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SPECTRUM OF THE PRODUCTS OF OPERATORS AND COMPACT PERTURBATIONS

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ABSTRACT. In this paper we will give a complete characterization of the operator B which satisfies the spectral condition $\sigma(AB) = \sigma(BA)$ (resp. $\sigma_e(AB) = \sigma_e(BA)$) for every A in $L(H)$ and also a spectral characterizations of the product of finitely many normal (resp. essentially normal) operators.

0. INTRODUCTION

Let H be an infinite-dimensional complex Hilbert space, $L(H)$ be the algebra of all bounded linear operators on H, and $K(H)$ be the compact operator ideal of $L(H)$. For B in $L(H)$, $\sigma(B)$ and $\sigma_e(B)$ denote the spectrum and the essential spectrum of B respectively.

For B in $L(H)$, we say that B is consistent in invertibility (with respect to multiplication) or, briefly, a CI operator if, for each A in $L(H)$, AB and BA are invertible or noninvertible together. By Jacobson's Theorem (for $A, B \in L(H)$, the nonzero elements of $\sigma(AB)$ and $\sigma(BA)$ are the same), B is a CI operator if and only if $\sigma(AB) = \sigma(BA)$ for every A in $L(H)$. Thus if A and B are CI operators, then so is AB. Our problem is: which elements in $L(H)$ are CI operators? Paper [6] caused the authors to think that this is a significant problem and remains to be considered.

The main purpose of this paper is to give a complete characterization of B in $L(H)$ which satisfies the spectral condition $\sigma(AB) = \sigma(BA)$ (resp. $\sigma_e(AB) = \sigma_e(BA)$) for every A in $L(H)$. We also give a complete characterization of the CI operators which are invariant under compact perturbations.

1. FUNDAMENTAL THEOREM

Every B in $L(H)$ must be included in one of the following five cases, and in each of them the problem is definitely answered. Theorem 11.2 in [7, p. 224] is useful in the proofs below.

Case 1. If B is invertible, then B is a CI operator.

Proof. It is sufficient to note that for every A in $L(H)$, $AB = B^{-1}(BA)B$.

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Case 2. If $\operatorname{ran} B$ is not closed, then B is a CI operator.

Proof. It follows from $\operatorname{ran} BA \subseteq \operatorname{ran} B \subset \overline{\operatorname{ran} B} \subseteq H$ for every A in $L(H)$ that BA is not invertible.

It is now to be proved that, for every A in $L(H)$, AB is also not invertible. If, for some $A \in L(H)$, AB were invertible, the expression $(AB)^{-1}AB = (AB)^{-1}(AB) = I$ indicates that B is bounded from below. Then $\operatorname{ran} B$ is closed, which contradicts the assumption.

Case 3. If $\ker B \neq 0$ and $\overline{\operatorname{ran} B} \subset H$, then B is a CI operator.

Proof. For each A in $L(H)$, $\ker AB \supseteq \ker B \neq 0$ implies that AB is not invertible and $\operatorname{ran} BA \subseteq \overline{\operatorname{ran} B} \subset H$ implies that BA is not invertible.

Case 4. If $\ker B = 0$ and $\operatorname{ran} B = \overline{\operatorname{ran} B} \subset H$, then B^*B is invertible while BB^* is not invertible, and so B is not a CI operator.

Proof. It follows from $\operatorname{ran} BB^* \subseteq \operatorname{ran} B \subset H$ that BB^* is not invertible.

Since B has closed range if and only if B^*B has (see [2]), this together with the fact that B^*B is one-to-one and has dense range implies that B^*B is invertible.

Case 5. If $\ker B \neq 0$ and $\operatorname{ran} B = \overline{\operatorname{ran} B} = H$, then B^*B is not invertible while BB^* is invertible, and so B is not a CI operator.

Proof. It follows from $\ker B \neq 0$ and $\operatorname{ran} B = \overline{\operatorname{ran} B} = H$ that $\ker B^* = 0$ and $\operatorname{ran} B^* = \overline{\operatorname{ran} B^*} \subset H$. Therefore, by replacing B by B^* in the proof of Case 4, we obtain that B^*B is not invertible and BB^* is invertible.

By the results just proved above we can conclude

Theorem 1.1. *$B \in L(H)$ is a CI operator if and only if one of the following three mutually disjoint cases occurs*:

(1) *B is invertible.*
(2) *$\operatorname{ran} B$ is not closed.*
(3) *$\ker B \neq 0$ and $\operatorname{ran} B = \overline{\operatorname{ran} B} \subset H$.*

Corollary 1.2. *$B \in L(H)$ is a CI operator if and only if B^*B and BB^* are invertible or noninvertible together, i.e., $\sigma(B^*B) = \sigma(BB^*)$.*

The following corollary is a natural complement to the above results and its proof is straightforward.

Corollary 1.3. *Let $B \in L(H)$. If $\ker B = 0 = \ker B^*$, then B is a CI operator.*

Remark 1.4. From the proofs above, we can also see that the CI operators can be classified into two parts: (1) there is an A in $L(H)$ such that AB and BA are invertible together (this is the case if and only if B is invertible); and (2) for all A in $L(H)$, AB and BA are always noninvertible (if and only if either $\operatorname{ran} B$ is nonclosed or $\ker B \neq 0$ and $\operatorname{ran} B = \overline{\operatorname{ran} B} \subset H$).

Remark 1.5. B is a CI operator if and only if so is B^*.

2. Examples

By the preceding results, normal, compact, and invertible operators are immediately examples of CI operators and so are their products. Next we shall consider their generalizations.

An operator $B \in L(H)$ such that $\|Bx\| \geq \|B^*x\|$ for each x in H is called hyponormal. Obviously, $\ker B \subseteq \ker B^*$.

2.1. *If $B \in L(H)$ is hyponormal and $\operatorname{ran} B$ is closed, then B is a CI operator if and only if* (1) $\ker B \neq 0$, *or* (2) $\ker B^* = 0$.

Note that if $\operatorname{ran} B$ is not closed, then, from Theorem 1.1, B is a CI operator.

Proof. The conclusion can be obtained, when $\ker B \neq 0$, from $\ker B^* \supseteq \ker B \neq 0$ and Theorem 1.1 (3), and, when $\ker B^* = 0$, from $\ker B \subseteq \ker B^* = 0$ and Corollary 1.3.

If B is a CI operator, then one of the two cases (1) and (3) in Theorem 1.1 must occur. Case (1) implies $\ker B^* = 0$ and case (3) implies $\ker B \neq 0$.

2.2. *If $B \in L(H)$ is hyponormal, then B is a CI operator if and only if either*:

(1) *BB^* is invertible, or*

(2) *B^*B is noninvertible.*

Proof. Necessity is trivial.

If BB^* is invertible, then it follows from $\operatorname{ran} B \supseteq \operatorname{ran} BB^* = H$ and $\ker B \subseteq \ker B^* = \ker BB^* = 0$ that B is invertible, hence B^*B is invertible. This also leads to that, if B^*B is noninvertible, then so is BB^*.

Remark 2.3. We say that $B \in L(H)$ is M-hyponormal if there is an $M > 0$ such that

$$\|(B-\lambda)^*x\| \leq M\|(B-\lambda)x\|$$

for all $x \in H$ and all complex number λ. Clearly, 2.1 and 2.2 remain true for M-hyponormal operator B with the proof unchanged.

Since $B \in L(H)$ is an isometry if and only if $B^*B = I$, we have

2.4. *If $B \in L(H)$ is an isometry, then B is a CI operator if and only if B is unitary.*

2.5. *If B is invertible, then $B+K$ is a CI operator for every $K \in K(H)$. This follows from* $\operatorname{ind}(B+K) = \operatorname{ind} B = 0$. *It should be noted that there exist Fredholm operators which are not CI operators.*

2.6. *If $B \in L(H)$ such that $\sigma(B)$ is a singleton, then $B+K$ is a CI operator for every K in $K(H)$. In particular, if B is a Riesz operator, then B is a CI operator.*

Recall that B is a Riesz operator if and only if $\sigma_e(B) = 0$.

Proof. If $\sigma(B) \neq 0$, then this follows from 2.5. Now suppose that $\sigma(B) = 0$. Then $\sigma_e(B+K) = 0 = \sigma_e(B)$, which implies that $B+K$ is not semi-Fredholm. By Theorem 2.5 in [4, p. 356], we have that either $\operatorname{ran}(B+K)$ is not closed or $\dim\ker(B+K) = \dim\ker(B+K)^* = \infty$. Thus, by Theorem 1.1, $B+K$ is a CI operator. If B is a Riesz operator, it is known from [5, Theorem 3.3] that $B = C+K$, where $\sigma(C) = 0$ and $K \in L(H)$; whence, B is a CI operator.

3. Compact perturbations and normality

In this section, we shall give a spectral characterization of the operators which are the products of finitely many normal operators (resp. essentially normal operators). In what follows, H will be a fixed separable complex Hilbert space.

The following Theorem 3.1 is due to [8, Theorem 1.1].

Theorem 3.1. *If $B \in L(H)$, then the following statements are equivalent:*

(1) *B is the product of finitely many normal operators.*

(2) $\dim\ker B = \dim\ker B^*$ *or* $\operatorname{ran} B$ *is not closed.*

(3) *B is the norm limit of a sequence of invertible operators.*

We shall prove

Theorem 3.2. *If $B \in L(H)$, then the following conditions are equivalent to those in Theorem* 3.1:

(4) *$\sigma(A(B+K)) = \sigma((B+K)A)$ for every A in $L(H)$ and K in $K(H)$, i.e., $B+K$ is a CI operator for every K in $K(H)$.*

(5) *$\sigma(A(B+F)) = \sigma((B+F)A)$ for every A in $L(H)$ and finite rank operator F.*

Proof. (5) $\Rightarrow$ (2). If $\operatorname{ran} B$ is closed, we claim that $\dim\ker B = \dim\ker B^*$. Otherwise, we may assume that $\dim\ker B < \dim\ker B^*$. Then, by Proposition 3.21 in [4, p. 366], there exists a finite rank operator F such that $\ker(B+F) = 0$ and $\operatorname{ind}(B+F) = \operatorname{ind} B \neq 0$. Therefore, by Theorem 1.1, $B+F$ is not a CI operator, which leads to a contradiction.

(4) $\Rightarrow$ (5) is obvious.

(2) $\Rightarrow$ (4). Let $K \in K(H)$. If $\operatorname{ran}(B+K)$ is not closed, then, by Theorem 1.1, $B+K$ is a CI operator. Now suppose that $\operatorname{ran}(B+K)$ is closed. If $\dim\ker(B+K) < \infty$, then $\dim\ker B < \infty$ and $\operatorname{ran} B$ is closed, thus $\dim\ker B = \dim\ker B^*$ which implies $\dim\ker(B+K) = \dim\ker(B+K)^*$. If $\dim\ker(B+K) = \infty$, a similar argument shows that $\dim\ker(B+K)^* = \infty$. Therefore, in these two cases, $B+K$ is a CI operator.

Corollary 3.3. *If $\sigma(B)$ is a singleton, then B satisfies the conditions in Theorem* 3.1.

Corollary 3.4. *Let $B \in L(H)$ be an essentially normal operator, i.e. $B^*B - BB^* \in K(H)$. Then the following statements are equivalent:*

(1) *There is a normal operator N and a compact operator K such that $B = N + K$.*

(2) *$p(B) + K$ is a CI operator for any polynomial p and any $K \in K(H)$.*

(3) *$p(B)$ is the product of finitely many normal operators for any polynomial p.*

This corollary may be regarded as a new version of the BDF theorem [3].

Proof. (1) $\Rightarrow$ (2) $\Leftrightarrow$ (3) follows from Theorem 3.2 immediately. Now we show that (2) implies (1). By the Brown-Douglas-Fillmore theorem [3], it is sufficient to show that $\operatorname{ind}(B - \lambda I) = 0$ whenever $\lambda \notin \sigma_e(B)$. If there exists $\lambda \notin \sigma_e(B)$ such that $\operatorname{ind}(B - \lambda I) \neq 0$, we may assume that $\dim\ker(B - \lambda I) < \dim\ker(B - \lambda I)^*$. Then, by Proposition 3.21 in [4, p. 366], there is $K \in K(H)$ such that $\ker(B - \lambda I + K) = 0$ and $\operatorname{ind}(B - \lambda I + K) = \operatorname{ind}(B - \lambda I) \neq 0$. Thus, by Theorem 1.1, $B - \lambda I + K$ is not a CI operator, which contradicts the assumption.

Remark 3.5. There exists a nonessentially normal operator which satisfies the conditions (2) and (3) of Corollary 3.4. For example, let $B \in L(H)$ such that $\sigma(B) = 0$ and $B \notin K(H)$. By $\sigma_e(B) = 0$ and $B \in K(H)$, B is not essentially normal; but, by Corollary 3.3, $p(B)$ is the product of finitely many normal operators for every polynomial p.

Lemma 3.6. *If $\sigma_e(AB) = \sigma_e(BA)$ for every A in $L(H)$, then one of the following statements holds*:

(a) *B is the product of finitely many normal operators.*
(b) *B is a Fredholm operator.*

Proof. From Theorem 3.1, it is sufficient to show that if B is semi-Fredholm, then B is Fredholm.

We may assume that $\dim\ker B < \infty$ and $\operatorname{ran} B$ is closed. Then there is $A \in L(H)$ such that $AB - I \in K(H)$; thus, $0 \notin \sigma_e(AB) = \sigma_e(BA)$, which implies that B is Fredholm.

The following theorem, an analogue of Theorem 3.2 in the Calkin algebra, gives a spectral characterization of compact perturbations of the product of finitely many essentially normal operators.

Theorem 3.7. *Let $B \in L(H)$. Then the following statements are equivalent*:

(1) *$\sigma_e(AB) = \sigma_e(BA)$ for every A in $L(H)$.*

(2) *B is a compact perturbation of the product of finitely many essentially normal operators*; *i.e., $\pi(B)$ is the product of finitely many normal elements in $L(H)/K(H)$.*

(3) *B is the norm limit of a sequence of Fredholm operators*; *i.e., $\pi(B)$ is the norm limit of invertible elements in $L(H)/K(H)$, where π is the canonical map of $L(H)$ onto $L(H)/K(H)$.*

Proof. From [1, Theorem 4] we have that (3) is equivalent to the condition that $\dim\ker B = \dim\ker B^*$, $\operatorname{ran} B$ is not closed, or B is Fredholm. Thus (1) implies (3) by Lemma 3.6.

(3) $\Rightarrow$ (2). It is sufficient to prove that, if B is Fredholm, then B must satisfy (2).

Let ϕ be a faithful unital *-representation of the Calkin algebra $L(H)/K(H)$ on a Hilbert space H'. Then $\phi(\pi(B))$ is invertible in $\operatorname{ran}\phi$. Let $R = (\phi(\pi(B))^*\phi(\pi(B)))^{1/2} \in \operatorname{ran}\phi$ and $S = \phi(\pi(B))R^{-1} \in \operatorname{ran}\phi$. Then $S^*S = SS^* = I$, and thus $\phi(\pi(B)) = SR$ is the product of normal operators. If $\pi(B_1) = \phi^{-1}(S)$, $\pi(B_2) = \phi^{-1}(R)$, then B_1 and B_2 are essentially normal operators and $\pi(B) = \pi(B_1B_2)$. Therefore, B satisfies (2),

(2) $\Rightarrow$ (1). Obviously, we may suppose that B is essentially normal. Let ϕ be as above. Then $\phi(\pi(B))$ is normal in $L(H')$, and thus $\sigma(\phi(\pi(B))C) = \sigma(C\phi(\pi(B)))$ for every $C \in L(H')$. In particular, $\sigma_e(AB) = \sigma(\pi(AB)) = \sigma(\phi(\pi(AB))) = \sigma(\phi(\pi(A))\phi(\pi(B))) = \sigma(\phi(\pi(B))\phi(\pi(A))) = \sigma(\pi(BA)) = \sigma_e(BA)$ for every A in $L(H)$. Thus (1) holds.

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References

1. R. Bouldin, *The essential minimum modulus*, Indiana Univ. Math. J. **30** (1981), 514–517.
2. ——, *The product of operators with closed range*, Tôhoku Math. J. **25** (1973), 359–363.
3. L. Brown, R. Douglas, and P. Fillmore, *Unitary equivalence modulo the compact operators and extensions of C^*-algebras*, Proc. Conf. Operator Theory, Lecture Notes in Math., vol. 345, Springer-Verlag, New York, 1973, pp. 58–128.

4. J. B. Conway, *A course in functional analysis*, Springer-Verlag, New York, 1985.

5. H. R. Dowson, *Spectral theory of linear operators*, London Math. Soc. Monographs (N.S.), vol. 12, Academic Press, New York, 1978.

6. M. Hladnik and M. Omladič, *Spectrum of the product of operators*, Proc. Amer. Math. Soc. **102** (1988), 300–302.

7. A. E. Taylor and D. C. Lay, *Introduction to functional analysis*, Wiley, New York, 1980.

8. P. Y. Wu, *Products of normal operators*, Canad. J. Math. **40** (1988), 1322–1330.

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