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DEGREE BOUNDS FOR INVERSES OF POLYNOMIAL AUTOMORPHISMS

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ABSTRACT. It is known that if k is a field and $\mathbf{F}: k[X_1, \dots, X_n] \to k[X_1, \dots, X_n]$ is a polynomial automorphism, then $\deg(\mathbf{F}^{-1}) \le (\deg \mathbf{F})^{n-1}$. We extend this result to the case where k is a reduced ring. Furthermore, if k is not a reduced ring, we show that for any integer $n \ge 1$ and any integer $\lambda \ge 0$ there exists a polynomial automorphism $\mathbf{F}$ such that $\deg(\mathbf{F}^{-1}) = \lambda + (\deg \mathbf{F})^{n-1}$.

Let k be a commutative ring with an identity element. Any k-algebra homomorphism $\mathbf{F}: k[X_1, \dots, X_n] \to k[X_1, \dots, X_n]$ is determined by the images of the X_i. Thus we denote $\mathbf{F}$ by $(F_1, \dots, F_n)$, where $F_i = \mathbf{F}(X_i)$ for all i. The *degree* of $\mathbf{F}$ is defined to be the maximum of the total degrees of the F_i.

Theorem 1. *Suppose k is a field and $\mathbf{F}: k[X_1, \dots, X_n] \to k[X_1, \dots, X_n]$ is a k-algebra automorphism. Then* $\deg(\mathbf{F}^{-1}) \le (\deg \mathbf{F})^{n-1}$. □

Corollary 2. *Suppose k is an integral domain and $\mathbf{F}: k[X_1, \dots, X_n] \to k[X_1, \dots, X_n]$ is a k-algebra automorphism. Then* $\deg(\mathbf{F}^{-1}) \le (\deg \mathbf{F})^{n-1}$. □

Remark. The inequality in Theorem 1 was first conjectured by Wang in [3, Degree Conjecture 63, p. 491] for the case where $\deg \mathbf{F} = 2$. The general case was first proved by Gabber and published in [2, Corollary (1.4) and Theorem (1.5), pp. 292–293], using algebraic geometry. A more elementary proof was given by Yu in [4, Theorem 2].

In the following we shall show that reduced rings are the best possible rings in which this inequality holds for all polynomial automorphisms.

A *reduced ring* k is a commutative ring whose nilradical (= intersection of all prime ideals) is zero or, equivalently [1, Propositions 1.7 and 1.8, p. 5], a commutative ring with no nonzero nilpotent elements. Given a prime ideal $\mathfrak{p}$ of k, we denote the quotient ring $k/\mathfrak{p}$ by $\overline{k}$. For each polynomial $H = \sum c_{i_1, \dots, i_n} X_1^{i_1} \cdots X_n^{i_n}$ in $k[X_1, \dots, X_n]$, let $\overline{H} = \sum (c_{i_1, \dots, i_n} + \mathfrak{p}) X_1^{i_1} \cdots X_n^{i_n}$ be the induced polynomial in $\overline{k}[X_1, \dots, X_n]$. Note that $\deg \overline{H} \le \deg H$. Also

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note that each k-algebra endomorphism $\mathbf{F} = (F_1, \dots, F_n) : k[X_1, \dots, X_n] \to k[X_1, \dots, X_n]$ induces a $\overline{k}$-algebra endomorphism $\overline{\mathbf{F}} = (\overline{F}_1, \dots, \overline{F}_n) : \overline{k}[X_1, \dots, X_n] \to \overline{k}[X_1, \dots, X_n]$. In this case $\deg \overline{\mathbf{F}} \leq \deg \mathbf{F}$.

Theorem 3. *Suppose k is a reduced ring and $\mathbf{F} : k[X_1, \dots, X_n] \to k[X_1, \dots, X_n]$ is a k-algebra automorphism. Then* $\deg(\mathbf{F}^{-1}) \leq (\deg \mathbf{F})^{n-1}$.

Proof. Let $\mathbf{F}^{-1} = \mathbf{G} = (G_1, \dots, G_n)$ be the inverse of F. Let $1 \leq i \leq n$ be such that $\deg G_i = \deg \mathbf{G}$. Then there exists a prime ideal $\mathfrak{p}$ of k such that $\deg \overline{G}_i = \deg G_i$. Otherwise, $\deg \overline{G}_i < \deg G_i$ for all prime ideals of k and so the nonzero coefficients of the highest degree terms of G_i are in the nilradical of k. Since k is reduced, this contradicts the fact that the nilradical of k is zero. By Corollary 2, $\deg \overline{\mathbf{G}} \leq (\deg \overline{\mathbf{F}})^{n-1}$. Thus,

$$\deg \mathbf{G} = \deg \overline{\mathbf{G}} \leq (\deg \overline{\mathbf{F}})^{n-1} \leq (\deg \mathbf{F})^{n-1}. \quad \square$$

Theorem 4. *Suppose k is not a reduced ring. Then, for any positive integer n and any nonnegative integer λ, there exists a k-algebra automorphism* $\mathbf{F}: k[X_1, \dots, X_n] \to k[X_1, \dots, X_n]$ *such that* $\deg(\mathbf{F}^{-1}) = \lambda + (\deg \mathbf{F})^{n-1}$.

Proof. Since k is not reduced, there exists a nonzero element $v \in k$ such that $v^2 = 0$. (If u is a nonzero nilpotent element in k such that $u^{d+1} = 0$, but $u^d \neq 0$ for some positive integer d, take $v = u^d$.) Thus, for any integer $m \geq 0$ and A, B in $k[X_1, \dots, X_n]$, as a direct consequence of the binomial theorem, we have that

$$(A \pm v\, B)^m = A^m \pm mv\, A^{m-1} B. \tag{1}$$

Now choose a positive integer e such that

(i) $e \geq \lambda + 1$, and
(ii) $e^{n-1} v \neq 0$ (so that none of v, ev, $e^2 v$, $\dots$, $e^{n-1} v$ is zero).

This can be done as follows. The annihilator of v in $\mathbb{Z}$, i.e., $\{l \in \mathbb{Z} \mid lv = 0\}$, is a principal ideal generated by a nonnegative integer μ. Take

$$e = \begin{cases} \lambda + 1, & \text{if } \mu = 0, \\ \lambda\mu + 1, & \text{if } \mu \geq 1. \end{cases}$$

Define $\mathbf{F} = (F_1, \dots, F_n): k[X_1, \dots, X_n] \to k[X_1, \dots, X_n]$ by

$$F_1 = X_1 - v\, X_1^{\lambda+1}, \qquad F_i = X_i - X_{i-1}^e \quad \text{for } i = 2, \dots, n.$$

We next show that the inverse of $\mathbf{F}$ is $\mathbf{G} = (G_1, \dots, G_n)$ where

$$G_1 = X_1 + v\, X_1^{\lambda+1}, \qquad G_i = X_i + G_{i-1}^e \quad \text{for } i = 2, \dots, n.$$

Note that

$$\begin{aligned} F_1(G_1, \dots, G_n) &= G_1 - v\, G_1^{\lambda+1} \quad \text{(by the definition of } F_1) \\ &= X_1 + v\, X_1^{\lambda+1} - v\, (X_1 + v\, X_1^{\lambda+1})^{\lambda+1} \quad \text{(by the definition of } G_1) \\ &= X_1 + v\, X_1^{\lambda+1} - v\, \left(X_1^{\lambda+1} + (\lambda+1) v\, X_1^{2\lambda+1}\right) \quad \text{(by (1))} \\ &= X_1. \end{aligned}$$

Similarly $G_1(F_1, \ldots, F_n) = X_1$. For $2 \le i \le n$,

$$\begin{aligned} F_i(G_1, \ldots, G_n) &= G_i - G_{i-1}^e \quad \text{(by the definition of } F_i\text{)} \\ &= X_i \quad \text{(by the definition of } G_i\text{)}, \\ G_i(F_1, \ldots, F_n) &= F_i + G_{i-1}(F_1, \ldots, F_n)^e \quad \text{(by the definition of } G_i\text{)} \\ &= F_i + X_{i-1}^e \quad \text{(by induction on } i\text{)} \\ &= X_i \quad \text{(by the definition of } F_i\text{)}. \end{aligned}$$

Therefore, $\mathbf{F}$ and $\mathbf{G}$ are inverses of each other.

We next compute $\deg \mathbf{G}$. If $e \ge 2$, then using (1) we can prove by induction on i that

$$\begin{aligned} G_i = &(X_1^{e^{i-1}} + \text{lower degree terms}) \\ &+ v(e^{i-1} X_1^{\lambda + e^{i-1}} + \text{lower degree terms}) \end{aligned}$$

where "lower degree terms" involves no v. Hence, $\deg G_i = \lambda + e^{i-1}$ by condition (ii). If $e = 1$, then we can prove by induction on i that $G_i = (X_1 + \cdots + X_i) + vX_1^{\lambda+1}$. Hence, $\deg G_i = \lambda + 1 = \lambda + e^{i-1}$ in this case. Therefore,

$$\deg \mathbf{G} = \max_{1 \le i \le n} \{\deg G_i\} = \max_{1 \le i \le n} \{\lambda + e^{i-1}\} = \lambda + e^{n-1},$$

i.e., $\deg \mathbf{G} = \lambda + (\deg \mathbf{F})^{n-1}$, since $\deg \mathbf{F} = e$ by condition (i). $\square$

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