GENERIC PROPERTIES OF THE NICOLETTI AND FLOQUET BOUNDARY VALUE PROBLEMS

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ABSTRACT. The existence of solutions and the existence and uniqueness of solutions are considered as generic properties of an abstract boundary value problem. Applications to the Nicoletti and the Floquet boundary value problems are given.

This note is concerned with the system of ordinary differential equations with the Nicoletti or the Floquet boundary value condition. In [4] Lasota and Olech presented two results concerning the uniqueness of solutions and the existence of solutions of the Nicoletti boundary value problem. The sufficient conditions they gave are the best possible in a certain class of the right-hand sides of the system. Similar results for the Floquet boundary value problem were obtained by Kasprzyk and Myjak in [3].

In this note we widen the classes of the right-hand sides of these systems. The mentioned results are not true for all maps from the extended classes; however, we show that they are true for 'most'. More precisely, the subset of maps for which the existence of solutions or the existence and uniqueness of solutions fails to be true is of the first Baire category in an appropriate complete metric space. To do this we first prove two theorems which deal with generic properties of abstract boundary value problems.

In [1] there are presented results showing that under similar assumptions the set of solutions of the Nicoletti or the Floquet boundary value problem is an R_δ-set.

Generic properties of functional and differential equations were studied by many authors. In Myjak's monograph [6] the reader can find several results and a list of references concerning this subject.

1. PRELIMINARIES

In this note E and E_1 denote two real Banach spaces (with norms $\|\cdot\|$ and $\|\cdot\|_1$ respectively). Let X be a linear subspace of E, and let $L\colon X \to E_1$ be a linear (not necessarily continuous) map such that $\operatorname{Im} L = E_1$ and $\operatorname{Ker} L$ is a finite-dimensional space. Moreover, let $P\colon E \to \operatorname{Ker} L$ be a continuous projection onto $\operatorname{Ker} L$. For $\varphi\colon E \to E_1$ and $\xi \in \operatorname{Ker} L$ consider the boundary

value problem

$$(1.1) \qquad \begin{cases} L(x) = \varphi(x), \\ P(x) = \xi, \end{cases}$$

where $x \in X$. Many interesting results concerning the problem (1.1) (in a more general setting) the reader can find in [7]. In this note we study the existence of solutions and the existence and uniqueness of solutions as generic properties of the problem (1.1). As applications we obtain some results about the Nicoletti and Floquet boundary value problems.

We can associate with L the right inverse $T: E_1 \to E$ given as

$$T(y) = x \quad \text{iff} \quad L(x) = y \text{ and } P(x) = 0.$$

This right-inverse can be defined by putting $T(y) = \tilde{x} - P(\tilde{x})$ for $y = L(\tilde{x})$. One can observe that T is well defined and linear. Now we can write the problem (1.1) in an equivalent form, namely,

$$(1.2) \qquad x = \xi + T \circ \varphi(x), \qquad x \in E.$$

In what follows by U we understand a neighbourhood of 0 in the space E and by $n(E)$ the family of all nonempty subsets of E. We say that a multivalued map $H: U \to n(E)$ satisfies the Lasota-Opial conditions if:

(i) the graph $\{(x, y): y \in H(x)\}$ of H is closed in $U \times E$,

(ii) $\overline{H(A)}$ is compact for every bounded $A \subset U$,

(iii) if $x \in U$ and $x \in H(x)$, then $x = 0$.

In [5] Lasota and Opial proved the following

Theorem A. *Suppose that $H: U \to n(E)$ satisfies the Lasota-Opial conditions. Then, for any continuous map $h: E \to E$ the condition*

$$h(x) - h(y) \in H(x - y) \quad \text{for } x - y \in U$$

implies that $I - h: E \to E$ is a homeomorphism ($I: E \to E$ denotes the identity map). Furthermore, for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $x \in E$

$$(I - h)(B(x, \varepsilon)) \supset B(x - h(x), \delta)$$

($B(x, \varepsilon)$ denotes the open ball about x of radius ε).

In what follows we use the domain invariance theorem [2, p. 65]:

Theorem B. *Let $\overline{B}(z, \varepsilon)$ be a closed ball in the space E, and let $g: \overline{B}(z, \varepsilon) \to E$ be a compact field (i.e., $I - g$ is a compact map). If g is injective and*

$$\|x - y\| = \varepsilon \Rightarrow \|g(x) - g(y)\| > \delta,$$

where δ is a positive constant, then

$$g(B(z, \varepsilon)) \supset B(g(z), \delta).$$

By $(\mathscr{Z}, \rho)$ we mean a complete linear subspace of the space of all continuous $\varphi: E \to E_1$ such that $\varphi(A)$ is bounded for every bounded $A \subset E$, endowed with the metric

$$(1.3) \qquad \rho(\varphi_1, \varphi_2) = \sum_{n=1}^{\infty} 2^{-n} \frac{\sup\{\|\varphi_1(x) - \varphi_2(x)\|_1 : \|x\| \le n\}}{1 + \sup\{\|\varphi_1(x) - \varphi_2(x)\|_1 : \|x\| \le n\}}.$$

2. GENERAL RESULTS

Theorem 1. *Let* $H: U \to n(E)$ *be a multivalued map with values starshaped with respect to* 0 *such that* $H_\lambda: U \to n(E)$ *given by* $H_\lambda(x) = \lambda H(x)$ *satisfies the Lasota-Opial conditions for every* $\lambda \in [0, 1)$. *Moreover, let* $(\mathscr{F}, \rho)$ *be the subspace of the space* $(\mathscr{Z}, \rho)$ *defined by*

$$(2.1) \qquad \mathscr{F} = \{\varphi \in \mathscr{Z} : x - y \in U \Rightarrow T(\varphi(x) - \varphi(y)) \in H(x - y)\}.$$

Suppose that $T: E_1 \to E$ *is continuous. Then* $\mathscr{F}^*$, *the set of all* $\varphi \in \mathscr{F}$ *such that for every* $\xi \in \operatorname{Ker} L$ *the problem* (1.1) *has exactly one solution, is residual in the complete metric space* $(\mathscr{F}, \rho)$.

Proof. Note that H has all values closed in E. Since the space $(\mathscr{Z}, \rho)$ is complete and $\mathscr{F}$ is a closed subset of $\mathscr{Z}$, the space $(\mathscr{F}, \rho)$ is also complete. Define

$$(2.2) \quad \mathscr{M} = \{\varphi \in \mathscr{F} : \exists \lambda \in [0, 1) \ (x - y \in U \Rightarrow T(\varphi(x) - \varphi(y)) \in \lambda H(x - y))\}.$$

Observe that $\mathscr{M}$ is dense in $\mathscr{F}$. Indeed, if $\varphi \in \mathscr{F}$ then for $\varphi_n: E \to E_1$ given by $\varphi_n = n\varphi/(n + 1)$ we have $\rho(\varphi_n, \varphi) \to 0$ as $n \to \infty$ and

$$x - y \in U \Rightarrow T(\varphi_n(x) - \varphi_n(y)) = \frac{n}{n + 1} T(\varphi(x) - \varphi(y)) \in \frac{n}{n + 1} H(x - y),$$

so from (2.2) we get $\varphi_n \in \mathscr{M}$ for $n \in \mathbf{N}$.

For every $\psi \in \mathscr{M}$ and $k \in \mathbf{N}$ we will assign a positive number $\varepsilon(\psi, k)$ in such a way that

$$(2.3) \qquad \bigcap_{k=0}^{\infty} \bigcup_{\psi \in \mathscr{M}} B(\psi, \varepsilon(\psi, k)) \subset \mathscr{F}^*.$$

From this inclusion it follows that $\mathscr{F}^*$ contains a dense G_δ-set, so $\mathscr{F}^*$ is residual in $\mathscr{F}$.

Suppose that $\psi \in \mathscr{M}$ and $k \in \mathbf{N}$. From (2.2) it follows that there exists $\lambda \in [0, 1)$ such that

$$x - y \in U \Rightarrow T \circ \psi(x) - T \circ \psi(y) \in H_\lambda(x - y).$$

Applying Theorem A to $T \circ \psi$ and H_λ we get that $I - T \circ \psi: E \to E$ is a homeomorphism and the problem (1.1) has exactly one solution if we replace φ by $\psi \in \mathscr{M}$. Furthermore, there exists $\delta(\psi, k) \in (0, \frac{1}{k+1})$ satisfying

$$(2.4) \qquad \|x - y\| \geq \frac{1}{k + 1} \Rightarrow \|x - T \circ \psi(x) - (y - T \circ \psi(y))\| > 3\delta(\psi, k).$$

From (1.3) and the continuity of linear T it follows that one can find a positive $\varepsilon(\psi, k)$ such that for any $\varphi \in B(\psi, \varepsilon(\psi, k))$ we have

$$(2.5) \qquad \|x\| \leq 2/\delta(\psi, k) \Rightarrow \|T(\varphi(x) - \psi(x))\| < \delta(\psi, k).$$

Now we are ready to show the inclusion (2.3). Suppose that φ belongs to the left side of this inclusion. This means that there exists a sequence $\{\psi_k\}$ such that $\psi_k \in \mathscr{M}$ and $\rho(\psi_k, \varphi) < \varepsilon(\psi_k, k)$ for $k \in \mathbf{N}$. Define $g: E \to E$ by $g = I - T \circ \varphi$. From (2.4) and (2.5) it follows that if $x, y \in E$ satisfy $\|x\|, \|y\| \leq 2/\delta(\psi_k, k)$ and $\|x - y\| \geq \frac{1}{k+1}$ then

$$\begin{aligned}
\|g(x) - g(y)\| &\geq \|x - y - T(\psi_k(x) - \psi_k(y))\| \\
(2.6) \qquad &\quad - (\|T(\psi_k(x) - \varphi(x))\| + \|T(\psi_k(y) - \varphi(y))\|) \\
&> \delta(\psi_k, k) > 0.
\end{aligned}$$

Since one can choose $k \in \mathbf{N}$ arbitrarily and $2/\delta(\psi_k, k) > 2k + 2$, the map $g = I - T \circ \varphi$ is injective. Therefore, the problem (1.2) (and so the equivalent problem (1.1)) has at most one solution.

The map $H_\lambda = \lambda H$ satisfies the Lasota-Opial conditions for $\lambda \in [0, 1)$, so $\overline{H(A)}$ is compact for every bounded $A \subset E$. Since $\varphi \in \mathscr{F}$, from (2.1) it follows that $T \circ \varphi$ is completely continuous.

Now consider the closed ball $\overline{B}(z, \frac{1}{k+1}) \subset E$, where $\|z\| \leq 1/\delta(\psi_k, k)$. Since g is injective and for $x, y \in \overline{B}(z, \frac{1}{k+1})$ such that $\|x - y\| = \frac{1}{k+1}$ (2.6) holds, by Theorem B we have

$$\|z\| \leq \frac{1}{\delta(\psi_k, k)} \Rightarrow g\left(B\left(z, \frac{1}{k+1}\right)\right) \supset B(g(z), \delta(\psi_k, k)).$$

From this and the inequality $\delta(\psi_k, k) < \frac{1}{k+1} \leq 1$ it follows that

$$g(B(0, 1/\delta(\psi_k, k))) \supset B(g(0), k).$$

Since one can choose $k \in \mathbf{N}$ arbitrarily, the map $g = I - T \circ \varphi$ is surjective. Therefore, the problem (1.2) (and so the equivalent problem (1.1)) has at least one solution, which finishes the proof.

Theorem 2. *Let $H: E \to n(E)$ be a multivalued map with values starshaped with respect to 0 such that*

$$(2.7) \qquad\qquad \lambda > 0, \qquad x \in E \Rightarrow H(\lambda x) = \lambda H(x)$$

and $H_\lambda: E \to n(E)$ given by $H_\lambda = \lambda H$ satisfy the Lasota-Opial conditions for every $\lambda \in [0, 1)$. Let $G: E \to n(E)$ be a multivalueld map with values closed and starshaped with respect to 0 and such that the set $\overline{\bigcup_{x \in E} G(x)}$ is compact. Moreover, let $(\mathscr{G}, \rho)$ be the subspace of the space $(\mathscr{Z}, \rho)$ defined by

$$(2.8) \qquad \mathscr{G} = \{\varphi \in \mathscr{Z} : x \in E \Rightarrow T \circ \varphi(x) \in H(x) + G(x)\}.$$

Suppose that $T: E_1 \to E$ is continuous. Then $\mathscr{G}^*$, the set of all $\varphi \in \mathscr{G}$ such that for every $\xi \in \operatorname{Ker} L$ the problem (1.1) has at least one solution, is residual in the complete space $(\mathscr{G}, \rho)$.

Proof. Since the space $(\mathscr{Z}, \rho)$ is complete and $\mathscr{G}$ is a closed subset of $\mathscr{Z}$, the space $(\mathscr{G}, \rho)$ is also complete. Define

$$\mathscr{N} = \{\varphi \in \mathscr{G} : \exists \lambda \in [0, 1)(x \in E \Rightarrow T \circ \varphi(x) \in H_\lambda(x) + G(x))\}.$$

Note that $\mathscr{N}$ is dense in $\mathscr{G}$. Indeed, if $\varphi \in \mathscr{G}$ then for $\varphi_n: E \to E_1$ given by $\varphi_n = n\varphi/(n + 1)$ we have $\rho(\varphi_n, \varphi) \to 0$ as $n \to \infty$ and

$$x \in E \Rightarrow T \circ \varphi_n(x) = \frac{n}{n + 1} T(\varphi(x)) \in \frac{n}{n + 1} H(x) + \frac{n}{n + 1} G(x),$$

so $\varphi_n \in \mathscr{N}$.

Suppose that $\psi \in \mathscr{N}$. Then there exists $\lambda \in [0, 1)$ such that

$$x \in E \Rightarrow T \circ \psi(x) \in H_\lambda(x) + G(x).$$

We claim that

$$(2.9) \qquad \begin{aligned} &\forall K \geq 0 \exists M > 0 \forall x, y \in E \forall \mu \geq 1 \\ &\quad (\|x\| = M \wedge \|y\| \leq K \Rightarrow \mu x \neq T \circ \psi(x) + y). \end{aligned}$$

To prove this, suppose the contrary. This means that there exist sequences $\{x_m\} \subset E$, $\{y_m\} \subset E$, $\{\mu_m\} \subset [1, \infty)$ such that $\|x_m\| \to \infty$ as $m \to \infty$, $\|y_m\| \leq K$ for $m \in \mathbf{N}$, and

$$\mu_m x_m = T \circ \psi(x_m) + y_m, \qquad m \in \mathbf{N}.$$

Using assumptions imposed on H_λ and G and putting $w_m = x_m/\|x_m\|$ we obtain

$$w_m - \frac{y_m}{\mu_m \|x_m\|} \in H_\lambda(w_m) + \frac{G(x_m)}{\|x_m\|}, \qquad m \in \mathbf{N}.$$

Since $y_m/(\mu_m \|x_m\|) \to 0$, $\sup\{\|z\|: z \in G(x_m)/\|x_m\|\} \to 0$ as $m \to \infty$, and $\overline{H_\lambda(\overline{B}(0, 1))}$ is compact, we have passing if necessary to a subsequence that $w_m \to w$ as $m \to \infty$. The graph of H_λ is closed, so $w \in H_\lambda(w)$, which implies that $w = 0$. But $\|w_m\| = 1$ for $m \in \mathbf{N}$ and $\|w\| = 1$, which leads to a contradiction.

From the condition (2.9) it follows that $I - T \circ \psi: E \to E$ is a surjective map. In fact, if $\|y\| = K$, then define a continuous retraction $\tau: E \to \overline{B}(0, M)$ by

$$\tau(x) = \begin{cases} x, & \|x\| \leq M, \\ Mx/\|x\|, & \|x\| > M, \end{cases}$$

where M is a constant satisfying the condition (2.9). Define also a compact map $h: E \to E$ by

$$h(x) = T \circ \psi \circ \tau(x) + y.$$

This map must have a fixed point x_0, but, since (2.9) holds, this point cannot lie outside the open ball $B(0, M)$. Thus, $\tau(x_0) = x_0$ and x_0 is also a fixed point of the map $T \circ \psi + y$, so $(I - T \circ \psi)(x_0) = y$.

For every $\psi \in \mathcal{N}$ and every $k \in \mathbf{N}$ we will assign a positive number $\eta(\psi, k)$ in such a way that

$$(2.10) \qquad \bigcap_{k=0}^{\infty} \bigcup_{\psi \in \mathcal{N}} B(\psi, \eta(\psi, k)) \subset \mathcal{G}^*.$$

Suppose that $\psi \in \mathcal{N}$ and $k \in \mathbf{N}$. Let $M(\psi, k)$ be such a positive constant that for all $\mu \geq 1$, $\|x\| = M(\psi, k)$, and $\|y\| \leq k$ we have $\mu x \neq T \circ \psi(x) + y$ (the existence of $M(\psi, k)$ is guaranteed by (2.9)). Since $T \circ \psi$ is completely continuous and $\operatorname{Ker} L$ is finite dimensional, one can find a positive $\delta(\psi, k)$ such that

$$(2.11) \qquad \begin{aligned} &\mu \geq 1, \ \xi \in \operatorname{Ker} L, \ \|\xi\| \leq k, \ \|x\| = M(\psi, k) \\ &\Rightarrow \|\mu x - T \circ \psi(x) - \xi\| > \delta(\psi, k). \end{aligned}$$

From (1.3) and the continuity of linear T it follows that one can find a positive $\eta(\psi, k)$ such that for any $\varphi \in B(\psi, \eta(\psi, k))$ we have

$$(2.12) \qquad \|x\| = M(\psi, k) \Rightarrow \|T(\varphi(x) - \psi(x))\| < \tfrac{1}{2}\delta(\psi, k).$$

Finally we show the inclusion (2.10). Suppose that φ belongs to the left side of (2.10). This means that there exists a sequence $\{\psi_k\}$ such that $\psi_k \in \mathcal{N}$ and $\rho(\psi_k, \varphi) < \eta(\psi_k, k)$ for $k \in \mathbf{N}$. From (2.11) and (2.12) it follows that for $\mu \geq 1$, $\|x\| = M(\psi_k, k)$, $\xi \in \operatorname{Ker} L$, and $\|\xi\| \leq k$ we have

$$\|\mu x - T \circ \varphi(x) - \xi\| > \tfrac{1}{2}\delta(\psi_k, k).$$

Since we can choose $k \in \mathbf{N}$ arbitrarily, we have

$$\forall K \geq 0 \, \exists M > 0 \, \forall x \in E \, \forall \xi \in \operatorname{Ker} L \, \forall \mu \geq 1$$
$$(\|x\| = M \wedge \|\xi\| \leq K \Rightarrow \mu x \neq T \circ \varphi(x) + \xi).$$

Using this condition one can show, similarly as for $\psi \in \mathcal{N}$, that $(I - T \circ \varphi)(E) \supset \operatorname{Ker} L$. Therefore, the problem (1.2) (and so the equivalent problem (1.1)) has at least one solution, which proves the inclusion (2.10). From this inclusion it follows that $\mathscr{G}^*$ contains a dense G_δ-set, so $\mathscr{G}^*$ is residual in $\mathscr{G}$, which finishes the proof.

3. The Nicoletti problem

We say that $f: [a, b] \times \mathbf{R}^d \to \mathbf{R}^d$ is a Carathéodory map if $f(\cdot, x): [a, b] \to \mathbf{R}^d$ is measurable for every $x \in \mathbf{R}^d$ and $f(t, \cdot): \mathbf{R}^d \to \mathbf{R}^d$ is continuous for every $t \in [a, b]$. Consider the Nicoletti problem

$$(3.1) \qquad \begin{cases} x'(t) = f(t, x(t)) & \text{a.e. on } [a, b], \\ N(x) = r, \end{cases}$$

where $f: [a, b] \times \mathbf{R}^d \to \mathbf{R}^d$ is a Carathéodory map, $r \in \mathbf{R}^d$, and $N: C([a, b], \mathbf{R}^d) \to \mathbf{R}^d$ is given by $N(x) = (x_1(t_1), x_2(t_2), \ldots, x_d(t_d))$, where $t_1, t_2, \ldots, t_d \in [a, b]$. By a solution of (3.1) we mean any absolutely continuous map $x: [a, b] \to \mathbf{R}^d$ satisfying (3.1).

In what follows by E we denote the space $C([a, b], \mathbf{R}^d)$ of all continuous maps from $[a, b]$ to $\mathbf{R}^d$ with the norm $\|x\| = \sup\{|x(t)|: t \in [a, b]\}$, where $|\cdot|$ denotes the Euclidean norm in $\mathbf{R}^d$. By X we mean the linear subspace of E of all absolutely continuous $x: [a, b] \to \mathbf{R}^d$. By E_1 we denote the space $L^1([a, b], \mathbf{R}^d)$ of all Lebesgue integrable maps from $[a, b]$ to $\mathbf{R}^d$ with the norm $\|x\|_1 = \int_a^b |x(t)| \, dt$.

Now consider the space $(\widetilde{\mathscr{X}}, \tilde{\rho})$ of all Carathéodory maps $f: [a, b] \times \mathbf{R}^d \to \mathbf{R}^d$ such that

$$\int_a^b \sup\{|f(t, u)|: |u| \leq n\} \, dt < \infty \quad \text{for } n \in \mathbf{N}$$

endowed with the metric $\tilde{\rho}$ given by

$$\tilde{\rho}(f_1, f_2) = \sum_{n=1}^{\infty} 2^{-n} \frac{\int_a^b \sup\{|f_1(t, u) - f_2(t, u)|: |u| \leq n\} \, dt}{1 + \int_a^b \sup\{|f_1(t, u) - f_2(t, u)|: |u| \leq n\} \, dt}.$$

We identify maps which differ only for t belonging to a set of measure zero. One can observe that this space is complete.

Let $(\mathscr{X}, \rho)$ be the space of all continuous maps $\varphi: E \to E_1$ such that there exists a Carathéodory map $f \in \widetilde{\mathscr{X}}$ satisfying

$$(3.2) \qquad x \in E \Rightarrow \varphi(x)(t) = f(t, x(t)) \quad \text{a.e. on } [a, b]$$

with the metric ρ given by (1.3).

Lemma. *Spaces $(\mathscr{X}, \rho)$ and $(\widetilde{\mathscr{X}}, \tilde{\rho})$ are isometric.*

Proof. We will show that $J: \widetilde{\mathscr{X}} \to \mathscr{X}$ defined by associating with $f \in \widetilde{\mathscr{X}}$ the map $\varphi \in \mathscr{X}$ by (3.2) is an isometry. It suffices to prove that for every $f \in \widetilde{\mathscr{X}}$

and $n \in \mathbf{N}$ we have

$$\int_a^b \sup\{|f(t, u)|: |u| \leq n\}\, dt = \sup\left\{\int_a^b |f(t, x(t))|\, dt: \|x\| \leq n\right\}.$$

Let $\{u_m\}$ be a sequence which is dense in $\overline{B}(0, n) \subset \mathbf{R}^d$. Notice that

$$\lim_{m \to \infty} \int_a^b \max\{|f(t, u)|: u = u_1, \ldots, u_m\}\, dt = \int_a^b \sup\{|f(t, u)|: |u| \leq n\}\, dt.$$

For $i = 1, \ldots, m$ define sets $A_{m,i}$ in such a way that $t \in [a, b]$ belongs to $A_{m,i}$ iff

$$|f(t, u_i)| = \max\{|f(t, u)|: u = u_1, \ldots, u_m\}$$

and

$$j < i \Rightarrow |f(t, u_i)| > |f(t, u_j)|.$$

One can observe that for a fixed m sets $A_{m,i}$ are measurable, disjoint, and $\bigcup_{i=1}^m A_{m,i} = [a, b]$. Define an integrable map $\tilde{x}_m: [a, b] \to \mathbf{R}^d$ by

$$\tilde{x}_m(t) = u_i \Leftrightarrow t \in A_{m,i}.$$

Now from Luzin's theorem we obtain that for every $m \in \mathbf{N}$ there exists a continuous map $x_m: [a, b] \to \mathbf{R}^d$ such that $x_m|_{D_m} = \tilde{x}_m|_{D_m}$, where D_m is closed in $[a, b]$ and $\lim_{m \to \infty} \mu(D_m) = b - a$. One can check that

$$\lim_{m \to \infty} \int_a^b |f(t, x_m(t))|\, dt = \int_a^b \sup\{|f(t, u)|: |u| \leq n\}\, dt.$$

Therefore, we have

$$\int_a^b \sup\{|f(t, u)|: |u| \leq n\}\, dt \leq \sup\left\{\int_a^b |f(t, x(t))|\, dt: \|x\| \leq n\right\}.$$

This completes the proof since the inverse inequality is obvious.

In [4] Lasota and Olech proved the following

Theorem C. *Suppose that $p, q: [a, b] \to [0, \infty)$ are Lebesgue integrable and p satisfies the inequality*

$$(3.3) \qquad \int_a^b p(t)\, dt < \frac{\pi}{2}.$$

Then:
   (i) *the map $x \equiv 0$ is the only one absolutely continuous of $[a, b]$ into $\mathbf{R}^d$ which satisfies the conditions*

$$N(x) = 0,$$
$$|x'(t)| \leq p(t)|x(t)| \quad \text{a.e. on } [a, b];$$

   (ii) *if a Carathéodory map $f: [a, b] \times \mathbf{R}^d \to \mathbf{R}^d$ satisfies the inequality*

$$|f(t, u) - f(t, v)| \leq p(t)|u - v| \quad \text{for } u, v \in \mathbf{R}^d, \ t \in [a, b],$$

and $f(\cdot, 0)$ is integrable, then the problem (3.1) has exactly one solution;

(iii) *if a Carathéodory map* $f\colon [a, b] \times \mathbf{R}^d \to \mathbf{R}^d$ *satisfies the inequality*

$$|f(t, u)| \leq p(t)|u| + q(t) \quad \text{for } u \in \mathbf{R}^d, \ t \in [a, b],$$

then the problem (3.1) *has at least one solution.*

An example in [4] shows that '$<$' in (3.3) cannot be replaced by '$\leq$'. However, such a situation is to be considered quite exceptional in view of the following results.

Theorem 3. *Assume that* $p\colon [a, b] \to [0, \infty)$ *is an integrable map satisfying the inequality*

$$(3.4) \qquad\qquad \int_a^b p(t)\,dt \leq \frac{\pi}{2}.$$

Let $\mathscr{F}_1$ *denote the subset of* $\widetilde{\mathscr{X}}$ *of all* f *satisfying*

$$|f(t, u) - f(t, v)| \leq p(t)|u - v| \quad \text{for } u, v \in \mathbf{R}^d, \ t \in [a, b].$$

Then $\mathscr{F}_1^*$, *the set of all* $f \in \mathscr{F}_1$ *such that for every* $r \in \mathbf{R}^d$ *the problem* (3.1) *has exactly one solution, is residual in the complete space* $(\mathscr{F}_1, \tilde{p})$.

Proof. If we put $\varphi = J(f)$, $L(x)(t) = x'(t)$, $\xi(t) = r$, and $P(x)(t) = N(x)$ for $t \in [a, b]$, then the problem (3.1) becomes equivalent to the problem (1.1). The continuous right inverse $T\colon E_1 \to E$ of L is given by

$$T(y)(t)_i = \int_{t_i}^t y_i(s)\,ds, \qquad i = 1, 2, \ldots, d.$$

Define the multivalued map $H\colon E \to n(E)$ by

$$
(3.5) \qquad
H(x) = \Bigg\{ z \in E\colon N(z) = 0, \ a \leq s \leq t \leq b \\
\Rightarrow |z(t) - z(s)| \leq \int_s^t p(u)|x(u)|\,du \Bigg\}.
$$

Note that H has values starshaped with respect to 0. Furthermore, $H_\lambda = \lambda H$ satisfies the Lasota-Opial conditions for every $\lambda \in [0, 1)$. In fact, if $x \in H_\lambda(x)$ then from (3.5) it follows that $N(x) = 0$ and

$$|x'(t)| \leq \lambda p(t)|x(t)| \quad \text{a.e. on } [a, b],$$

so by Theorem C we have that $x = 0$. Two remaining conditions also follows from the definitions of H and H_λ.

Now using Theorem 1 we have that $\mathscr{F}^*$, the set of all $\varphi \in \mathscr{F}$ such that for every $\xi \in \operatorname{Ker} L$ the problem (1.1) has exactly one solution, is residual in the complete space $(\mathscr{F}, \rho)$, where $\mathscr{F}$ is given by (2.1).

Observe that $J(\mathscr{F}_1) = \mathscr{F}$. In fact, it is obvious that $J(\mathscr{F}_1) \subset \mathscr{F}$. Now suppose that $J(f) \in \mathscr{F}$, where $f \in \widetilde{\mathscr{X}}$. This says that

$$(3.6) \qquad \left| \int_s^t (f(u, x(u)) - f(u, y(u)))\,du \right| \leq \int_s^t p(u)|x(u) - y(u)|\,du$$

for all $a \leq s \leq t \leq b$ and $x, y \in E$. Choose a sequence $\{(v_m, w_m)\}$ dense in $\mathbf{R}^d \times \mathbf{R}^d$. Substituting v_m and w_m for $x(u)$ and $y(u)$ in (3.6), dividing this inequality by $|t - s|$, and passing $t \to s$ yields that the set of s satisfying

$$(3.7) \qquad\qquad |f(s, v_m) - f(s, w_m)| \leq p(s)|v_m - w_m|$$

for a fixed $m \in \mathbf{N}$ has measure $b - a$. Therefore, the measure of the set of s satisfying (3.7) for all $m \in \mathbf{N}$ has also measure $b - a$. This means that $f \in \mathscr{F}_1$ because $f(s, \cdot)$ is continuous for every $s \in [a, b]$. Since $J(\mathscr{F}_1) = \mathscr{F}$ and J is an isometry, we find that $\mathscr{F}_1^* = J^{-1}(\mathscr{F}^*)$ is residual in $\mathscr{F}_1$, which completes the proof.

Theorem 4. *Assume that* $p, q: [a, b] \to [0, \infty)$ *are integrable maps and* p *satisfies the inequality* (3.4). *Let* $\mathscr{G}_1$ *denote the subset of* $\widetilde{\mathscr{X}}$ *of all* f *satisfying*

$$|f(t, u)| \le p(t)|u| + q(t) \quad \text{for } u \in \mathbf{R}^d, \ t \in [a, b].$$

Then $\mathscr{G}_1^*$, *the set of all* $f \in \mathscr{G}_1$ *such that for every* $r \in \mathbf{R}^d$ *the problem* (3.1) *has at least one solution, is residual in the complete space* $(\mathscr{G}_1, \tilde{p})$.

Proof. As before, by putting $\varphi = J(f)$, $L(x)(t) = x'(t)$, $\xi(t) = r$, and $P(x)(t) = N(x)$ for $t \in [a, b]$ we consider problem (1.1) instead of the problem (3.1). We define the maps $T: E_1 \to E$ and $H: E \to n(E)$ as in the proof of Theorem 3. The multivalued map $G: E \to n(E)$ we define by

$$G(x) = \left\{ z \in E : N(z) = 0, \ a \le s \le t \le b \Rightarrow |z(t) - z(s)| \le \int_s^t q(u)\, du \right\}.$$

Note that G has closed values starshaped with respect to 0 and compact image. Furthermore, H satisfies (2.7). Now using Theorem 2 we have that $\mathscr{G}^*$, the set of all $\varphi \in \mathscr{G}$ such that for every $r \in \operatorname{Ker} L$ the problem (1.1) has at least one solution, is residual in the space $(\mathscr{G}, \rho)$, where $\mathscr{G}$ is given by (2.8). Note that $J(\mathscr{G}_1) = \mathscr{G}$. In fact, if $f \in \mathscr{G}_1$, then define $\varphi_1, \varphi_2 : E \to E_1$ by

$$\varphi_1(x)(t) = \frac{p(t)|x(t)|f(t, x(t))}{p(t)|x(t)| + q(t)}, \qquad \varphi_2(x)(t) = \frac{q(t)f(t, x(t))}{p(t)|x(t)| + q(t)}$$

(if the denominator is 0, we set $\varphi_1(x)(t) = 0$ and $\varphi_2(x)(t) = f(t, x(t))$). One can see that $J(f) = \varphi_1 + \varphi_2$ and $T \circ \varphi_1(x) \in H(x)$, $T \circ \varphi_2(x) \in G(x)$ for $x \in E$, so $J(\mathscr{G}_1) \subset \mathscr{G}$. The inverse inclusion can be proved similarly as the inclusion $J(\mathscr{F}_1) \supset \mathscr{F}$ in the proof of Theorem 3. Since J is an isometry, the set $\mathscr{G}_1^* = J^{-1}(\mathscr{G}^*)$ is residual in $\mathscr{G}_1$, which finishes the proof.

4. The Floquet problem

Consider the Floquet problem

(4.1)
$$\begin{cases} x'(t) = f(t, x(t)) & \text{a.e. on } [a, b], \\ F(x) = r, \end{cases}$$

where $f: [a, b] \times \mathbf{R}^d \to \mathbf{R}^d$ is a Carathéodory map, $r \in \mathbf{R}^d$, and $F: C([a, b], \mathbf{R}^d) \to \mathbf{R}^d$ is given by $F(x) = x(a) + \zeta x(b)$, where ζ is a positive constant. By a solution of (4.1) we mean any absolutely continuous map $x: [a, b] \to \mathbf{R}^d$ satisfying (4.1).

Spaces $E, X, E_1, (\mathscr{X}, \rho), (\widetilde{\mathscr{X}}, \tilde{p})$ and the isometry $J: \widetilde{\mathscr{X}} \to \mathscr{X}$ we define in the same way as in the §3. In [3] Kasprzyk and Myjak proved the following.

Theorem D. *Suppose that* $p, q: [a, b] \to [0, \infty)$ *are Lebesgue integrable and* p *satisfies the inequality*

(4.2)
$$\int_a^b p(t)\, dt < \sqrt{\pi^2 + \ln^2 \zeta}.$$

Then

(i) *the map $x \equiv 0$ is the only one absolutely continuous of $[a, b]$ into $\mathbf{R}^d$ which satisfies the conditions*

$$F(x) = 0,$$
$$|x'(t)| \leq p(t)|x(t)| \quad a.e. \ on \ [a, b];$$

(ii) *if a Carathéodory map $f: [a, b] \times \mathbf{R}^d \to \mathbf{R}^d$ satisfies the inequality*

$$|f(t, u) - f(t, v)| \leq p(t)|u - v| \quad for \ u, v \in \mathbf{R}^d, \ t \in [a, b],$$

and $f(\cdot, 0)$ is integrable, then the problem (4.1) has exactly one solution;

(iii) *if a Carathéodory map $f: [a, b] \times \mathbf{R}^d \to \mathbf{R}^d$ satisfies the inequality*

$$|f(t, u)| \leq p(t)|u| + q(t) \quad for \ u \in \mathbf{R}^d, \ t \in [a, b],$$

then the problem (4.1) has at least one solution.

An example in [3] shows that '$<$' in (4.2) cannot be replaced by '$\leq$'.

Theorem 5. *Assume that $p: [a, b] \to [0, \infty)$ is an integrable map satisfying the inequality*

$$(4.3) \qquad\qquad \int_a^b p(t)\,dt \leq \sqrt{\pi^2 + \ln^2 \zeta}.$$

Let $\mathscr{F}_2$ denote the subset of $\widetilde{\mathscr{X}}$ of all f satisfying

$$|f(t, u) - f(t, v)| \leq p(t)|u - v| \quad for \ u, v \in \mathbf{R}^d, \ t \in [a, b].$$

Then $\mathscr{F}_2^*$, the set of all $f \in \mathscr{F}_2$ such that for every $r \in \mathbf{R}^d$ the problem (4.1) has exactly one solution, is residual in the complete space $(\mathscr{F}_2, \tilde{p})$.

Proof. This theorem can be proved similarly to Theorem 3; the continuous projection $P: E \to \operatorname{Ker} L$ is given by $P(x)(t) = \frac{1}{1+\zeta} F(x)$ and the continuous right inverse $T: E_1 \to E$ of L by

$$T(y)(t) = \frac{1}{1+\zeta} \int_a^t y(s)\,ds - \frac{\zeta}{1+\zeta} \int_t^b y(s)\,ds.$$

Theorem 6. *Assume that $p, q: [a, b] \to [0, \infty)$ are integrable maps and p satisfies the inequality (4.3). Let $\mathscr{G}_2$ denote the subset of $\widetilde{\mathscr{X}}$ of all f satisfying*

$$|f(t, u)| \leq p(t)|u| + q(t) \quad for \ u \in \mathbf{R}^d, \ t \in [a, b].$$

Then $\mathscr{G}_2^*$, the set of all $f \in \mathscr{G}_2$ such that for every $r \in \mathbf{R}^d$ the problem (4.1) has at least one solution, is residual in the complete space $(\mathscr{G}_2, \tilde{p})$.

The proof is similar to the proof of Theorem 4.

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