*(e_m)\) and then define \(v_{m,n} = \chi_{A_{mn}}U(\phi_m r_k(m, n)) \in Y\) and \(v_{m,n}^* = \chi_{A_{mn}}V^*(\psi_m r_k(m, n))\in Y^*\). Now suppose \((a_{mn})\) is a finitely nonzero double sequence. Then

\[
\left\|\sum_{m,n} a_{mn}v_{mn}\right\|_Y \leq \left\| \left(\sum_{m,n} |a_{mn}|^2|U(\phi_m r_k(m, n))|^2\right)^{1/2}\right\|_Y \\
\leq K_G\|U\|\left\| \left(\sum_{m,n} |a_{mn}|^2|\phi_m r_k(m, n)|^2\right)^{1/2}\right\|_X \\
= K_G\|U\|\left\| \left(\sum_m \left(\sum_n |a_{mn}|^2\right) |\phi_m|^2\right)^{1/2}\right\|_X \\
= K_G\|U\|\left\| \left(\sum_n \left(\sum_m |a_{mn}|^2\right) |V e_m|^2\right)^{1/2}\right\|_X \\
\leq K_G^2\|U\||V|\left\| \left(\sum_n \left(\sum_m |a_{mn}|^2\right) |e_m|^2\right)^{1/2}\right\|_Y \\
= K_G^2\|U\||V|\left\| \sum_{m,n} a_{mn}e_{mn}\right\|_{Y(\ell_2)}.\]
Here we have used Krivine’s theorem twice. It follows that we can define a
linear operator \( S : Y(\ell_2) \to Y \) by \( S e_{mn} = v_{mn} \) and then \( \|S\| \leq K_0^2 \|U\|\|V\| \).

Similar calculations yield that for any finitely nonzero double sequence \((b_{mn})\) we have
\[
\left\| \sum_{m,n} b_{mn} v_{mn}^* \right\|_{Y^*} \leq K_0^2 \|U\|\|V\| \left\| \sum_m \left( \sum_n |b_{mn}|^2 \right)^{1/2} e_m \right\|_{Y^*}.
\]

Suppose then \( y \in Y \) and set \( a_{mn} = \langle y, v_{mn}^* \rangle \). Let \( F \) be a finite subset of \( \mathbb{N}^2 \).
Let \( \alpha_m = (\sum_n \chi_F(m,n) |a_{mn}|^2)^{1/2} \), and suppose the finitely nonzero sequence
\((\beta_m)\) is chosen so that \( \| \sum \beta_m e_m \|_{Y^*} = 1 \) and \( \sum \beta_m \alpha_m = \| \sum \alpha_m e_m \|_Y \). Then,
with the convention that \( 0/0 = 0 \),
\[
\left\| \sum_{(m,n) \in F} a_{mn} e_{mn} \right\|_{Y(\ell_2)} = \sum_m \beta_m \alpha_m
\]
\[
= \sum_{(m,n) \in F} \beta_m \alpha_m^{-1} |a_{mn}|^2 = \left\langle y, \sum_{(m,n) \in F} \beta_m \alpha_m^{-1} a_{mn} v_{mn}^* \right\rangle
\]
\[
\leq \|y\|_Y \left\| \sum_{(m,n) \in F} \beta_m \alpha_m^{-1} a_{mn} v_{mn}^* \right\|_{Y^*} \leq K_0^2 \|U\|\|V\|\|y\|_Y.
\]

Thus for each \( F \) the map \( T_F : Y \to Y(\ell_2) \) given by
\[
T_F y = \sum_{(m,n) \in F} \langle y, v_{mn}^* \rangle e_{mn}
\]
has norm at most \( K_0^2 \|U\|\|V\| \). More generally, we have
\[
\|T_F y\| \leq K_0^2 \|U\|\|V\|\|X_{AF,y}\|
\]
where \( A_F = \bigcup_{(m,n) \in F} A_{mn} \).

It follows that for each \( y \in Y \) the series \( \sum_{m,n} \langle y, v_{mn}^* \rangle e_{mn} \) converges (unconditionally) in \( Y(\ell_2) \). We can thus define an operator \( T : Y \to Y(\ell_2) \) by
\( Ty = \sum_{m,n} \langle y, v_{mn}^* \rangle e_{mn} \) and \( \|T\| \leq K_0^2 \|U\|\|V\| \).

Now notice that \( TS(e_{mn}) = c_{mn} e_{mn} \) where \( c_{mn} = \langle v_{mn}, v_{mn}^* \rangle \). But
\[
\langle v_{mn}, v_{mn}^* \rangle = \langle v_{mn}, V^* \psi_m r_{k(m,n)} \rangle \geq \langle U(\phi_m r_{k(m,n)}), V^* (\psi_m r_{k(m,n)}) \rangle - \eta \|V\|\|\psi_m\|_X^* \sim \langle \phi_m, \psi_m \rangle - \eta \|V\|\|\psi_m\|_X \geq 1 - \eta \|V\|\|U\| \geq 1/2.
\]

Thus \( TS \) is invertible, so it follows that \( Y(\ell_2) \) is isomorphic to a complemented
subspace of \( Y \). It then follows from the Pelczyński decomposition technique
that \( Y \sim Y(\ell_2) \); more precisely, \( Y \sim Y(\ell_2) \oplus W \) for some \( W \) and so \( Y \sim Y(\ell_2) \oplus (Y(\ell_2) \oplus W) \sim Y(\ell_2) \oplus Y \sim Y(\ell_2) \). The conclusion follows from Corollary
2.2. □

Remark. The order continuity of the Banach lattice \( X \) is essential. In [14]
a nonatomic Banach lattice \( X \) (actually an M-space) was constructed which
is isomorphic to \( c_0 \). In particular, \( X \) has an unconditional basis but is not
isomorphic to \( X(\ell_2) \).
Theorem 2.4. Let $Y$ be a Banach space with an unconditional basis. Then $Y$ is isomorphic to an order-continuous nonatomic Banach lattice if and only if $Y \sim Y(\ell_2)$.

Remark. Here again we regard $Y$ as an order-continuous Banach lattice.

Proof. One direction follows immediately from Theorem 2.3 and Corollary 2.2. For the other direction, it is only necessary to show that if $Y \sim Y(\ell_2)$ then $Y$ is isomorphic to order-continuous nonatomic Banach lattice. To this end we introduce the space $Y(L_2)$; this is the space of sequences of functions $(f_n)$ in $L_2[0, 1]$ such that $\sum \|f_n\|_2 e_n$ converges in $Y$. We set $\|(f_n)\|_{Y(L_2)} = \|\sum \|f_n\|_2 e_n\|_Y$. It is clear that $Y(L_2)$ is an order-continuous Banach lattice. Now if $(g_n)$ is an orthonormal basis of $L_2$, we define $W : Y(\ell_2) \rightarrow Y(L_2)$ by $W(\sum_{m,n} a_{mn} e_{mn}) = (\sum_n a_{mn} g_n)_{m=1}^\infty$, and it is easy to see that $W$ is an isometric isomorphism. □

Proposition 2.5. If $X$ is a nonatomic order-continuous Banach lattice with unconditional basis, then $X \sim X \oplus X$ and $X \sim X \oplus \mathbb{R}$.

Proof. Both facts follow from Theorem 2.3. □

Note that for spaces with unconditional basis both properties do not hold in general (see [5, 6]).

Proposition 2.6. Let $X$ be an order continuous nonatomic Banach lattice with an unconditional basis, and let $Y$ be a complemented subspace of $X$. Assume that $Y$ contains a complemented subspace isomorphic to $X$. Then $X \sim Y$.

Proof. The proof is a repetition of the proof of Proposition 2.5. of [16]. □

3. Hardy spaces

We recall that $H^1(T^n)$ is defined to be the space of boundary values of functions $f$ holomorphic in the unit disk $D$ and such that

$$\sup_{0<r<1} \int_{T^n} |f(re^{it_1}, re^{it_2}, \ldots, re^{it_n})| \, dt_1 \, dt_2 \cdots \, dt_n < \infty.$$  

The basic theory of such spaces is explained in [18].

Let us consider first the case $n = 1$. Then $\mathbb{R}H^1$ is defined be the space of real functions $f \in L_1(T)$ such that for some $F \in H^1(T)$ we have $\mathbb{R}F = f$. $\mathbb{R}H^1$ is normed by $\|f\|_1 + \min\{\|F\|_{H^1} : \mathbb{R}F = f\}$. Then $H^1$ is isomorphic to the complexification of $\mathbb{R}H^1$ and, further, when considered as a real space is isomorphic to $\mathbb{R}H^1$. Further it was shown in [17] that $\mathbb{R}H^1$ has an unconditional basis and is isomorphic to a space of martingales $H^1(\delta)$. To define the space $H^1(\delta)$ let $(h_n)_{n \geq 1}$ be the usual enumeration of the Haar functions on $I = [0, 1]$ normalized so that $\|h_n\|_\infty = 1$. Then suppose $f \in L_1$ is of the form $f = \sum a_n h_n$. We define $\|f\|_{H^1(\delta)} = \int (\sum_n |a_n|^2 h_n^2)^{1/2} \, dt$ and $H^1(\delta) = \{f : \|f\|_{H^1(\delta)} < \infty\}$.

These considerations can be extended to the case $n > 1$. In a similar way, $H^1(T^n)$ is isomorphic to the complexification of, and is also real-isomorphic to, a martingale space $H^1(\delta^n)$. Here we define for $\alpha \in \mathcal{M} = \mathbb{N}^n$ the function $h_\alpha \in L_1(T^n)$ by $h_\alpha(t_1, \ldots, t_n) = \prod h_{\alpha_1}(t_k)$. Then $H^1(\delta^n)$ consists of all $f = \sum_{\alpha \in \mathcal{M}} a_\alpha h_\alpha$ such that $\|f\|_{H^1(\delta^n)} = \int (\sum |a_\alpha|^2 h_\alpha^2)^{1/2} \, dt < \infty$.  

It is clear from the definition that the system \((h_a)_{a \in \mathcal{A}}\) is an unconditional basis of \(H_1(\delta^n)\). We can thus define a space \(H_1(\delta^n, \ell_2) = H(\delta^n)(\ell_2)\) as in §1; since \(H_1(\delta^n)\) has cotype two, this space is isomorphic to \(\text{Rad} H_1(\delta^n)\). The following theorem is due to Bourgain [2]:

**Theorem 3.1.** \(H_1(\delta, \ell_2)\) is not isomorphic to a complemented subspace of \(H_1(\delta)\).

In a subsequent paper [3] Bourgain implicitly extended this result to higher dimensions.

**Theorem 3.2.** For every \(n = 1, 2, \ldots\) the space \(H_1(\delta^n, \ell_2)\) is not isomorphic to any complemented subspace of \(H_1(\delta^n)\).

**Sketch of proof.** For \(n = 1\) this theorem is proved in detail in [2]. The subsequent paper [3] states only the weaker fact that \(H_1(\delta^n)\) is not isomorphic to \(H_1(\delta^{n+1})\). His proof, however, gives Theorem 3.2 as well. All that is needed is to change in §3 of [3] condition \((m+1)\) and Lemma 4. Before we formulate the appropriate condition we need some further notation. By \(BMO(\delta^n)\) we will denote the dual of \(H_1(\delta^n)\) and by \(BMO(\delta^n, \ell_2)\) we will denote the dual of \(H_1(\delta^n, \ell_2)\). The space \(H_1(\delta^n, \ell_2)\) has an unconditional basis given by \((h_a \otimes e_k)_{a \in \mathcal{A}, k \in \mathbb{N}}\). In our notation from §2 \(h_a \otimes e_k\) is a sequence of \(H_1(\delta^n)\)-functions which consists of zero functions except at the \(k\)th place where there is \(h_a\). The same element can be treated as an element of the dual space. Note that the natural duality gives

\[
\langle h_a \otimes e_k, h_{a'} \otimes e_{k'} \rangle = \begin{cases} \int_{\mathbb{N}} |h_a| & \text{when } a = a' \text{ and } k = k', \\ 0 & \text{otherwise.} \end{cases}
\]

Now we are ready to state the new condition \((m+1)\):

Let \(\Phi : H_1(\delta^n, \ell_2) \rightarrow H_1(\delta^n)\) and \(\Phi^* : BMO(\delta^n, \ell_2) \rightarrow BMO(\delta^n)\) be bounded linear operators (note that \(\Phi^*\) is not the adjoint of \(\Phi\)). Then for every \(\epsilon > 0\) there exists a set \(A \subset \mathcal{A}\) such that \(\sum_{\alpha \in A} |h_\alpha| = 1\) and integers \(k_\alpha\) for \(\alpha \in A\) such that

\[
\sum_{\alpha \in A} \int_{\mathbb{N}} |\Phi(h_\alpha \otimes e_{k_\alpha})| \cdot |\Phi^*(h_\alpha \otimes e_{k_\alpha})| < \epsilon.
\]

With this condition one can repeat the proof from [3] and obtain the theorem. □

**Corollary 3.3.** We have

\[
\ell_2 \subseteq H_1(\delta) \subseteq H_1(\delta, \ell_2) \subseteq H_1(\delta^2) \subseteq H_1(\delta^2, \ell_2) \subseteq \cdots
\]

where \(X \subseteq Y\) means that \(X\) is isomorphic to a complemented subspace of \(Y\) but \(Y\) is not isomorphic to a complemented subspace of \(X\).

**Proof.** It is well known and easy to check that the map \(h_\alpha \otimes e_k \mapsto h_\alpha(t_1, \ldots, t_n) \cdot r_{k}(t_{n+1})\) where \(r_k\) is the \(k\)th Rademacher function gives the desired complemented embedding. That no smaller space is isomorphic to a complemented subspace of a bigger one is the above theorem of Bourgain. □

**Corollary 3.4.** The spaces \(H_1(\delta^n)\) is not isomorphic to a nonatomic Banach lattice for \(n = 1, 2, \ldots\). The spaces \(H_1(\delta^n, \ell_2)\) are each isomorphic to a nonatomic Banach lattice.
**Proof.** The first claim follows directly from Theorems 3.1, 3.2, and 2.3. We only have to observe that (since \( H_1(\delta^n) \) does not contain any subspace isomorphic to \( c_0 \) and indeed has cotype two) any Banach lattice isomorphic as a Banach space to \( H_1(\delta^n) \) is order continuous (see [16, Theorem 1.c.4]). The second claim follows from Corollary 2.4. \( \Box \)

**Remark.** For \( H_p(T^n) \) with \( 0 < p < \infty \) we have the following situation. When \( 1 < p < \infty \) the orthogonal projection from \( L_p(T^n) \) onto \( H_p(T^n) \) is bounded so then \( H_p(T^n) \) is isomorphic to \( L_p(T^n) \). This implies in particular that these spaces are isomorphic to nonatomic lattices. When \( 0 < p < 1 \) then \( H_p(T^n) \) admit only purely atomic orders as a \( p \)-Banach lattices. To see this observe that if \( X \) is not a purely atomic \( p \)-Banach lattice then its Banach envelope (for definition and properties see [11]) is a Banach lattice which is not purely atomic. On the other hand it is known that the Banach envelope of \( H_p(T^n) \) is isomorphic to \( \ell_1 \). For \( n = 1 \) this can be found in [11, Theorem 3.9], for \( n > 1 \) the proof uses [19, Theorem 2'] but otherwise is the same; alternatively see [11, Theorem 3.5] for a proof using bases. When we compare it with the observation from [1] mentioned in the Introduction, that \( \ell_1 \) is not isomorphic to any nonatomic Banach lattice, we conclude that the spaces \( H_p(T^n) \) cannot be isomorphic to any nonatomic \( p \)-Banach lattice.

**Remark.** For the dual spaces \( H_1(T^n)^* = BMO(T^n) \) the situation is rather different. We first observe the following proposition:

**Proposition 3.5.** For any Banach space \( X \) the spaces \( \ell_1(X)^*(= \ell_{\infty}(X^*)) \) and \( L_1([0, 1], X)^* \) are isomorphic.

**Proof.** Clearly \( \ell_1(X)^* \) is isomorphic to a one-complemented subspace of \( L_1(X)^* \). Now let \( x_n,k = x_{(k-1)2^{-n}, k2^{-n}} \) for \( 1 \leq k \leq 2^n \) and \( n = 0, 1, \ldots \). Let \( T: \ell_1(X) \rightarrow L_1(X) \) be defined by \( T((x_n)) = \sum x_n x_{m,k} \) where \( n = 2^m + k - 1 \). Let \( L_1(\mathcal{S}_N, X) \) be the subspace of all functions measurable with respect to the finite algebra generated by the sets \( (k-1)2^{-N}, k2^{-N} \) for \( 1 \leq k \leq 2^N \), and define \( S_N: L_1(\mathcal{S}_N; X) \rightarrow \ell_1(X) \) by setting \( S(x \otimes x_{N,k}) \) to be the element with \( x \) in position \( 2^N + k - 1 \) and zero elsewhere. Then applying [22, IIE, Exercise 7] (cf. [8, Proposition 1]), we obtain that \( L_1(X)^* \) is isomorphic to a complemented subspace of \( \ell_1(X)^* \). Then by the Pelczyński decomposition technique we obtain the proposition. \( \Box \)

Now from Proposition 3.5, observe that, since \( H_1(T^n) \sim \ell_1(H_1(T^n)) \), we have \( L_1(H_1(T^n))^* \sim BMO(T^n) \), and clearly this isomorphism induces a nonatomic (but not order-continuous) lattice structure on \( BMO(T^n) \). (It is easy to see that a space which contains a copy of \( \ell_{\infty} \) cannot have an order-continuous lattice structure, because it fails the separable complementation property.)

**4. Rad \( H_1 \) AND TENT SPACES**

The space \( H_1(\delta, \ell_2) \) is, as observed in \( \S 2 \), isomorphic to \( \text{Rad} H_1 \) and has a structure as a nonatomic Banach lattice. The complex space \( \text{Rad} H_1 \) is easily seen to be isomorphic to the vector-valued space \( H_1(T, \ell_2) \) consisting of the boundary values of the space of all functions \( F \) analytic in the unit disk \( \mathbb{D} \).
with values in a Hilbert space $\ell_2$ and such that
\[
\sup_{0<r<1} \int_0^{2\pi} \|F(re^{i\theta})\| \frac{d\theta}{2\pi} = \|F\| < \infty.
\]
To see this isomorphism just note that $H_1(T, \ell_2)$ can be identified with the space of sequences $(f_n)$ in $H_1$ such that
\[
\|(f_n)\| = \int_0^{2\pi} \left( \sum_{n=1}^{\infty} |f_n(e^{i\theta})|^2 \right)^{1/2} \frac{d\theta}{2\pi} < \infty.
\]
This is in turn easily seen to be equivalent to the norm of $\sum r_n f_n$ in $L_2([0, 1]; H_1)$ (see [16, Theorem 1.d.6]).

We now show that a nonatomic Banach lattice isomorphic to $\text{Rad } H_1$ arises naturally in harmonic analysis. More precisely we will show that tent space $T^1$, which was introduced and studied by Coifman, Meyer, and Stein in [4], is isomorphic to $\text{Rad } H_1$. Tent spaces are useful in some questions of harmonic analysis (cf. [7] or [21]). They can be defined over $\mathbb{R}^n$, but for the sake of simplicity we will consider them only over $\mathbb{R}$.

Let us fix $\alpha > 0$. For $x \in \mathbb{R}$ we define
\[
\Gamma_\alpha(x) = \{(y, t) \in \mathbb{R} \times \mathbb{R}^+: |x - y| < \alpha t\}.
\]
Given a function $f(y, t)$ defined on $\mathbb{R} \times \mathbb{R}^+$ we put
\[
\|f\|_\alpha = \int_{\mathbb{R}} \left( \int_{\Gamma_\alpha(x)} |f(y, t)|^2 t^{-2} \, dy \, dt \right)^{1/2} \, dx.
\]
It was shown in [4, Proposition 4] that for different $\alpha$'s the norms $\| \cdot \|_\alpha$ are equivalent; i.e., for $0 < \alpha < \beta < \infty$ there is a $C = C(\alpha, \beta)$ such that for every $f$ we have
\[
(4.1) \quad \|f\|_\alpha \leq \|f\|_\beta \leq C\|f\|_\alpha.
\]
This implies that the space $T^1 = \{f(y, t): \|f\|_\alpha < \infty\}$ does not depend on $\alpha$. Observe that $T^1$ is clearly a nonatomic Banach lattice.

The main result of this section is

**Theorem 4.1.** The space $T^1$ is lattice-isomorphic to $H_1(\delta, L_2)$ and, hence, isomorphic to $\text{Rad } H_1$.

Actually for the proof of this theorem it is natural to work with the dyadic $H_1$ space on $\mathbb{R}$. This space, which we denote $H_1(\delta_\infty)$, can be defined as follows:

Let $I_{nk} = [k \cdot 2^n, (k + 1) \cdot 2^n]$ for $n, k = 0, \pm 1, \pm 2, \ldots$, and let $h_{nk}$ be the function which is equal to 1 on the left-hand half of $I_{nk}$, -1 on the right-hand half of $I_{nk}$, and zero outside $I_{nk}$. In other words, $h_{nk}$ is the Haar system on $\mathbb{R}$. The system $\{h_{nk}\}_{n,k=0,\pm 1,\pm 2,\ldots}$ is a complete orthogonal system. For a function $f = \sum_{n,k} a_{nk} h_{nk}$ we define its $H_1(\delta_\infty)$-norm by
\[
(4.2) \quad \|f\| = \int_{\mathbb{R}} \left( \sum_{n,k} |a_{nk}|^2 |h_{nk}|^2 \right)^{1/2} \, dt.
\]
That this space is isomorphic to the space $H_1(\delta)$ follows from the work of Sjölin and Stromberg [20]. However, slightly more is true:
Lemma 4.2. The atomic Banach lattices $H_1(\delta)$ and $H_1(\delta_\infty)$ are lattice-isomorphic (or, equivalently the natural normalized unconditional bases of these spaces are permutatively equivalent).

Proof. For any subset $\mathcal{A}$ of $\mathbb{Z}^2$ write $H_\mathcal{A}$ for the closed linear span of $\{h_{nk}: (n, k) \in \mathcal{A}\}$ in $H_1(\delta_\infty)$. For $m \in \mathbb{Z}$ let $\mathcal{A}_m = \{(n, k): I_{nk} \subset [2^{-m-1}, 2^{-m}]\}$ and $\mathcal{B}_m = \{(n, k): I_{nk} \subset [-2^{-m}, -2^{-m-1}]\}$. Let $\mathcal{D} = \bigcup_{m \in \mathbb{Z}} (\mathcal{A}_m \cup \mathcal{B}_m)$ and $\mathcal{D}_+ = \bigcup_{m \geq 0} \mathcal{A}_m$. Then it is clear that $H_\mathcal{A}$ and $H_\mathcal{D}$ are each lattice isomorphic to $\ell_1(H_1(\delta))$. Now $H_1(\delta_\infty)$ is lattice isomorphic to $H_\mathcal{D} \oplus H_\mathcal{G}$ where $\mathcal{G} = \{(m, 0), (m, -1): m \in \mathbb{Z}\}$. It is easy to show that $H_\mathcal{G}$ is lattice isomorphic to $\ell_1$. Similarly $H_1(\delta)$ is lattice-isomorphic to $H_1(\mathcal{D}_+) \oplus \ell_1$, and this completes the proof of the lemma. □

Remark. Note also that $H_1(\delta)$ is lattice-isomorphic to $\ell_1(H_1(\delta))$.

Proof of Theorem 4.1. We will prove that $T^1$ is lattice-isomorphic to $H_1(\delta_\infty, L_2)$. Let us introduce squares $A_{nk} \subset \mathbb{R} \times \mathbb{R}^+$ defined as $A_{nk} = I_{nk} \times [2^n, 2^{n+1}]$ for $n, k = 0, \pm 1, \pm 2, \ldots$. It is geometrically clear that squares $\{A_{nk}\}_{n, k = 0, \pm 1, \pm 2, \ldots}$ are essentially disjoint and that they cover $\mathbb{R} \times \mathbb{R}^+$. For $j = 0, 1, 2$ we define

$$A_{nk}^j = \left[\left(k + j/3\right)2^n, \left(k + (j + 1)/3\right)2^n\right] \times [2^n, 2^{n+1}].$$

Note that in this way we divide each $A_{nk}$ into three essentially disjoint rectangles. Let $D_j = \bigcup_{n, k} A_{nk}^j$. Let $T^1_j$ be the subspace of $T^1$ consisting of all functions whose support is contained in $D_j$. Clearly $T^1 = T^1_0 \oplus T^1_1 \oplus T^1_2$, so it is enough to show that $T^1_j$ is lattice-isomorphic to $H_1(\delta_\infty, L_2)$.

We write $f^j \in T^1_j$ as $f^j = \sum_{n, k} f_{nk}^j$, where $f_{nk}^j = f^j \cdot \chi_{A_{nk}^j}$. We start with $j = 1$. For any $\alpha > 0$ we have

$$\|f^1\|_\alpha = \int_{\mathbb{R}} \left( \int_{\Gamma_\alpha(x)} |f^1(y, t)|^2 t^{-2} dy dt \right)^{1/2} dx$$

(4.3)

$$= \int_{\mathbb{R}} \left( \int_{\Gamma_\alpha(x)} \sum_{n, k} |f_{nk}^1(y, t)|^2 t^{-2} dy dt \right)^{1/2} dx$$

$$= \int_{\mathbb{R}} \left( \sum_{n, k} \int_{\Gamma_\alpha(x)} |f_{nk}^1(y, t)|^2 t^{-2} dy dt \right)^{1/2} dx.$$  

If we now take $\alpha = \frac{3}{2}$ we have $\Gamma_\alpha(x) \supset A_{nk}^1$ for all $x \in I_{nk}$, so from (4.3) we get

$$\|f^1\|_\alpha \geq \int_{\mathbb{R}} \left( \sum_{n, k} \chi_{I_{nk}}(x) \int_{A_{nk}^1} |f_{nk}^1(y, t)|^2 t^{-2} dy dt \right)^{1/2} dx.$$  

(4.4)

On the other hand, when we take $\alpha = \frac{1}{8}$ we have $\Gamma_\alpha(x) \cap A_{nk}^1 = \emptyset$ for all $x \notin I_{nk}$, so from (4.3) we get

$$\|f^1\|_\alpha \leq \int_{\mathbb{R}} \left( \sum_{n, k} \chi_{I_{nk}}(x) \int_{A_{nk}^1} |f_{nk}^1(y, t)|^2 t^{-2} dy dt \right)^{1/2} dx.$$  

(4.5)
For each \((n, k)\) the subspace of \(T^1\) consisting of functions supported on \(A_{nk}\) is easily seen to be isometric to the Hilbert space. If we fix an isometry between this space and \(\ell_2\), we obtain from (4.2)-(4.4) that \(T^1\) is lattice-isomorphic to \(H_1(\delta_\infty, L_2)\). In order to complete the proof of Theorem 4.1 it is enough to show that \(T_0^1\) and \(T_1^1\) are lattice-isomorphic to \(T^1\). This isomorphism can be given by \(\sum nk f_{nk}^1 \mapsto \sum nk f_{nk}^1\). The fact that this map is really an isomorphism follows from

**Lemma 4.3.** Let \(\phi(t)\) be a uniformly bounded measurable function on \(\mathbb{R}^+\). For a function \(f\) defined on \(\mathbb{R} \times \mathbb{R}^+\) we define

\[ A_\phi(f)(y, t) = f(y + t\phi(t), t). \]

Then \(A_\phi : T^1 \to T^1\) is a continuous linear operator.

**Proof of Lemma 4.3.** Since

\[
\int_{\Gamma_{a}(x)} |A_\phi(f)(y, t)|^2 t^{-2} dy dt = \int_{\mathbb{R}^+} \left( t^{-2} \int_{x-at}^{x+at} |A_\phi(f)(y, t)|^2 dy \right) dt
\]

\[
= \int_{\mathbb{R}^+} \left( t^{-2} \int_{x-at}^{x+at} t^{-1}\phi(t) \right) \left| f(y, t) \right|^2 dy dt
\]

\[
\leq \int_{\mathbb{R}^+} \left( t^{-2} \int_{x-\|\phi\|_\infty+at}^{x+\|\phi\|_\infty+at} \left| f(y, t) \right|^2 dy \right) dt
\]

\[
= \int_{\Gamma_{\|\phi\|_\infty}(x)} \left| f(y, t) \right|^2 t^{-2} dy dt,
\]

the lemma follows. \(\square\)

**References**


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