SPECTRUM OF THE PRODUCTS OF OPERATORS AND COMPACT PERTURBATIONS

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Abstract. In this paper we will give a complete characterization of the operator $B$ which satisfies the spectral condition $\sigma(AB) = \sigma(BA)$ (resp. $\sigma_e(AB) = \sigma_e(BA)$) for every $A$ in $L(H)$ and also a spectral characterizations of the product of finitely many normal (resp. essentially normal) operators.

0. Introduction

Let $H$ be an infinite-dimensional complex Hilbert space, $L(H)$ be the algebra of all bounded linear operators on $H$, and $K(H)$ be the compact operator ideal of $L(H)$. For $B$ in $L(H)$, $\sigma(B)$ and $\sigma_e(B)$ denote the spectrum and the essential spectrum of $B$ respectively.

For $B$ in $L(H)$, we say that $B$ is consistent in invertibility (with respect to multiplication) or, briefly, a CI operator if, for each $A$ in $L(H)$, $AB$ and $BA$ are invertible or noninvertible together. By Jacobson's Theorem (for $A, B \in L(H)$, the nonzero elements of $\sigma(AB)$ and $\sigma(BA)$ are the same), $B$ is a CI operator if and only if $\sigma(AB) = \sigma(BA)$ for every $A$ in $L(H)$. Thus if $A$ and $B$ are CI operators, then so is $AB$. Our problem is: which elements in $L(H)$ are CI operators? Paper [6] caused the authors to think that this is a significant problem and remains to be considered.

The main purpose of this paper is to give a complete characterization of $B$ in $L(H)$ which satisfies the spectral condition $\sigma(AB) = \sigma(BA)$ (resp. $\sigma_e(AB) = \sigma_e(BA)$) for every $A$ in $L(H)$. We also give a complete characterization of the CI operators which are invariant under compact perturbations.

1. Fundamental theorem

Every $B$ in $L(H)$ must be included in one of the following five cases, and in each of them the problem is definitely answered. Theorem 11.2 in [7, p. 224] is useful in the proofs below.

Case 1. If $B$ is invertible, then $B$ is a CI operator.

Proof. It is sufficient to note that for every $A$ in $L(H)$, $AB = B^{-1}(BA)B$.

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Case 2. If \( \text{ran}\, B \) is not closed, then \( B \) is a CI operator.

Proof. It follows from \( \text{ran}\, B A \subseteq \text{ran}\, B \subseteq \overline{\text{ran}\, B} \subseteq H \) for every \( A \) in \( L(H) \) that \( B A \) is not invertible.

It is now to be proved that, for every \( A \) in \( L(H) \), \( AB \) is also not invertible. If, for some \( A \in L(H) \), \( AB \) were invertible, the expression \( (AB)^{-1} AB = (AB)^{-1}(AB) = I \) indicates that \( B \) is bounded from below. Then \( \text{ran}\, B \) is closed, which contradicts the assumption.

Case 3. If \( \ker B \neq 0 \) and \( \overline{\text{ran}\, B} \subseteq H \), then \( B \) is a CI operator.

Proof. For each \( A \) in \( L(H) \), \( \ker AB \supseteq \ker B \neq 0 \) implies that \( AB \) is not invertible and \( \text{ran}\, B A \subseteq \overline{\text{ran}\, B} \subseteq H \) implies that \( B A \) is not invertible.

Case 4. If \( \ker B = 0 \) and \( \text{ran}\, B = \overline{\text{ran}\, B} \subseteq H \), then \( B^* B \) is invertible while \( B B^* \) is not invertible, and so \( B \) is not a CI operator.

Proof. It follows from \( \text{ran}\, B B^* \subseteq \text{ran}\, B \subset H \) that \( B B^* \) is not invertible.

Since \( B \) has closed range if and only if \( B^* B \) has \( \sigma(B^* B) = \sigma(BB^*) \).

Case 5. If \( \ker B \neq 0 \) and \( \overline{\text{ran}\, B} \subseteq \text{ran}\, B \subseteq H \), then \( B^* B \) is not invertible while \( B B^* \) is invertible, and so \( B \) is not a CI operator.

Proof. It follows from \( \ker B \neq 0 \) and \( \overline{\text{ran}\, B} \subseteq \text{ran}\, B \subseteq H \) that \( \ker B^* = 0 \) and \( \text{ran}\, B^* \subseteq \overline{\text{ran}\, B} \subseteq H \). Therefore, by replacing \( B \) by \( B^* \) in the proof of Case 4, we obtain that \( B^* B \) is not invertible and \( BB^* \) is invertible.

By the results just proved above we can conclude

**Theorem 1.1.** \( B \in L(H) \) is a CI operator if and only if one of the following three mutually disjoint cases occurs:

1. \( B \) is invertible.
2. \( \text{ran}\, B \) is not closed.
3. \( \ker B \neq 0 \) and \( \overline{\text{ran}\, B} \subseteq H \).

**Corollary 1.2.** \( B \in L(H) \) is a CI operator if and only if \( B^* B \) and \( BB^* \) are invertible or noninvertible together, i.e., \( \sigma(B^* B) = \sigma(BB^*) \).

The following corollary is a natural complement to the above results and its proof is straightforward.

**Corollary 1.3.** Let \( B \in L(H) \). If \( \ker B = 0 = \ker B^* \), then \( B \) is a CI operator.

**Remark 1.4.** From the proofs above, we can also see that the CI operators can be classified into two parts: (1) there is an \( A \) in \( L(H) \) such that \( AB \) and \( BA \) are invertible together (this is the case if and only if \( B \) is invertible); and (2) for all \( A \) in \( L(H) \), \( AB \) and \( BA \) are always noninvertible (if and only if either \( \text{ran}\, B \) is nonclosed or \( \ker B \neq 0 \) and \( \overline{\text{ran}\, B} \subseteq H \)).

**Remark 1.5.** \( B \) is a CI operator if and only if so is \( B^* \).

2. **Examples**

By the preceding results, normal, compact, and invertible operators are immediately examples of CI operators and so are their products. Next we shall consider their generalizations.
An operator \( B \in L(H) \) such that \( \|Bx\| \geq \|B^*x\| \) for each \( x \) in \( H \) is called hyponormal. Obviously, \( \ker B \subseteq \ker B^* \).

2.1. If \( B \in L(H) \) is hyponormal and \( \text{ran} \ B \) is closed, then \( B \) is a CI operator if and only if (1) \( \ker B \neq 0 \), or (2) \( \ker B^* = 0 \).

Note that if \( \text{ran} \ B \) is not closed, then, from Theorem 1.1, \( B \) is a CI operator.

Proof. The conclusion can be obtained, when \( \ker B \neq 0 \), from \( \ker B^* \supseteq \ker B \neq 0 \) and Theorem 1.1 (3), and, when \( \ker B^* = 0 \), from \( \ker B \subseteq \ker B^* = 0 \) and Corollary 1.3.

If \( B \) is a CI operator, then one of the two cases (1) and (3) in Theorem 1.1 must occur. Case (1) implies \( \ker B^* = 0 \) and case (3) implies \( \ker B \neq 0 \).

2.2. If \( B \in L(H) \) is hyponormal, then \( B \) is a CI operator if and only if either:

(1) \( BB^* \) is invertible, or
(2) \( B^*B \) is noninvertible.

Proof. Necessity is trivial.

If \( BB^* \) is invertible, then it follows from \( \text{ran} \ B \supseteq \text{ran} \ BB^* = H \) and \( \ker B \subseteq \ker B^* = \ker BB^* = 0 \) that \( B \) is invertible, hence \( B^*B \) is invertible. This also leads to that, if \( B^*B \) is noninvertible, then so is \( BB^* \).

Remark 2.3. We say that \( B \in L(H) \) is \( M \)-hyponormal if there is an \( M > 0 \) such that

\[ \| (B - \lambda)^* x \| \leq M \| (B - \lambda) x \| \]

for all \( x \in H \) and all complex number \( \lambda \). Clearly, 2.1 and 2.2 remain true for \( M \)-hyponormal operator \( B \) with the proof unchanged.

Since \( B \in L(H) \) is an isometry if and only if \( B^*B = I \), we have

2.4. If \( B \in L(H) \) is an isometry, then \( B \) is a CI operator if and only if \( B \) is unitary.

2.5. If \( B \) is invertible, then \( B + K \) is a CI operator for every \( K \in K(H) \).
This follows from \( \text{ind} (B + K) = \text{ind} B = 0 \). It should be noted that there exist Fredholm operators which are not CI operators.

2.6. If \( B \in L(H) \) such that \( \sigma(B) \) is a singleton, then \( B + K \) is a CI operator for every \( K \) in \( K(H) \). In particular, if \( B \) is a Riesz operator, then \( B \) is a CI operator.

Recall that \( B \) is a Riesz operator if and only if \( \sigma_e(B) = 0 \).

Proof. If \( \sigma(B) \neq 0 \), then this follows from 2.5. Now suppose that \( \sigma(B) = 0 \).
Then \( \sigma_e(B + K) = 0 = \sigma_e(B) \), which implies that \( B + K \) is not semi-Fredholm. By Theorem 2.5 in [4, p. 356], we have that either \( \text{ran} (B + K) \) is not closed or \( \text{dim ker} (B + K) = \text{dim ker} (B + K)^* = \infty \). Thus, by Theorem 1.1, \( B + K \) is a CI operator. If \( B \) is a Riesz operator, it is known from [5, Theorem 3.3] that \( B = C + K \), where \( \sigma(C) = 0 \) and \( K \in L(H) \); whence, \( B \) is a CI operator.

3. Compact perturbations and normality

In this section, we shall give a spectral characterization of the operators which are the products of finitely many normal operators (resp. essentially normal operators). In what follows, \( H \) will be a fixed separable complex Hilbert space.

The following Theorem 3.1 is due to [8, Theorem 1.1].
Theorem 3.1. If $B \in L(H)$, then the following statements are equivalent:
(1) $B$ is the product of finitely many normal operators.
(2) $\dim \ker B = \dim \ker B^*$ or $\text{ran } B$ is not closed.
(3) $B$ is the norm limit of a sequence of invertible operators.

We shall prove

Theorem 3.2. If $B \in L(H)$, then the following conditions are equivalent to those in Theorem 3.1:
(4) $\sigma(A(B + K)) = \sigma((B + K)A)$ for every $A$ in $L(H)$ and $K$ in $K(H)$, i.e., $B + K$ is a CI operator for every $K$ in $K(H)$.
(5) $\sigma(A(B + F)) = \sigma((B + F)A)$ for every $A$ in $L(H)$ and finite rank operator $F$.

Proof. (5) $\Rightarrow$ (2). If $\text{ran } B$ is closed, we claim that $\dim \ker B = \dim \ker B^*$. Otherwise, we may assume that $\dim \ker B < \dim \ker B^*$. Then, by Proposition 3.21 in [4, p. 366], there exists a finite rank operator $F$ such that $\ker(B+F) = 0$ and $\text{ind}(B+F) = \text{ind } B \neq 0$. Therefore, by Theorem 1.1, $B + F$ is not a CI operator, which leads to a contradiction.

(4) $\Rightarrow$ (5) is obvious.

(2) $\Rightarrow$ (4). Let $K \in K(H)$. If $\text{ran}(B + K)$ is not closed, then, by Theorem 1.1, $B + K$ is a CI operator. Now suppose that $\text{ran}(B + K)$ is closed. If $\dim \ker(B + K) < \infty$, then $\dim \ker B < \infty$ and $\text{ran } B$ is closed, thus $\dim \ker B = \dim \ker B^*$ which implies $\dim \ker(B + K) = \dim \ker(B + K)^*$. If $\dim \ker(B + K) = \infty$, a similar argument shows that $\dim \ker(B + K)^* = \infty$. Therefore, in these two cases, $B + K$ is a CI operator.

Corollary 3.3. If $\sigma(B)$ is a singleton, then $B$ satisfies the conditions in Theorem 3.1.

Corollary 3.4. Let $B \in L(H)$ be an essentially normal operator, i.e. $B^*B - BB^* \in K(H)$. Then the following statements are equivalent:
(1) There is a normal operator $N$ and a compact operator $K$ such that $B = N + K$.
(2) $p(B) + K$ is a CI operator for any polynomial $p$ and any $K \in K(H)$.
(3) $p(B)$ is the product of finitely many normal operators for any polynomial $p$.

This corollary may be regarded as a new version of the BDF theorem [3].

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) follows from Theorem 3.2 immediately. Now we show that (2) implies (1). By the Brown-Douglas-Fillmore theorem [3], it is sufficient to show that $\text{ind}(B - \lambda I) = 0$ whenever $\lambda \notin \sigma_e(B)$. If there exists $\lambda \notin \sigma_e(B)$ such that $\text{ind}(B - \lambda I) \neq 0$, we may assume that $\dim \ker(B - \lambda I) < \dim \ker(B - \lambda I)^*$. Then, by Proposition 3.21 in [4, p. 366], there is $K \in K(H)$ such that $\ker(B - \lambda I + K) = 0$ and $\text{ind}(B - \lambda I + K) = \text{ind}(B - \lambda I) \neq 0$. Thus, by Theorem 1.1, $B - \lambda I + K$ is not a CI operator, which contradicts the assumption.

Remark 3.5. There exists a nonessentially normal operator which satisfies the conditions (2) and (3) of Corollary 3.4. For example, let $B \in L(H)$ such that $\sigma(B) = 0$ and $B \notin K(H)$. By $\sigma_e(B) = 0$ and $B \in K(H)$, $B$ is not essentially normal; but, by Corollary 3.3, $p(B)$ is the product of finitely many normal operators for every polynomial $p$. 

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Lemma 3.6. If $\sigma_e(AB) = \sigma_e(BA)$ for every $A$ in $L(H)$, then one of the following statements holds:

(a) $B$ is the product of finitely many normal operators.
(b) $B$ is a Fredholm operator.

Proof. From Theorem 3.1, it is sufficient to show that if $B$ is semi-Fredholm, then $B$ is Fredholm.

We may assume that $\dim \ker B < \infty$ and $\text{ran} B$ is closed. Then there is $A \in L(H)$ such that $AB - I \in K(H)$; thus, $0 \notin \sigma_e(AB) = \sigma_e(BA)$, which implies that $B$ is Fredholm.

The following theorem, an analogue of Theorem 3.2 in the Calkin algebra, gives a spectral characterization of compact perturbations of the product of finitely many essentially normal operators.

Theorem 3.7. Let $B \in L(H)$. Then the following statements are equivalent:

1. $\sigma_e(AB) = \sigma_e(BA)$ for every $A$ in $L(H)$.
2. $B$ is a compact perturbation of the product of finitely many essentially normal operators; i.e., $\pi(B)$ is the product of finitely many normal elements in $L(H)/K(H)$.
3. $B$ is the norm limit of a sequence of Fredholm operators; i.e., $\pi(B)$ is the norm limit of invertible elements in $L(H)/K(H)$, where $\pi$ is the canonical map of $L(H)$ onto $L(H)/K(H)$.

Proof. From [1, Theorem 4] we have that (3) is equivalent to the condition that $\dim \ker B = \dim \ker B^*$, $\text{ran} B$ is not closed, or $B$ is Fredholm. Thus (1) implies (3) by Lemma 3.6.

(3) $\Rightarrow$ (2). It is sufficient to prove that, if $B$ is Fredholm, then $B$ must satisfy (2).

Let $\phi$ be a faithful unital *-representation of the Calkin algebra $L(H)/K(H)$ on a Hilbert space $H'$. Then $\phi(\pi(B))$ is invertible in $\text{ran} \phi$. Let $R = (\phi(\pi(B))^*\phi(\pi(B)))^{1/2} \in \text{ran} \phi$ and $S = \phi(\pi(B))R^{-1} \in \text{ran} \phi$. Then $S^*S = SS^* = I$, and thus $\phi(\pi(B)) = SR$ is the product of normal operators. If $\pi(B_1) = \phi^{-1}(S), \pi(B_2) = \phi^{-1}(R)$, then $B_1$ and $B_2$ are essentially normal operators and $\pi(B) = \pi(B_1B_2)$. Therefore, $B$ satisfies (2).

(2) $\Rightarrow$ (1). Obviously, we may suppose that $B$ is essentially normal. Let $\phi$ be as above. Then $\phi(\pi(B))$ is normal in $L(H')$, and thus $\sigma(\phi(\pi(B))C) = \sigma(C\phi(\pi(B)))$ for every $C \in L(H')$. In particular, $\sigma_e(AB) = \sigma(\pi(AB)) = \sigma(\phi(\pi(AB))) = \sigma(\phi(\pi(A))\phi(\pi(B))) = \sigma(\phi(\pi(B))\phi(\pi(A))) = \sigma(\pi(BA)) = \sigma_e(BA)$ for every $A$ in $L(H)$. Thus (1) holds.

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References


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